Time evolution of an anharmonic oscillator in an initial Holstein-Primakoff SU(1,1) coherent state

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(Received 11 October 1988)

We have studied the interaction of the squeezed light with the nonabsorbing nonlinear medium in the framework of the model of an anharmonic oscillator initially in a Holstein-Primakoff SU(1,1) coherent state. We have shown that the initial squeezing reappears periodically. Furthermore, due to the interaction with the nonlinear medium, the squeezing can be enhanced.

I. INTRODUCTION

Today, when it has become possible to generate the squeezed states of the electromagnetic field (for reviews on the subject see Refs. 1–3) in the laboratory,[4–7] new perspectives have opened in quantum optics—theoretical as well as experimental. In this situation it seems meaningful to analyze the influence of the material media on squeezing properties of the light field.

The recent paper by Gerry[8] has been devoted to the problem of the interaction of matter with the squeezed light. He has studied the solvable model of the nonabsorbing medium modeled as an anharmonic oscillator with the Hamiltonian[9–14] ($\hbar=1$)

$$H=\omega a^{\dagger}a+\frac{\lambda}{2}(a^{\dagger})^2a^2 \tag{1.1}$$

interacting with the squeezed light described as an SU(1,1) coherent state (CS).[15,16] Gerry has found that the Hamiltonian (1.1) can be rewritten in terms of the same SU(1,1) generators as those on which the SU(1,1) CS has been built (see below). In the framework of such a model it has been shown that the nonlinear medium tends to revoke the squeezing of an initially squeezed SU(1,1) CS. Moreover, the greater the initial squeezing the more rapidly it is revoked and at longer times the variances of the quadrature operators tend to oscillate, but seem to never become squeezed again.

In the present paper we will analyze the model of the anharmonic oscillator very similar to that considered by Gerry the typical feature of which are the periodical revivals of the initial squeezing. We will also show that due to the interaction with the nonlinear medium the initial squeezing gets enhanced.

II. MODEL

Gerry has studied in his paper[8] the dynamics of the nonlinear oscillator with the Hamiltonian (1.1) rewritten in terms of the generators K_0 and $K_\pm$ of the Lie algebra of the SU(1,1) group given in terms of the bosonic operators a and $a^\dagger$ ($[a,a^\dagger]=1$),

$$K_0=\tfrac{1}{4}(a^\dagger a+aa^\dagger),\quad K_+=\tfrac{1}{2}(a^\dagger)^2,\quad K_-=\tfrac{1}{2}(a)^2\ , \tag{2.1}$$

which obey the commutation relations

$$[K_0,K_\pm]=\pm K_\pm,\quad [K_-,K_+]=2K_0\ . \tag{2.2}$$

Using (2.1) the Hamiltonian (1.1) can be rewritten (up to constant factors) as follows:

$$H=wK_0+\lambda K_+K_-\ . \tag{2.3}$$

The SU(1,1) CS corresponding to the realization (2.1) of the generators K_0 and $K_\pm$ is given by the relation[8,15–16]

$$|\xi\rangle=(1-|\xi|^2)^{1/4}\sum_{n=0}^{\infty}\left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})n!}\right]^{1/2}\xi^n|2n\rangle\ , \tag{2.4}$$

where $\xi=|\xi|e^{i\varphi}$ $(0\le|\xi|\le1)$.

In our analysis we will also consider the Hamiltonian in the form (2.3), but with a different realization of the generators K. Namely, we will use the Holstein-Primakoff realization of the SU(1,1) Lie algebra,[17]

$$K_0=\tfrac{1}{2}(a^\dagger a+aa^\dagger),\quad K_+=\sqrt{N}a^\dagger,\quad K_-a\sqrt{N}\ , \tag{2.5}$$

where $N=a^\dagger a$ is the photon-number operator.

The SU(1,1) CS corresponding to the generators (2.5) is the special (one-photon) case of the multiboson Holstein-Primakoff SU(1,1) CS as discussed by Katriel and coworkers[18–20] which in the number state representation has the following form:

$$|\xi\rangle=(1-|\xi|^2)^{1/2}\sum_{n=0}^{\infty}\xi^n|n\rangle=\sum_{n=0}^{\infty}Q_n|n\rangle\ , \tag{2.6}$$

where $\xi=|\xi|e^{i\varphi}$ and $0\le|\xi|\le1$, $0\le\varphi\le2\pi$. This state is not a minimum-uncertainty state (its wave function is not a Gaussian one), nevertheless, it is a squeezed state with the variance in one quadrature less than the one associated with the coherent field (for details see Ref. 21).

The evolution of the system under consideration is given by the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle=H|\psi(t)\rangle\ . \tag{2.7}$$

Following the idea of Gerry, we will assume the initial state of the system (at $t=0$) to be an SU(1.1) CS (2.6) and for the state vector $|\psi(t)\rangle$ given by the Eq. (2.7) we obtain

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} Q_n \exp[-i(\omega n + \lambda n^2)t]|n\rangle \ . \quad (2.8)$$

For the purposes of the following calculations we will write down the mean values of the photon-number operator $\langle N\rangle$, photon amplitude $\langle a\rangle$, and the squared photon amplitude $\langle a^2\rangle$,

$$A_0 \equiv \langle N\rangle = \sum_{n=0}^{\infty} nP_n = |\xi|^2(1-|\xi|^2)^{-1} \equiv \bar{n} \ ,$$

$$A_1 \equiv \langle a\rangle e^{i(\omega t-\varphi)} = |\xi| \sum_{n=0}^{\infty} P_n e^{-i(2n+1)\tau}\sqrt{n+1} \ ,$$

$$A_2 = \langle a^2\rangle e^{2i(\omega t-\varphi)} = |\xi|^2 \sum_{n=0}^{\infty} P_n e^{-4i(n+1)\tau}\sqrt{(n+1)(n+2)} \ , \quad (2.9)$$

where $\tau=\lambda t$ and P_n is the distribution of the SU(1,1) CS (2.6),

$$P_n = |Q_n|^2 = (1-|\xi|^2)|\xi|^{2n} \ . \quad (2.10)$$

At the end of the present section we mention that the average photon number $\bar{n}$ is the integral of motion, which is the consequence of the commutation relation $[K_0,H]=0$.

III. LIGHT SQUEEZING

To analyze the squeezing properties of light we introduce two Hermitian time-dependent quadrature operators

$$a_1(t) = \tfrac{1}{2}(ae^{i(\omega t-\delta)} + a^\dagger e^{-i(\omega t-\delta)}) \ , \quad (3.1a)$$

$$a_2(t) = \frac{1}{2i}(ae^{i(\omega t-\delta)} - a^\dagger e^{-i(\omega t-\delta)}) \ , \quad (3.1b)$$

where δ is an arbitrary phase chosen to be equal to φ (the phase of the squeezing parameter ξ). One of the consequences of the commutation relation for the quadrature operators $a_i(t)$,

$$[a_1(t), a_2(t)] = \frac{i}{2} \ , \quad (3.2)$$

is the uncertainty relation

$$V_1(t)V_2(t) \geq \tfrac{1}{16} \ , \quad (3.3)$$

where $V_i(t)$ are the variances of the quadrature operators $a_i(t)$,

$$V_i(t) = \langle a_i^2(t)\rangle - \langle a_i(t)\rangle^2 \ . \quad (3.4)$$

Since the squeezed states are defined as the states with a smaller uncertainty in one quadrature of the field than that associated with the coherent field, the squeezing condition can be written as

$$V_i(t) < \tfrac{1}{4} \ \text{ for } i=1 \text{ or } 2 \ . \quad (3.5)$$

The variances of the quadrature operators a_i can be expressed through the mean values of the photon operators (2.9),

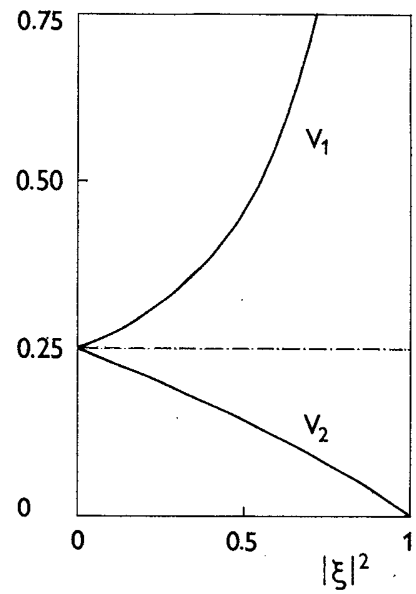

FIG. 1. Functions $V_i(t)$ at $t=0$ vs the squeezing parameter $|\xi|^2$.

$$V_1(t) = \tfrac{1}{2}[A_0 + \tfrac{1}{2} + \mathrm{Re}A_2 - 2(\mathrm{Re}A_1)^2] \ , \quad (3.6a)$$

$$V_2(t) = \tfrac{1}{2}[A_0 + \tfrac{1}{2} - \mathrm{Re}A_2 - 2(\mathrm{Im}A_1)^2] \ . \quad (3.6b)$$

IV. DISCUSSION

From the expressions for the functions $V_i(t)$ several properties of the variances of the quadrature operators follow.

(1) For the particular value of the phase δ ($\delta=\varphi$) the variance in the second quadrature is squeezed at $t=0$ $[V_2(t=0)<\frac{1}{4}]$. The functions $V_i(t)$ at $t=0$ versus the parameter $|\xi|^2$ are plotted in Fig. 1. It is typical for the

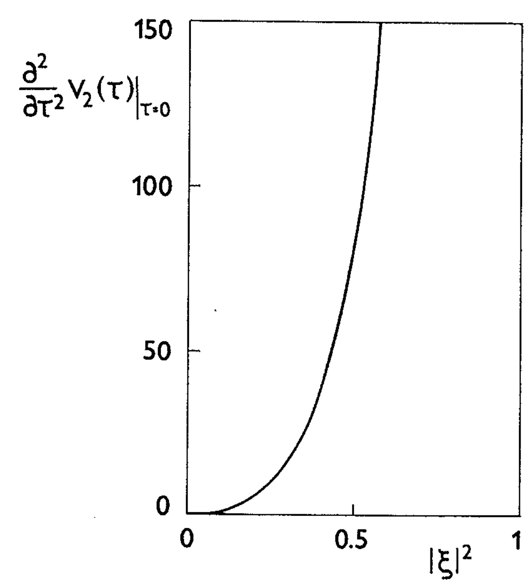

FIG. 2. Second derivative of the function $V_2(\tau)$ at $\tau=0$ vs $|\xi|^2$.

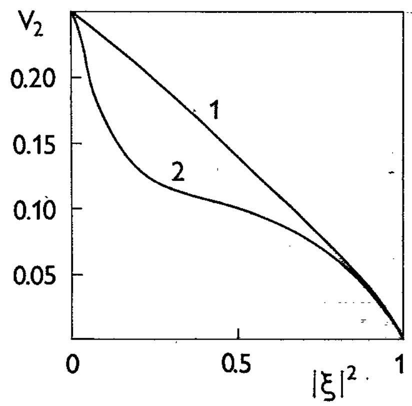

FIG. 3. Functions $V_2(\tau=0)$ (curve 1) and $V_2(\tau=\pi/2)$ (curve 2) vs $|\xi|^2$.

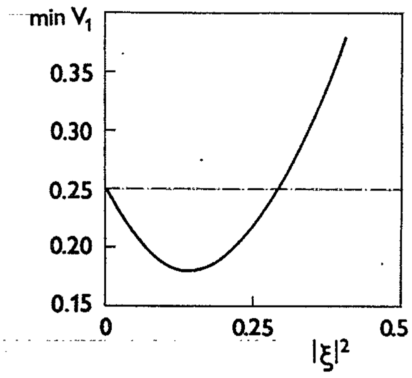

FIG. 5. Minimal values of the function $V_1(\tau)$ reached during the evolution vs $|\xi|^2$.

Holstein-Primakoff SU(1,1) CS (Ref. 21) that the degree of squeezing increases with increasing the intensity of the field (or, what is the same, with increasing $|\xi|^2$). The maximum (100%) squeezing is obtained for $\bar{n}\to\infty$ ($|\xi|^2\to 1$).

(2) The time evolution of the variances of the quadrature operators is described by the functions $V_i(t)$. These functions are strictly periodical with the period $T=\pi/\lambda$,

$$V_i(t)=V_i(t+\pi/\lambda) . \tag{4.1}$$

This means that if at $t=0$ one quadrature is squeezed, then this initial squeezing reappears periodically at later moments.

(3) One can find that in the initial moments of the evolution the squeezing becomes revoked. To determine how rapidly it is revoked we evaluate the time derivatives of the function $V_2(t)$ (the second quadrature is initially squeezed) at $t=0$ and examine their dependence on the parameter $|\xi|^2$. Since the first derivative at $t=0$ vanishes, we have to calculate the second one. This derivative is strictly positive and it increases with increasing the initial photon number; this means that (see Fig. 2)

$$\frac{\partial}{\partial|\xi|^2}\frac{\partial^2 V_2(\tau)}{\partial\tau^2}>0 . \tag{4.2}$$

We can say that the more photons in the initial state the more rapidly squeezing is revoked (this is the situation identical to that in Gerry's[8] case). Nevertheless, it should be stressed once more that the initial squeezing is periodically restored in the long-time scale.

(4) Furthermore, one can find that the global minimum of the function $V_2(t)$ is obtained not for $t=0$, but for $t=T/2$. Consequently, due to the interaction with a nonlinear medium squeezing of the light can be enhanced. Functions $V_2(\tau=0)$ and $V_2(\tau=\pi/2)$ versus $|\xi|^2$ are plotted in Fig. 3. It is seen that

$$V_2(\tau=0)\leq V_2(\tau-\pi/2) \tag{4.3}$$

for any value of the initial squeezing.

(5) In Fig. 4 time evolutions of the functions $V_1(t)$ and $V_2(t)$ for $|\xi|^2=0.1$ are plotted. From this figure it is seen that for weak intensities ($|\xi|^2\lesssim 0.3$) the first quadrature becomes squeezed during the evolution in spite of the fact that at $t=0$ this quadrature is unsqueezed. In Fig. 5 the minimum of the function $V_1(\tau)$ reached during the evolution versus the parameter $|\xi|^2$ is plotted. The maximum squeezing in this quadrature is obtained for $\tau\cong 28\pi/100$

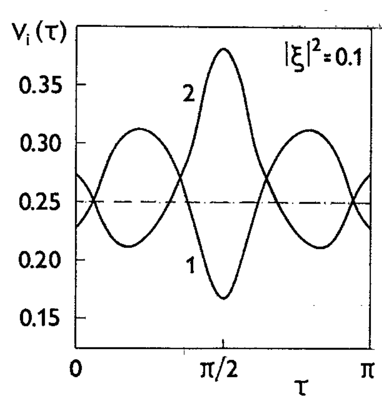

FIG. 4. Time evolution of the functions $V_2(\tau)$ (curve 1) and $V_1(\tau)$ (curve 2) for $|\xi|^2=0.1$.

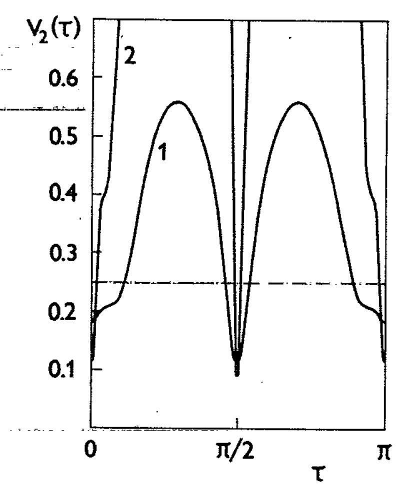

FIG. 6. Time evolution of the function $V_2(\tau)$ for $|\xi|^2=0.3$ (curve 1) and $|\xi|^2=0.6$ (curve 2).

and $|\xi|^2 \simeq 0.14$.

The time evolution of the function $V_2(\tau)$ for $|\xi|^2=0.3$ and 0.6 is given in Fig. 6. Typical features (1)–(4) of the functions $V_2(\tau)$ are clearly seen in this picture.

V. CONCLUSIONS

We can conclude that in the present model of the anharmonic oscillator interacting with the Holstein-Primakoff SU(1,1) CS, squeezing of the variances exhibits periodical revivals for any value of the initial squeezing. Furthermore, the interaction with a nonlinear medium leads to the enhancement of the initial squeezing.

The periodicity of the model described above [see Eq. (4.1)] is preserved for any initial state. In particular, if we suppose the light field to be in Glauber's coherent state at $t=0$, then during the evolution of the system one of the quadratures becomes squeezed and this squeezing reappears periodically.[22]

In the present paper we have analyzed the interaction of the squeezed light with the nonlinear medium without dissipations. To make the problem more realistic, the dissipations should be taken into account. This problem will be discussed elsewhere.

ACKNOWLEDGMENTS

The author is indebted to Prof. C. C. Gerry for turning his attention to the problem of the Holstein-Primakoff SU(1,1) CS interacting with the nonlinear medium.

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