Light in gravitational field.

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From Einstein equation we can derive following equation for the propagation of light in gravitational field

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\,\phi^2} + u = \frac{3}{2}u^2, \text{ for } r_g = 1, \tag{1}$$

where $u = \frac{1}{r}$, r, ϕ are polar coordinates in plane in which the propagation occurs and r_g is Schwarzschild radius. Since we are not interested in quantitative description we are free to consider simplified model with $r_g = 1$. This equation is, however, not analytically solvable and hence we will try to solve it numerically. The situation which interests us most is the one in which the light converges to one point, or equivalently emerges from one point.

1 Propagation of light

To do the proposed we need to consider initial conditions for the light. For light starting in distance r_0 from the source of gravitational field we need to have

$$u_0 = \frac{1}{r_0}. \tag{2}$$

Next, we will use following for setting the initial value for first derivative of u:

$$\left.\frac{\mathrm{d}\,u}{\mathrm{d}\,\phi}\right|_0 = \left.\frac{\mathrm{d}}{\mathrm{d}\,\phi}\left(\frac{1}{r}\right)\right|_0 = -\frac{1}{r^2}\left.\frac{\mathrm{d}\,r}{\mathrm{d}\,\phi}\right|_0 = -u_0^2\left.\frac{\mathrm{d}\,r}{\mathrm{d}\,\phi}\right|_0. \tag{3}$$

From Fig.1 we now see, that following holds:

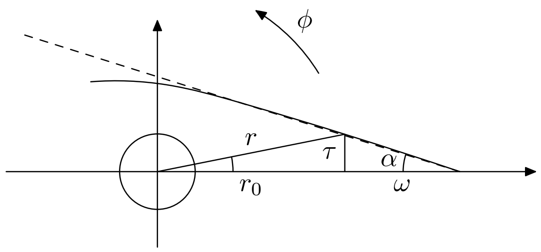

Figure 1: The situation considered needed to express the initial value of first derivative of u.

$$\operatorname{tg}\alpha = \frac{\tau}{\omega} \tag{4}$$

$$\tau = (r_0 - \omega)\,\mathrm{d}\,\phi \approx r_0\,\mathrm{d}\,\phi, \tag{5}$$

and hence

$$\operatorname{tg}\alpha = \frac{r_0\,\mathrm{d}\,\phi}{\omega}. \tag{6}$$

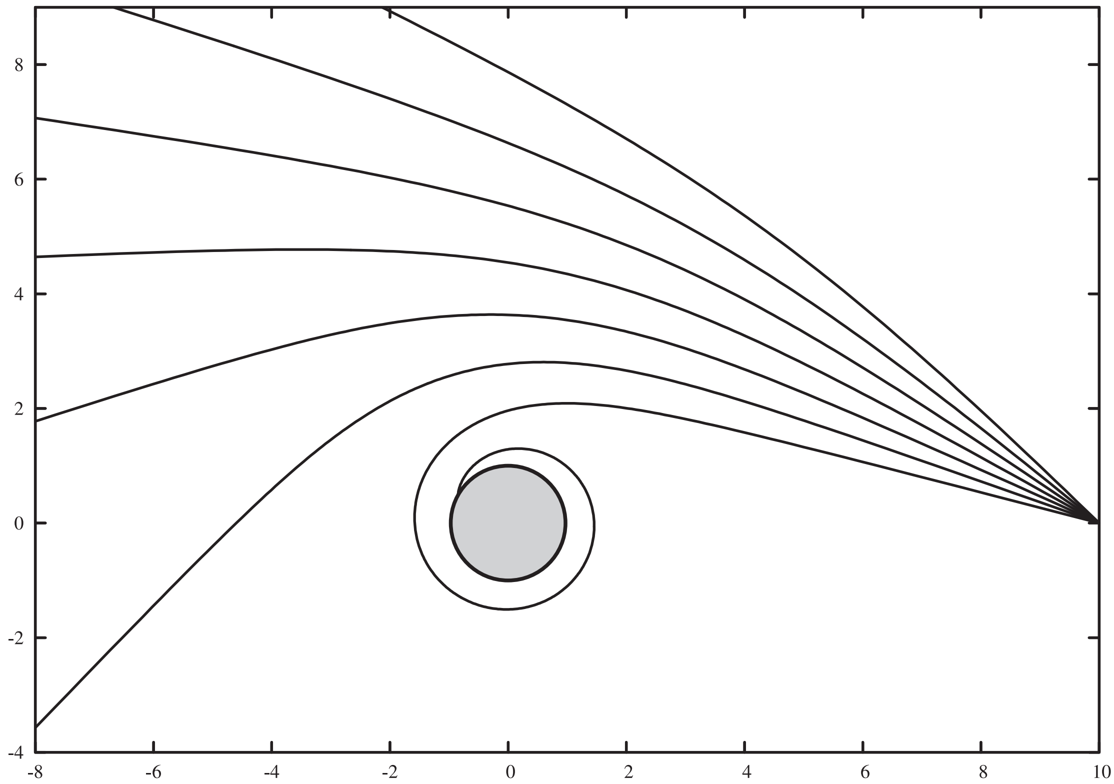

Figure 2: Propagation of light in gravitational field for various starting angles (15°, 20°, 25°, 30°, 35°, 40° and 45°). The light passing more closely to the source of field is curved more.

By using this Eq.(6) we get approximation

$$\frac{\mathrm{d}\, r}{\mathrm{d}\, \phi} \approx -\frac{\omega}{\mathrm{d}\, \phi} = -\frac{r_0}{\mathrm{tg}\, \alpha}. \tag{7}$$

To summarize obtained, we now have to solve

$$\frac{\mathrm{d}^2\, u}{\mathrm{d}\, \phi^2} = \frac{3}{2}u^2 - u \tag{8}$$

$$u_0 = \frac{1}{r_0} \tag{9}$$

$$\left.\frac{\mathrm{d}\, u}{\mathrm{d}\, \phi}\right|_0 = \frac{u_0}{\mathrm{tg}\, \alpha} \tag{10}$$

This second-order differential equation can be integrated using Stoermer's rule (from Numerical Recipes in C, `http://www.nr.com`). For various starting angles α we have obtatined results depicted in Fig.2. As we can see, the light is curved more for beams passing closer to the source of field. Several pathological situations may occur. Some of them are depicted in Fig.3. These situatins will be described in following section.

2 Seeing it...

To see the effect of the source of field we started with computation of Fig.4 depicting the situation, where light's position (in figure called height) is obtained in the moment, when it crosses the plane

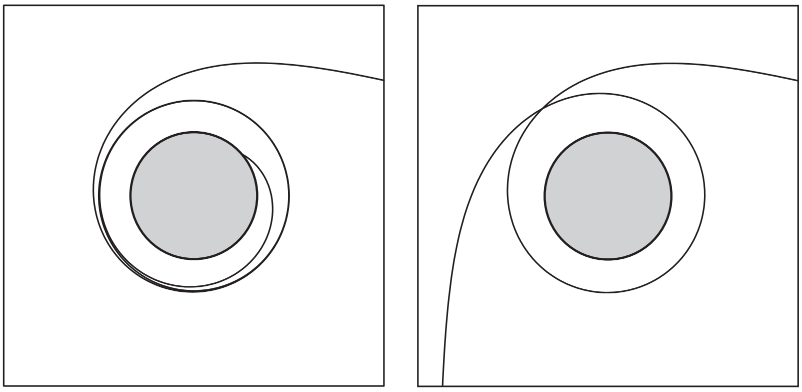

Figure 3: Propagation of light in gravitational field. On the left is a situation when light after cirling the source of gravitation field finally reaches the event horizon and on the right is the situation when the starting angle is only a little larger and the light after circling the source of field finally escapes.

behind the source of field, i.e. on the other side as the observer (the point of origin of light). Inverting this dependence (numerically) we can determine the angle, under which we will see different points originating on the plane. This will show us the deformation of different objects located behind the source of gravitational fieled. How an orthogonal grid is affected is depicted on the Fig.5.

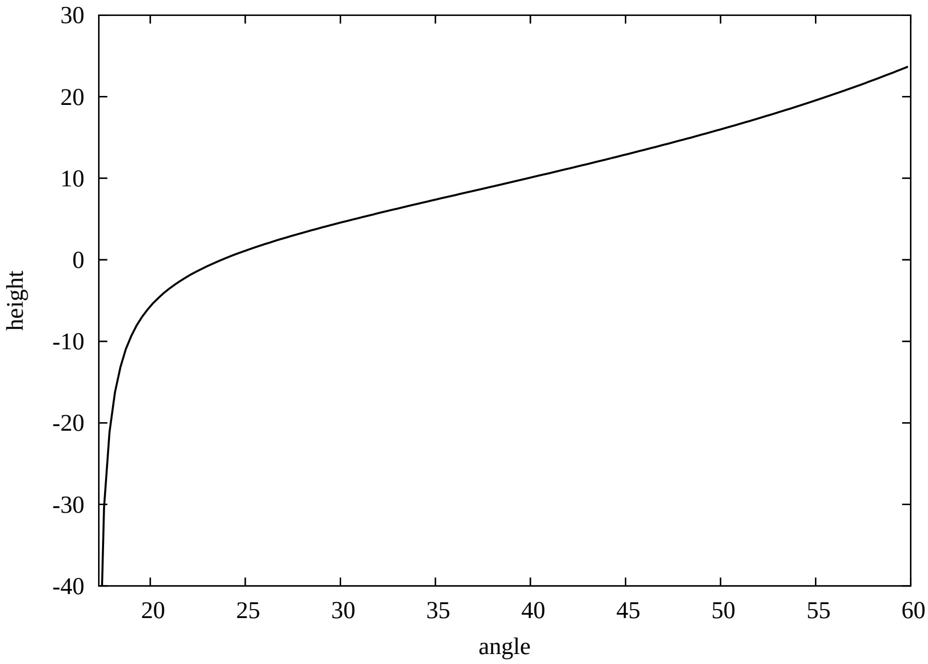

Figure 4: The figure depicts the y-position (height) of light passing the plane $x = -r_0$ when originating from point with $x = r_0$ in dependence on the starting angle (in degrees).

There are several features of deformation worth describing:

1. Yellow dashed line shows the Schwarzschild radius of source of the gravitational field, however light cannot come from nowhere within the black circle.

2. Red dashed line shows the direction under which on could “see" oneself as in mirror, i.e. the light originating in the point where the observer stands finally ends in the same point.

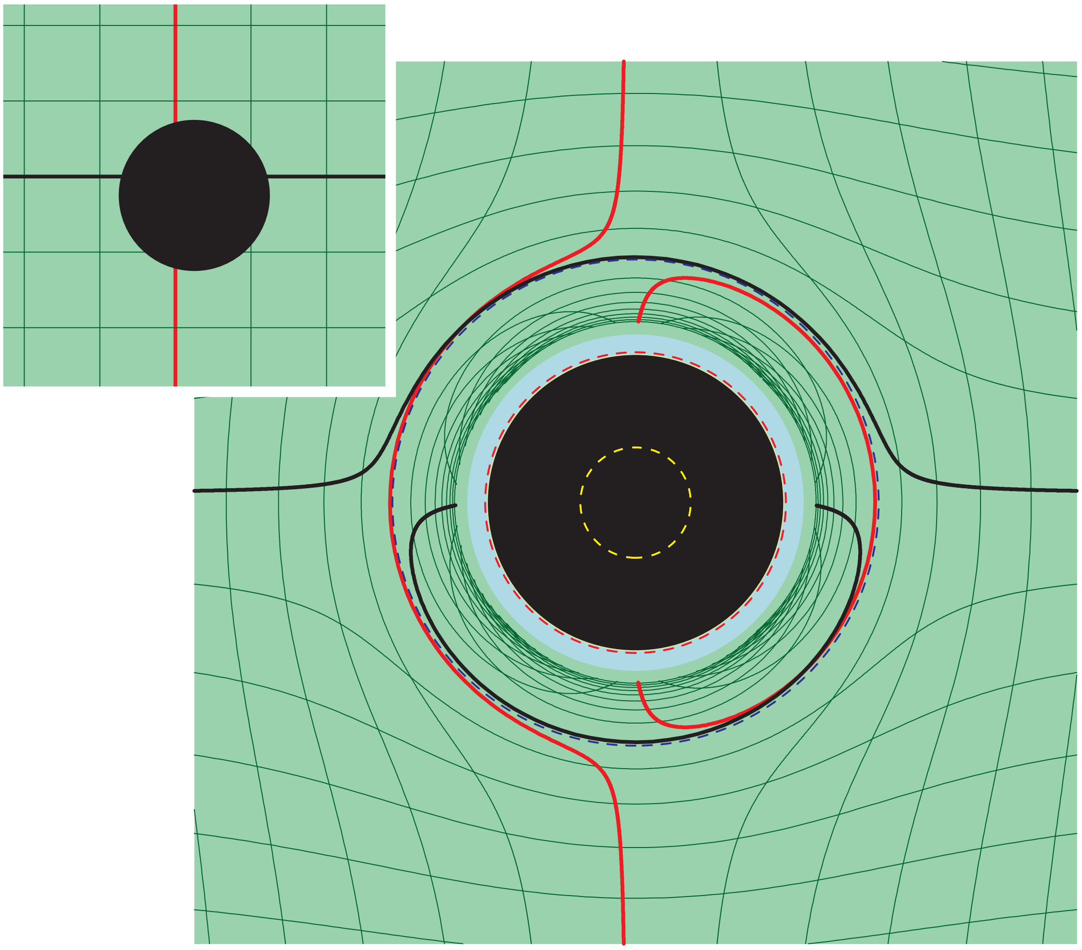

Figure 5: The deformation of othogonal grid located behind the source of gravitational field as seen by the observer located on the other side of source as the grid. (This picture was inspired by a similar image created by t'Hooft that he had on his web-page)

3. Between red dashed line and black circle is area from which originates light that has curved more than 180°.
4. Light coming from within the blue ring has curved more than 90° but less than 180°.
5. Blue dashed line depicts the point being opposite the observer.
6. Everything between blue dashed line and blue circle is the image created by curving light more than it is on the blue dashed line and creates center symmetric deformated image of the grid, e.g. the red axis ending below the blue ring is in fact going on the other end to infinity.
7. Everything beyond the blue dashed line is just deformed image.