very entangled spin chains

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very entangled spin chains

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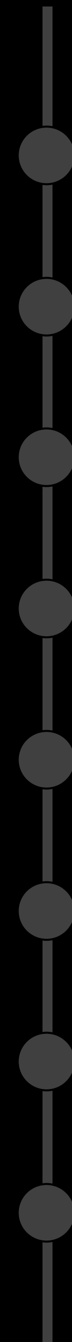
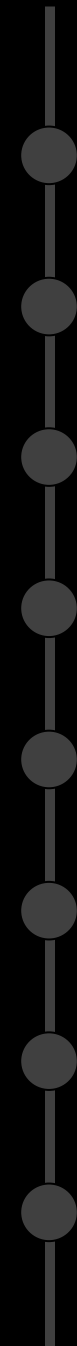
very
simple
spin
chains



classical

very
entangled
spin
chains



quantum

very
entangled
spin
chains

small qudits

translationally invariant

unique g. state

fun to play with

projections, perturbations…

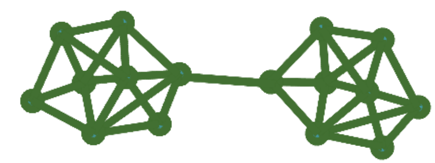
Motzkin & Dyck paths, congestion, matchings…

1 entangled chains

the area law, the gap & the entanglement

2 rules & states

unique, invariant & entangled

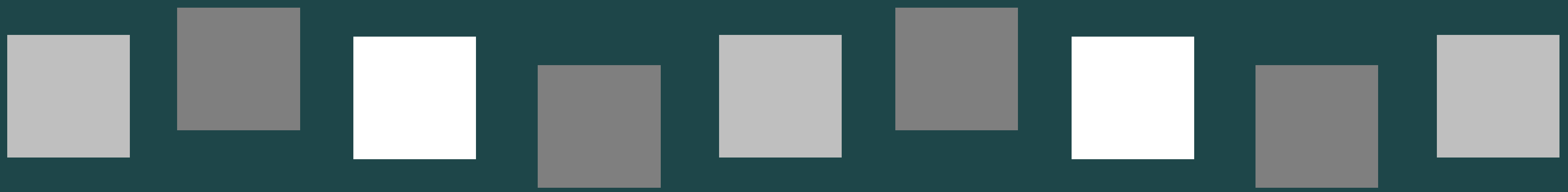


3 new models

with hope: families & complexity

pc, d=4 model

What can happen in 1D?



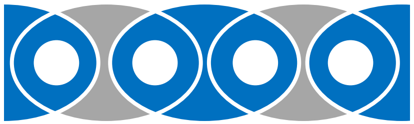
1 Spin chains & their ground states

- many interesting (standard) models

$$-J\sum_i Z_i Z_{i+1} + B\sum_i X_i$$

$$\sum_i (X_i X_{i+1} + Y_i Y_{i+1} + Z_i Z_{i+1})$$

- gapped systems: theorems
  - the area law [Hastings'07]
  - the algorithm [LanVazVid'13]
- in practice: numerics
  - DMRG [White'92], MPS+ [HaegOsbVers'13]
- hard or easy in general?
  - consistency of (qubit) RDM's?
  - translational invariance?

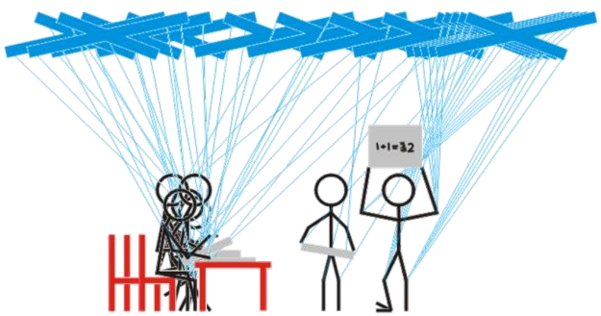


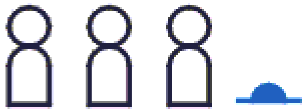
1 Spin chain ground states can be complex

- history of a computation, qudits

  QMA [AGIK'06, HNN'13]

$$\frac{1}{2}\big(|ab\rangle - |cd\rangle\big)\big(\langle ab| - \langle cd|\big)$$





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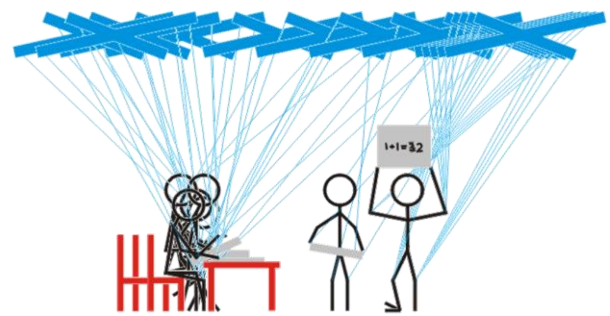
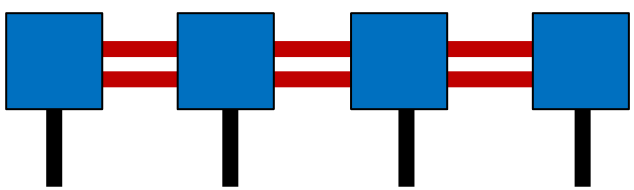


$$\tfrac{1}{2}\big(|ab\rangle - |cd\rangle\big)\big(\langle ab| - \langle cd|\big)$$

  NP-hard for poly-size MPS [SchCirVer'08]

- a ground state with poly(n) MPS entries?

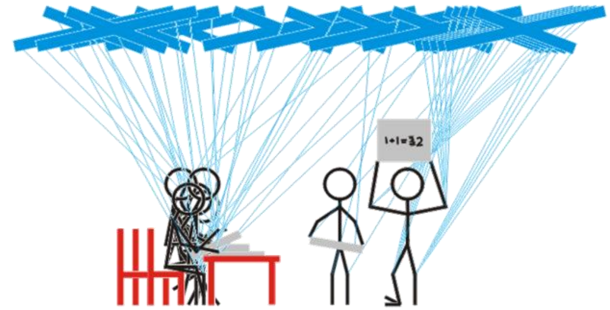
  smells like NP



$$c_{stuv} = \sum_{a,b,c=1}^{\chi} A^s_a B^t_{ab} C^u_{bc} D^v_c \qquad |\psi\rangle = \sum_{s,t,u,v=0}^{1} c_{stuv} |stuv\rangle$$

1 Spin chain ground states can be complex

- history of a computation, qudits

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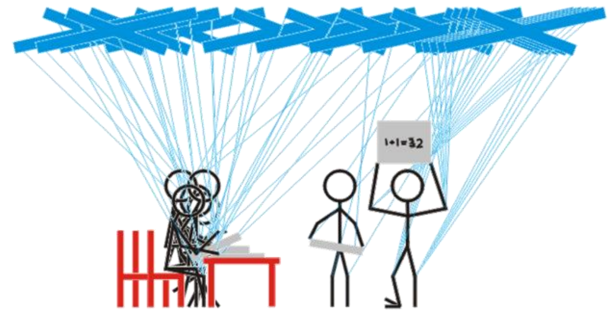
  any states natural for QCMA (MQA)?

- dissipative models [JohTicViola'15]

1 Spin chain ground states can be complex



- history of a computation, qudits
  - QMA [AGIK'06, HNN'13]
  - $\frac{1}{2}\left(|ab\rangle - |cd\rangle\right)\left(\langle ab| - \langle cd|\right)$
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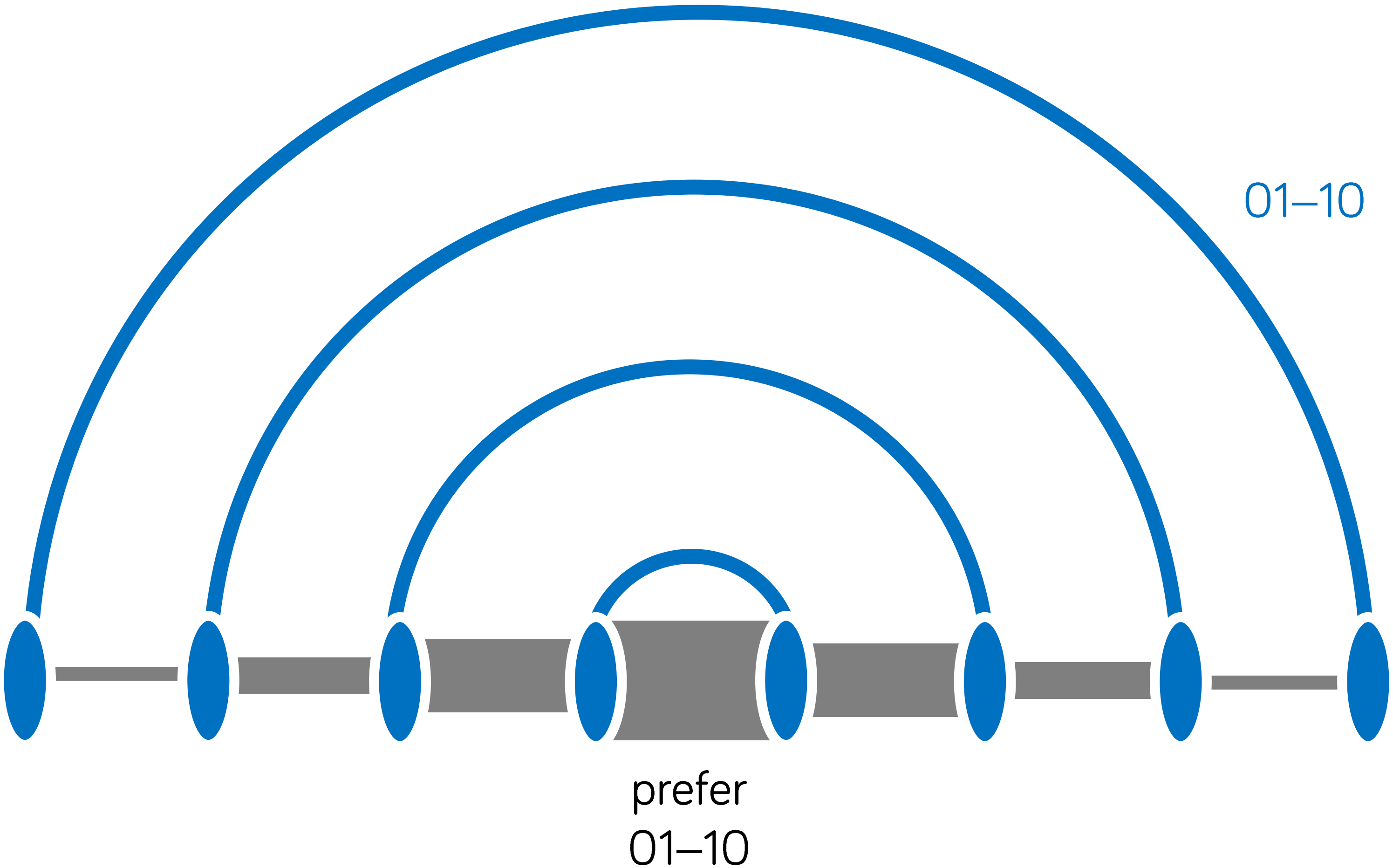
- a ground state with poly(n) MPS entries?
  - smells like NP
  - any states natural for QCMA (MQA)?
- dissipative models [JohTicViola'15] [Kastoryano]
- translationally invariant models?
  - high d: $\mathrm{QMA}_{\mathrm{EXP}}$ tiling [GottIrani'09] [Ozols+'15?]
  - entanglement & gap [GottHas'09] [Irani'09]

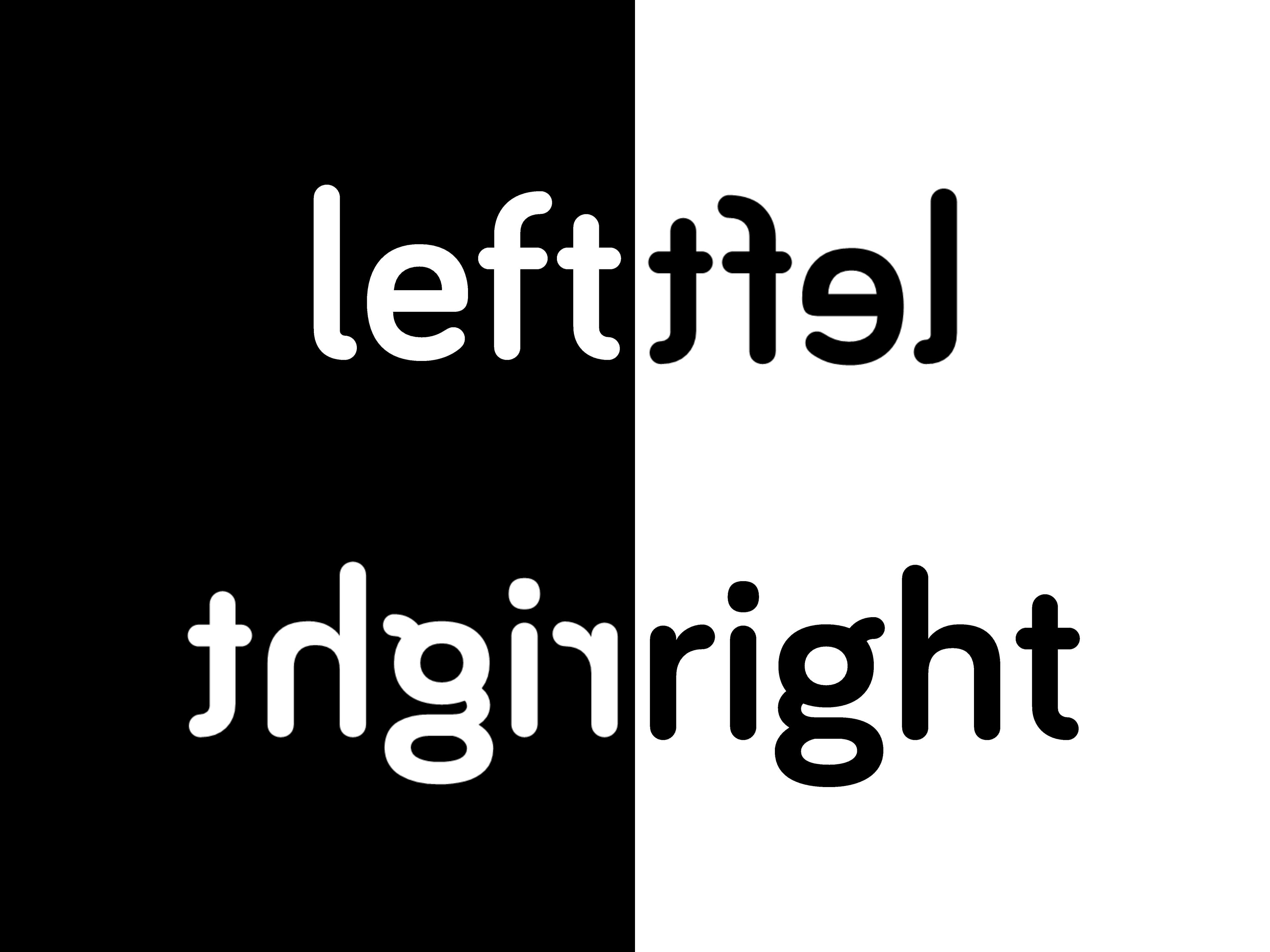
gap

enta
ngle
ment

1 Entanglement vs. gap

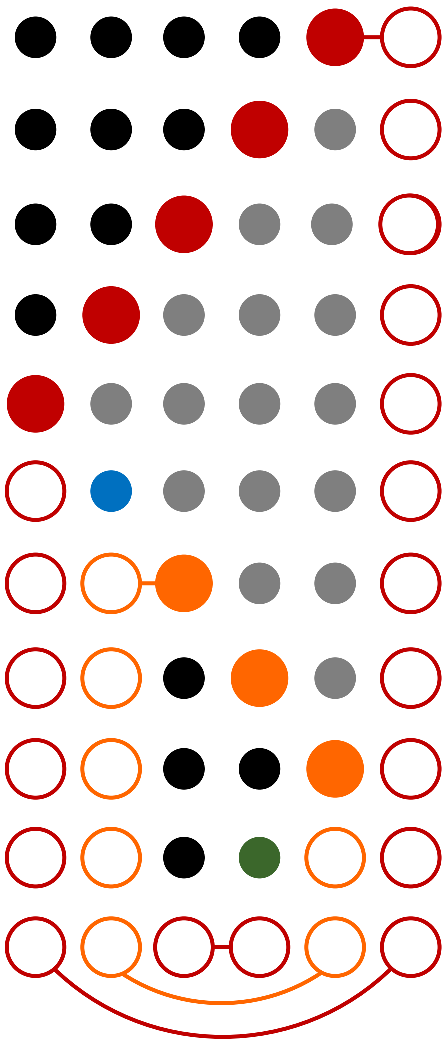


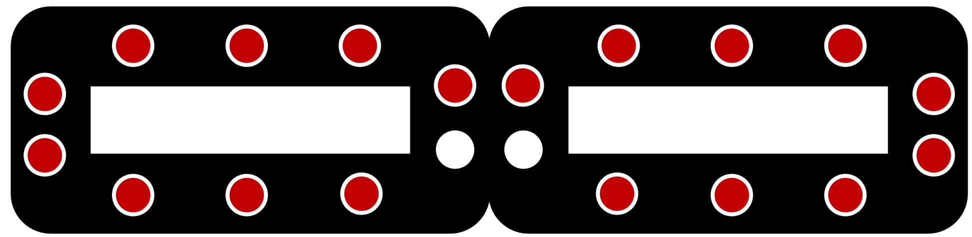
- exponentially small gap, renormalization [Latorre+]


left
right

1 Translationally invariant chains & the area/volume law

create/distribute EPRs [Irani'09]





synchronized wheels [GotHas'09]

(not trans. invariant)

1 Translationally invariant chains & the area/volume law

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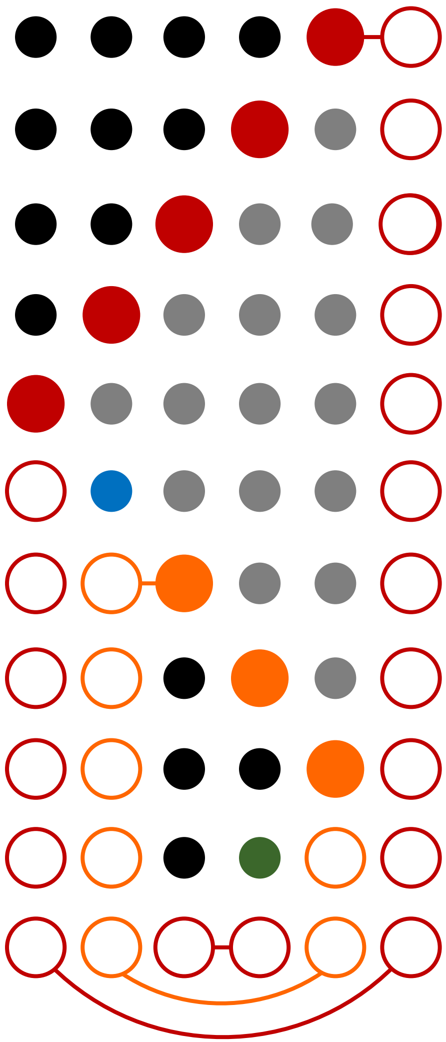

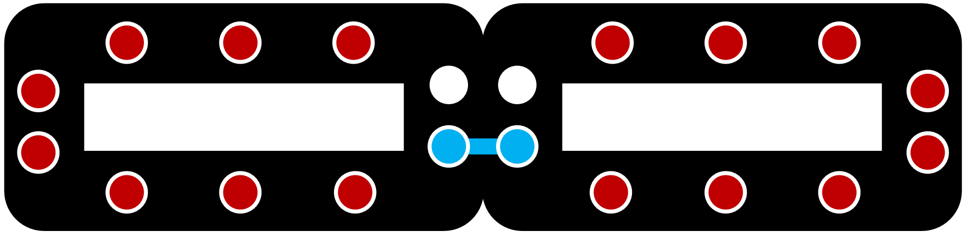

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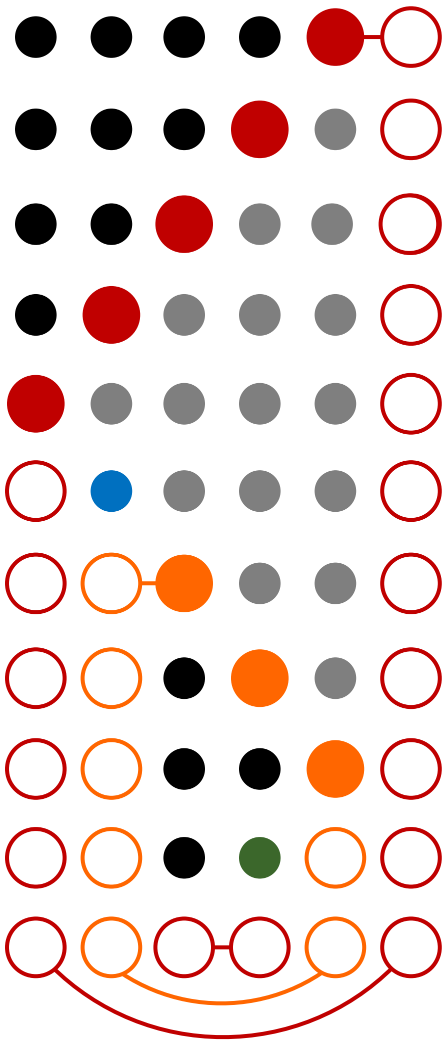

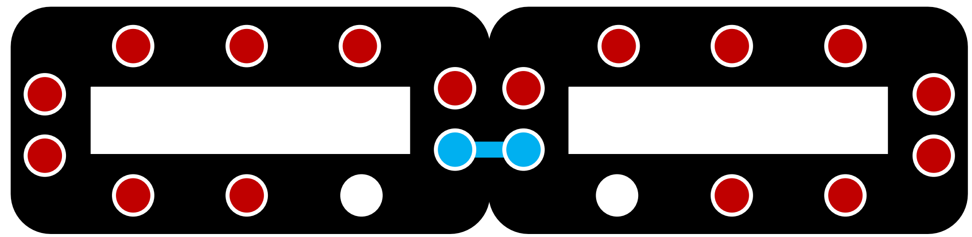

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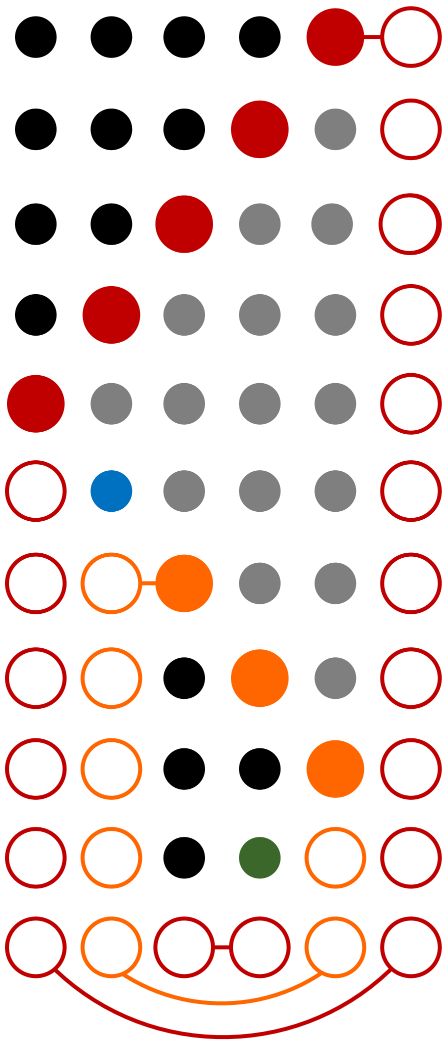

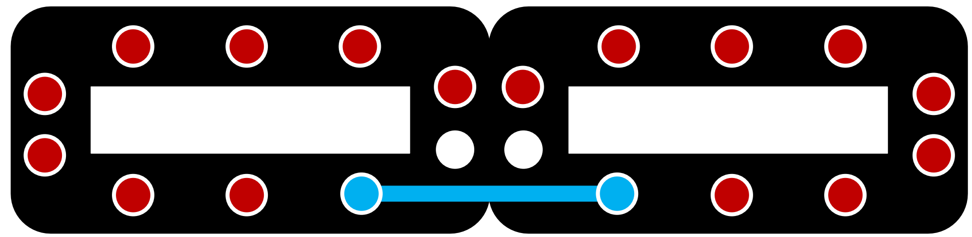

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Translationally invariant chains & the area/volume law

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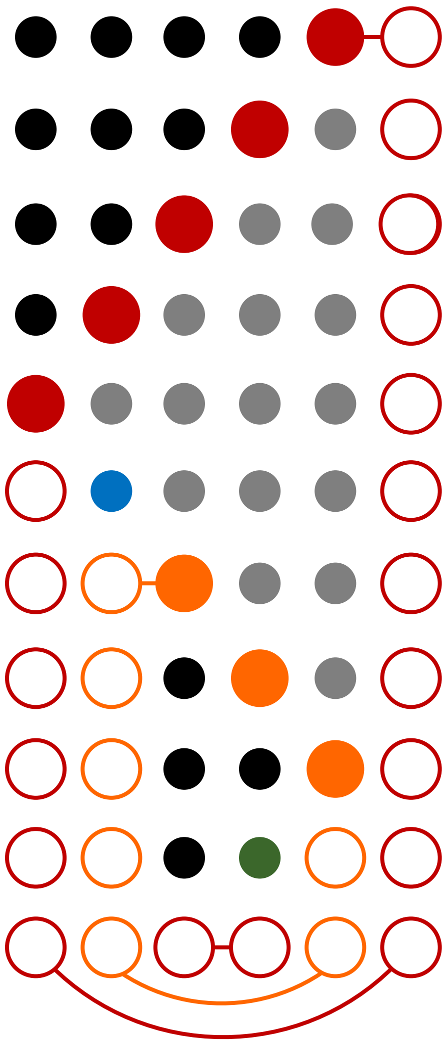
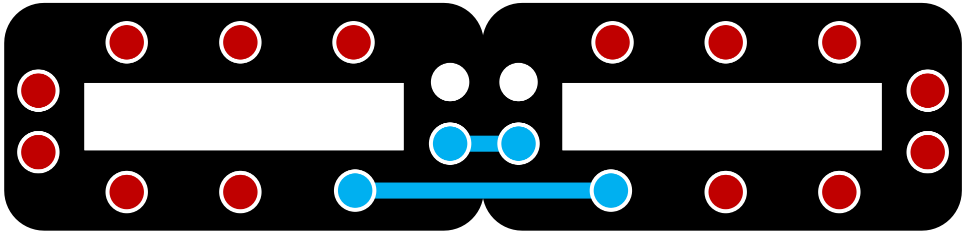




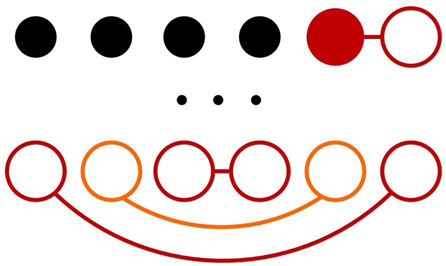
synchronized wheels [GotHas'09]

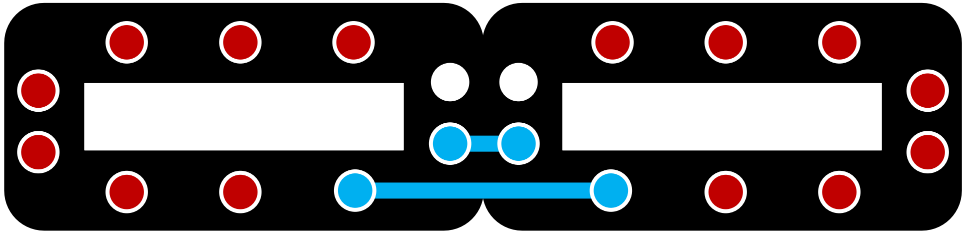
(not trans. invariant)

1 Translationally invariant chains & the area/volume law

create/distribute EPRs [Irani'09]

$$S \propto \Delta^{-\frac{1}{4}}$$





$$S \propto \Delta^{-\frac{1}{12}}$$

synchronized wheels [GotHas'09]

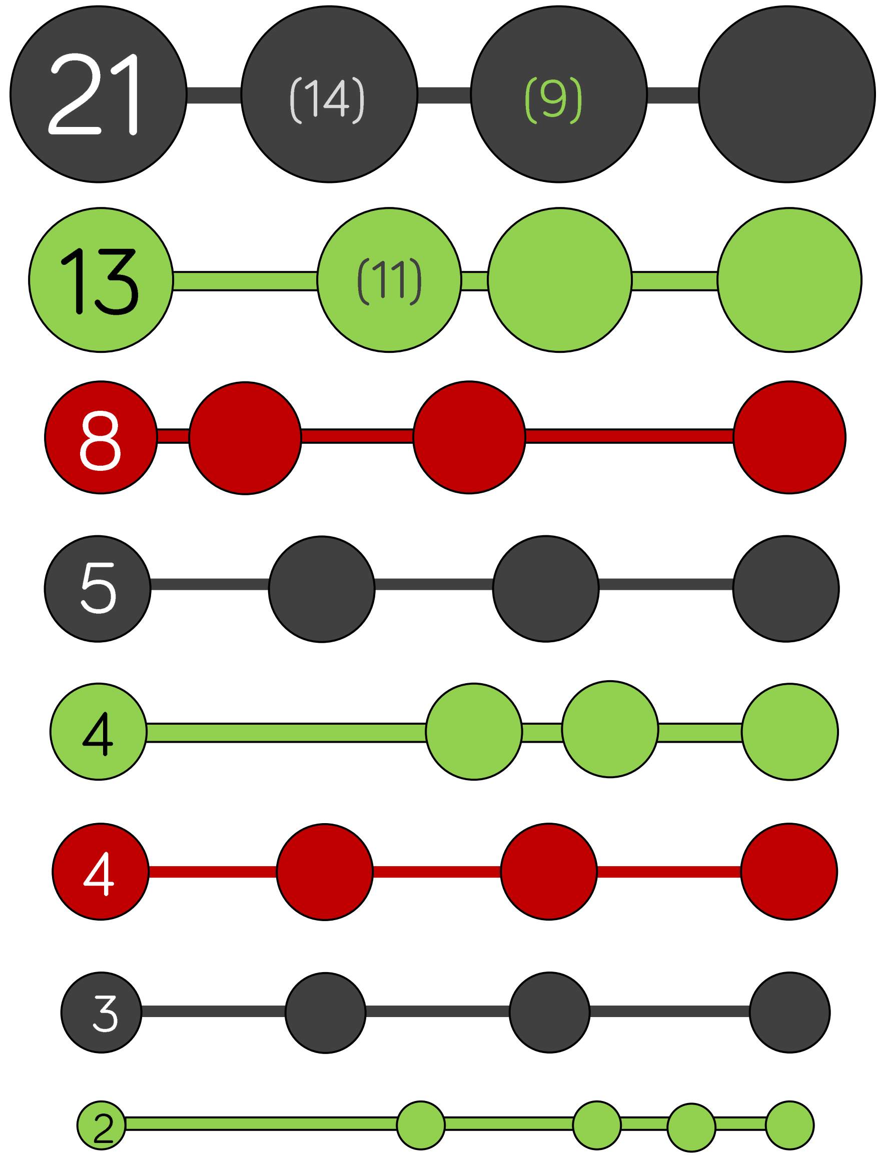
(not trans. invariant)

$$S \propto n$$

$$\Delta \propto n^{-\beta}$$

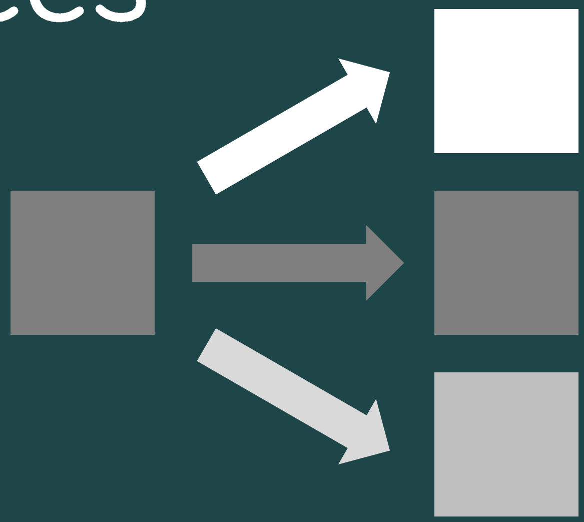
1 Ground states in 1D

How hard is it to find/describe them?



entropy ~ L	trans. invariant [Irani'09, GoHa'09]
QMA_1-comp.	history state [AGIK'06, N'08]
QMA-comp.	frustrated [HalNagNar+'12]
entropy ~ $\sqrt{L}$	trans. invariant [MovSho'14]
very entangled	random projectors [Mov+'10]
■ entropy ~ $\sqrt{L}$	trans. invariant [this talk]
entropy ~ $\log L$	trans. invariant [Bravyi+'12]
product states	Q 2-SAT in P [Bravyi'06, Chen+'11]

forbidden states
transition rules
invariant subspaces



2 Hamiltonians related to projectors and rules

- example: ferro Heisenberg

$$-(XX + YY + ZZ)$$
$$= -XX(\mathbb{I} - ZZ) + (\mathbb{I} - ZZ) - \mathbb{I}$$
$$\propto |01 - 10\rangle\langle 01 - 10| - \ldots$$

forbidden state,
punishes antisymmetry,
prefers symmetry

particle hopping

$$|01\rangle\langle 10| + |10\rangle\langle 01|$$
$$= XX\left(\frac{\mathbb{I} - ZZ}{2}\right)$$

$$\begin{matrix} \langle \ldots 01 \ldots | \\ +\langle \ldots 10 \ldots | \end{matrix} \; H \; \begin{matrix} | \ldots 01 \ldots \rangle \\ +| \ldots 10 \ldots \rangle \end{matrix} = 0$$

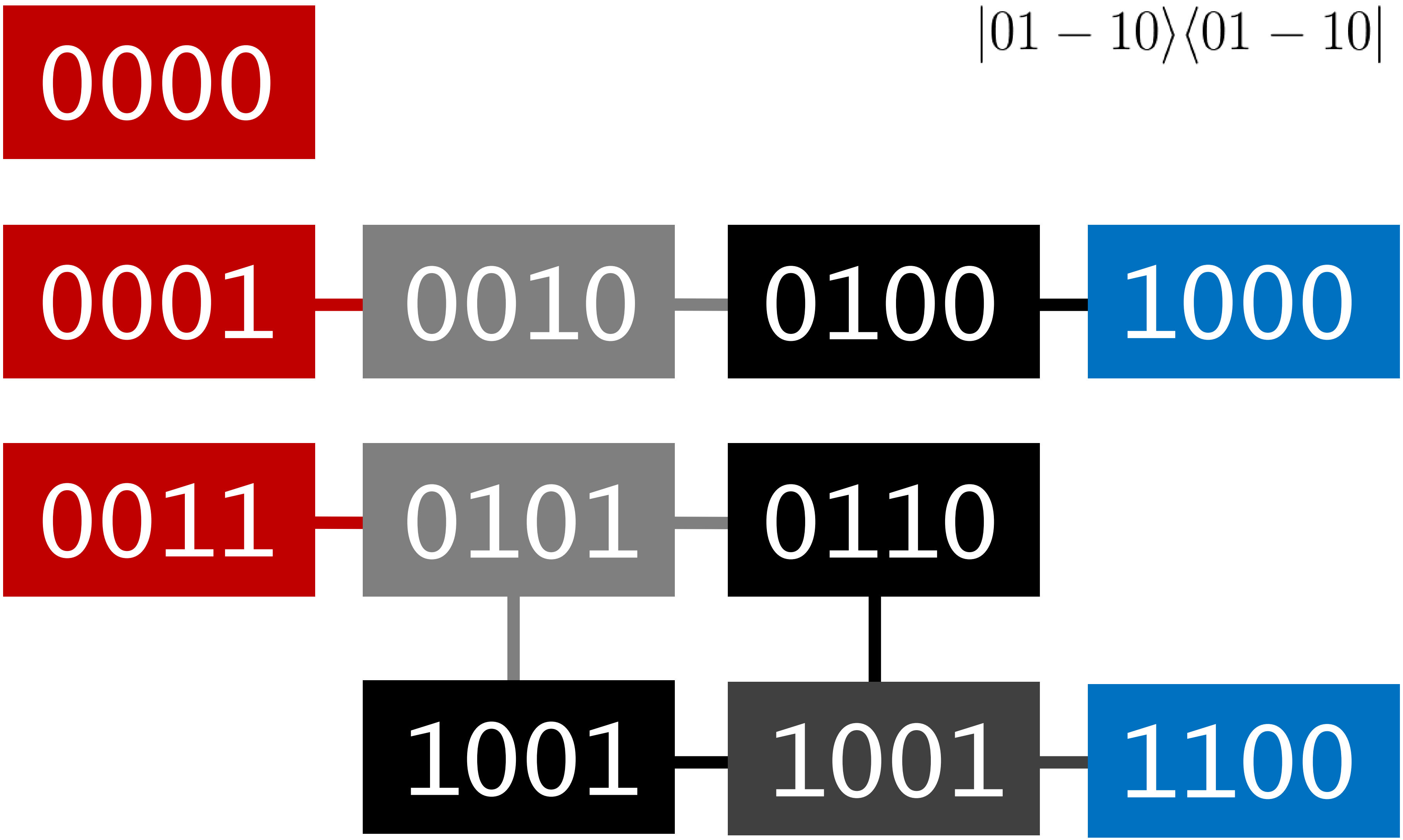
- low energy: obey the "rule"

$$01 \leftrightarrow 10$$

- ground states: uniform superposition over…

01↔10 particles that move

$$|01-10\rangle\langle 01-10|$$



2 Hamiltonians from projectors, rules & inv. subspaces

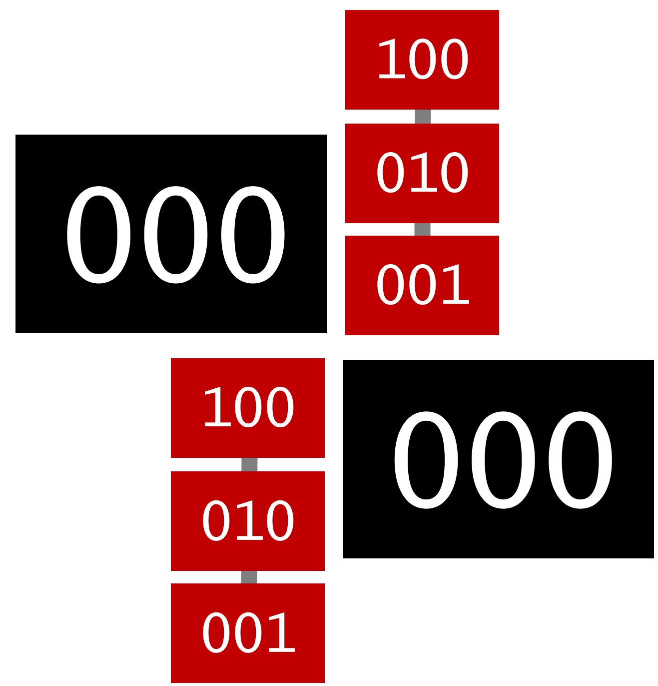
- a “forbidden state”

$$|01 - 10\rangle$$

a rule to build superpositions

$$01 \leftrightarrow 10$$

- Schmidt decomposition using half-chain invariant subspaces

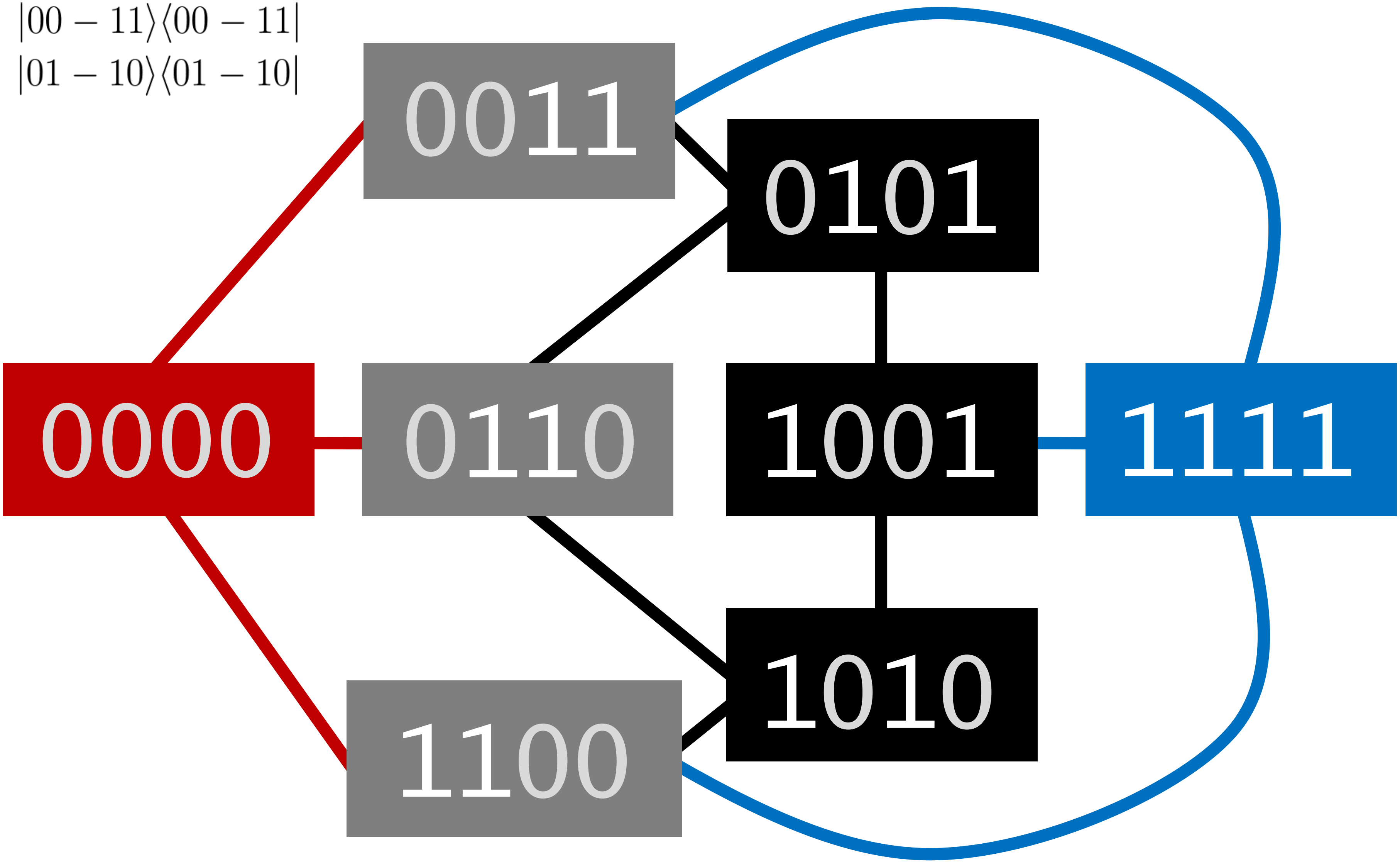


$$\chi = 2$$

independent of the length

00↔11 01↔10 pair creation

$|00-11\rangle\langle 00-11|$
$|01-10\rangle\langle 01-10|$



00↔[]   [0↔0[   ]0↔0]   brackets

"forbidden" states

$|00 - [\,]\rangle$

$|0[- [0\rangle$

$|0] -]0\rangle$

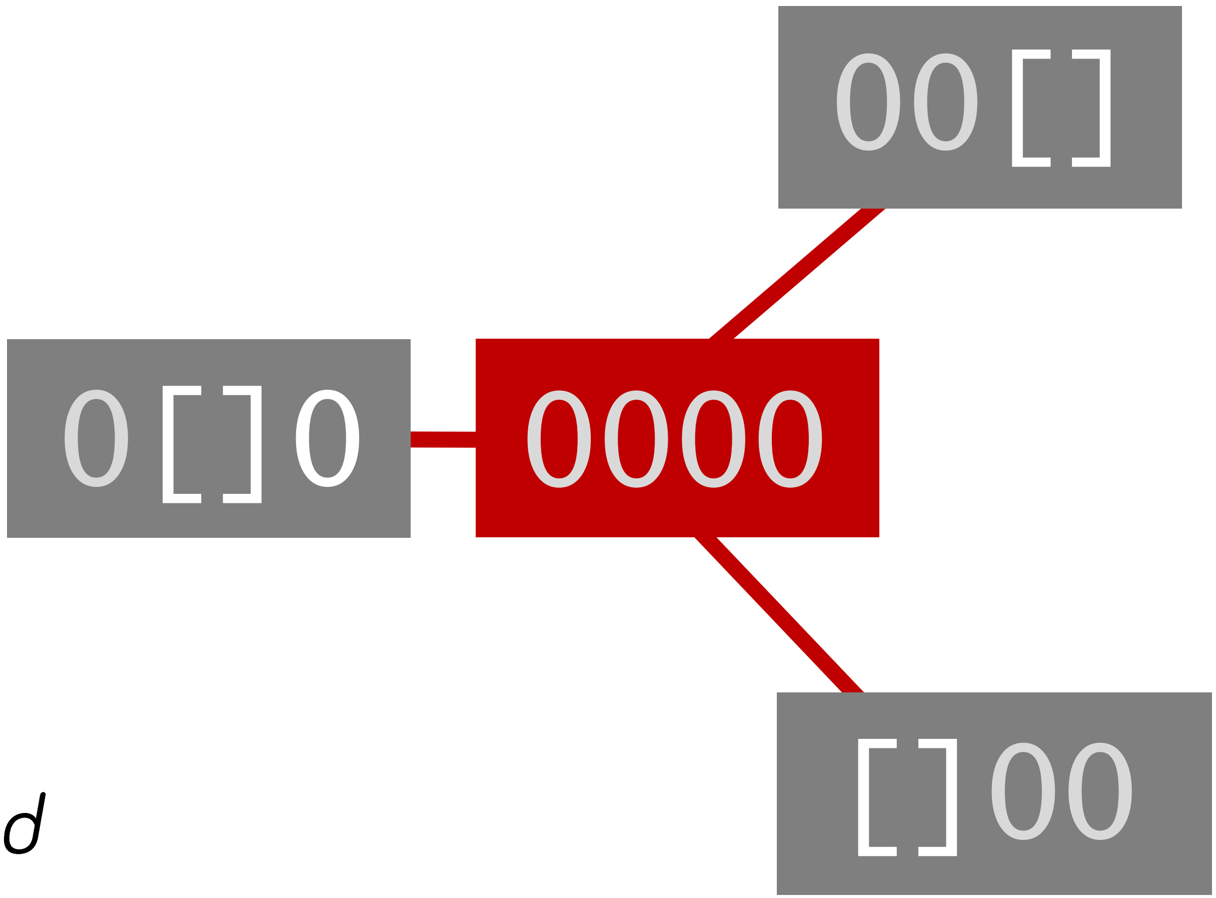
0000

well-bracketed subspace

$00 \leftrightarrow [\,]$ $[0 \leftrightarrow 0[$ $]0 \leftrightarrow 0]$ brackets

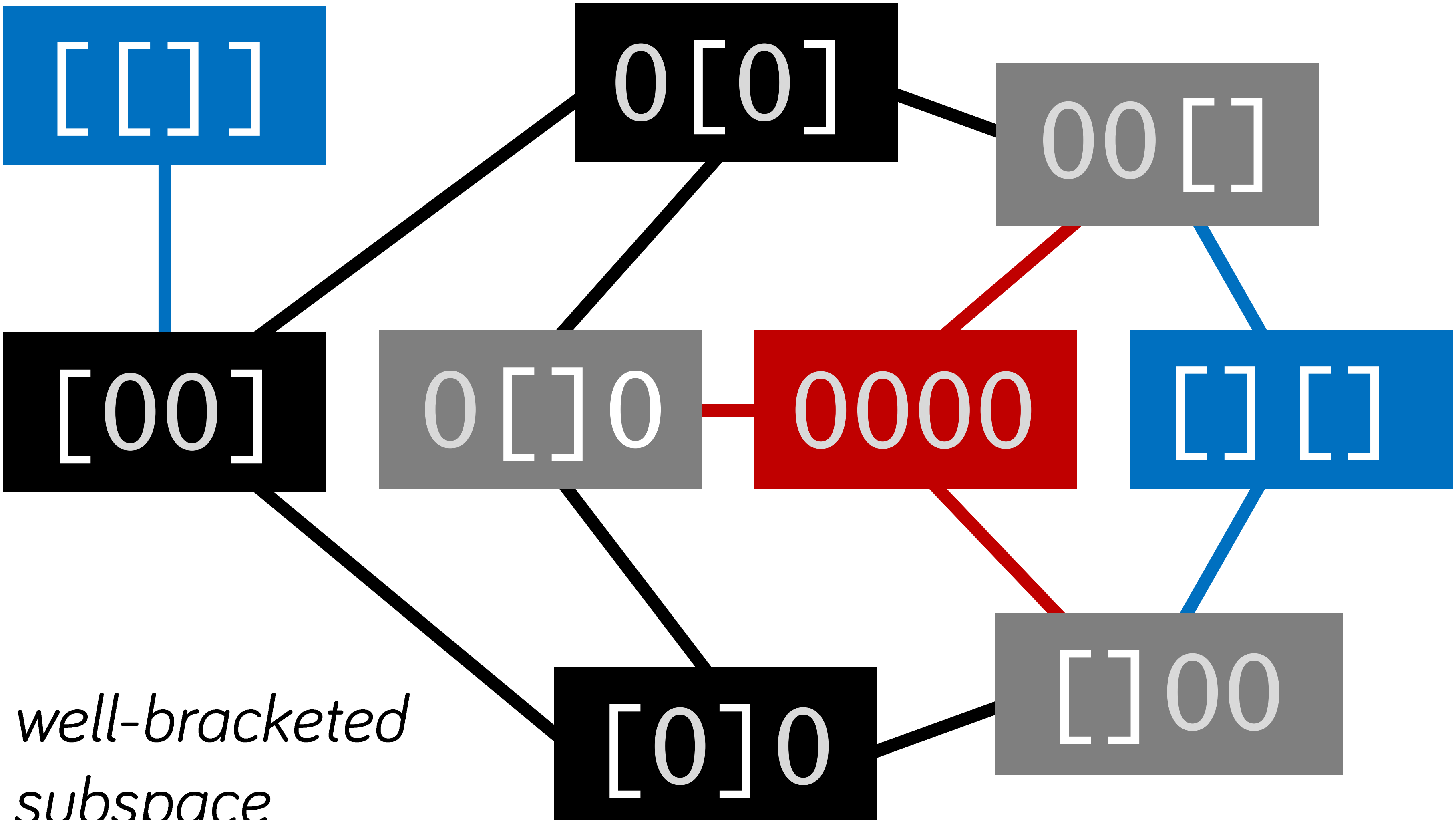
"forbidden" states

$$|00 - [\,]\rangle$$



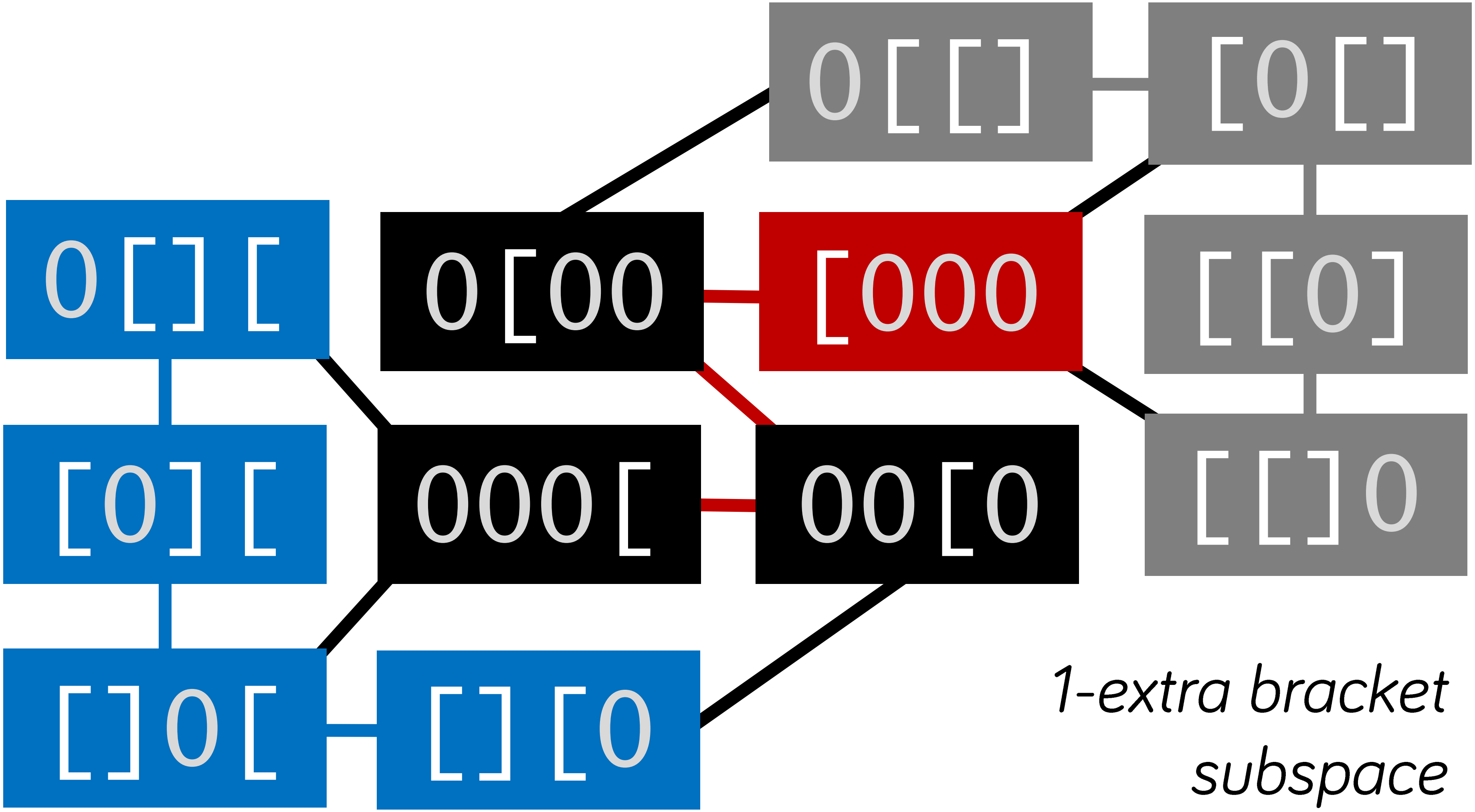
well-bracketed subspace

00 ↔ [] [0 ↔ 0[ ]0 ↔ 0] brackets



well-bracketed subspace

00↔[]   [0↔0[   ]0↔0]   brackets



1-extra bracket subspace

2 The well-bracketed states: how are they built?

00000000000000000000

0[0[]0]0[]

0[[[]]]000

2 The well-bracketed states: how are they built?

00000000000000000000

0000000000

0000000000

000000000[

0[0]0]0000

2 The well-bracketed states: how are they built?

00000000000000000000

- the left brackets match the right, OG, ON, … Schmidt

0000000000	0000000000
000000000[	]000000000
00000000[[	]]00000000
0000000[[[	]]]0000000

2 The well-bracketed states: how are they built?

well-bracketed state

- the left brackets match the right, OG, ON, … Schmidt

$$|\mathcal{M}_n\rangle = \frac{1}{\sqrt{M_n}} \sum_{s \in S^n_{0,0}} |s\rangle$$

the Motzkin state
(uniform, well-bracketed)

$$= \sum_{m=0}^{n/2} \sqrt{p_m}\, |\hat{C}_{0,m}\rangle_A \otimes |\hat{C}_{m,0}\rangle_B$$

uniform, length *n*/2, *m* extra [*m* extra]

- Schmidt rank $\chi = \frac{n}{2} + 1$

2 (Well-)bracketed words & Motzkin paths

```
0[0[0[[]0]0]0[000]0]
```

- Dyck paths:
  mountains (above 0)
  Catalan #'s

```
   /\
  /  \
 /    \/\
/        \
[[[[]]][]]
```

- Motzkin paths:
  mountains with plateaus
  Motzkin #'s

```
      /\_
    _/   \_   ___
  _/       \_/   \_
_/                 \
0[0[0[[]0]0]0[000]0]
```

2 (1-extra-)bracketed words & Motzkin paths

```
0[0[0[[]0]0]0[000]00
```

- Dyck paths:
  mountains, end at 1
  Catalan #'s

```
   /\
  /  \
 /    \/\
/
[[[[]]][]
```

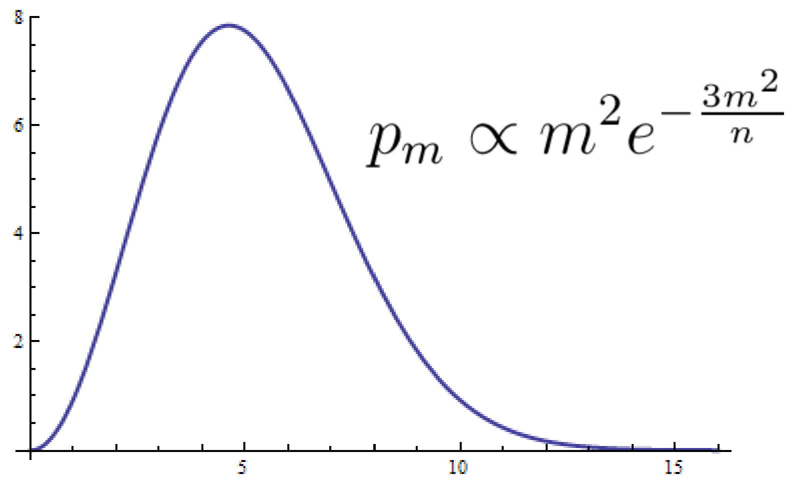
- Motzkin paths:
  mountains + plateaus
  Motzkin #'s

```
      /\_
    _/   \_   ___
  _/       \_/   \__
_/
0[0[0[[]0]0]0[000]00
```

2 Well-bracketed: entanglement entropy

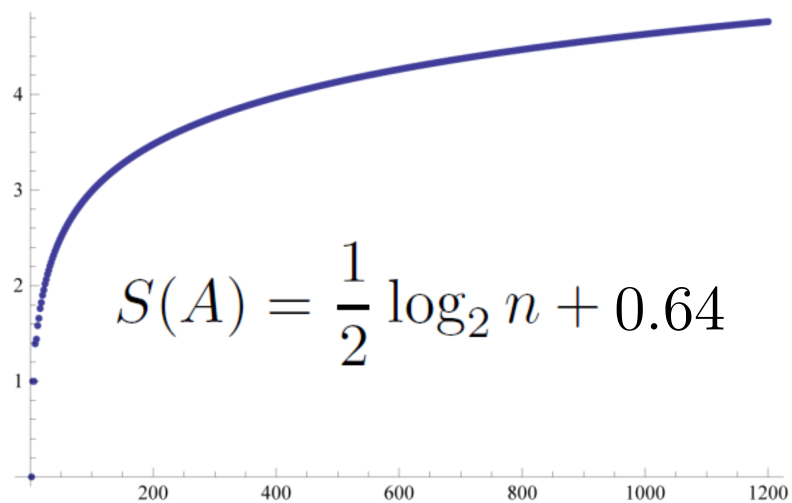
- many significant Schmidt coefficients

$$\sum_{m=0}^{n/2} \sqrt{p_m}\, |\hat{C}_{0,m}\rangle_A \otimes |\hat{C}_{m,0}\rangle_B$$



- logarithmic entanglement entropy

$$S(A) = -\operatorname{Tr} \rho_A \log_2 \rho_A$$

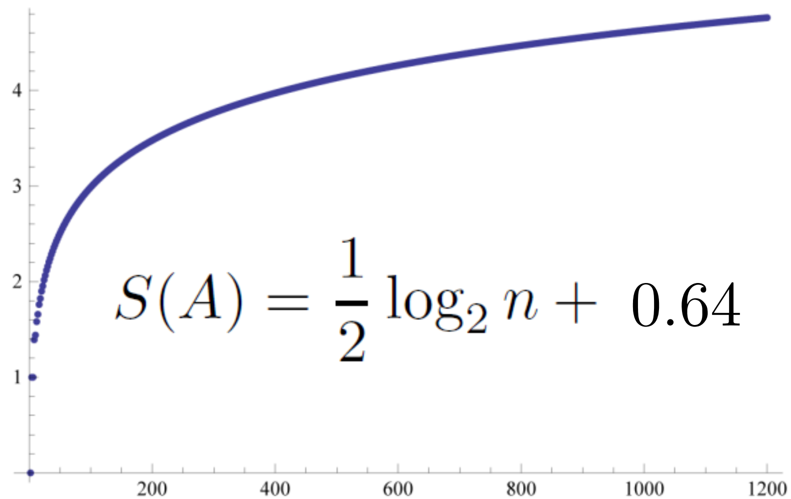


2 The bracket model: degeneracy, frustration & gap?

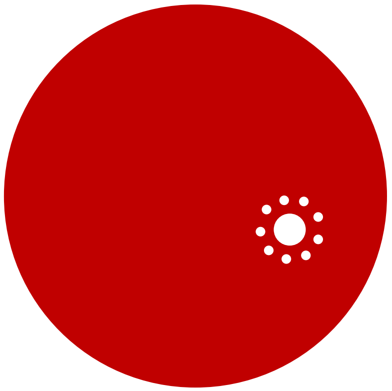
- add endpoints: the well-bracketed subspace wins

- logarithmic entanglement entropy

$$S(A) = -\operatorname{Tr} \rho_A \log_2 \rho_A$$



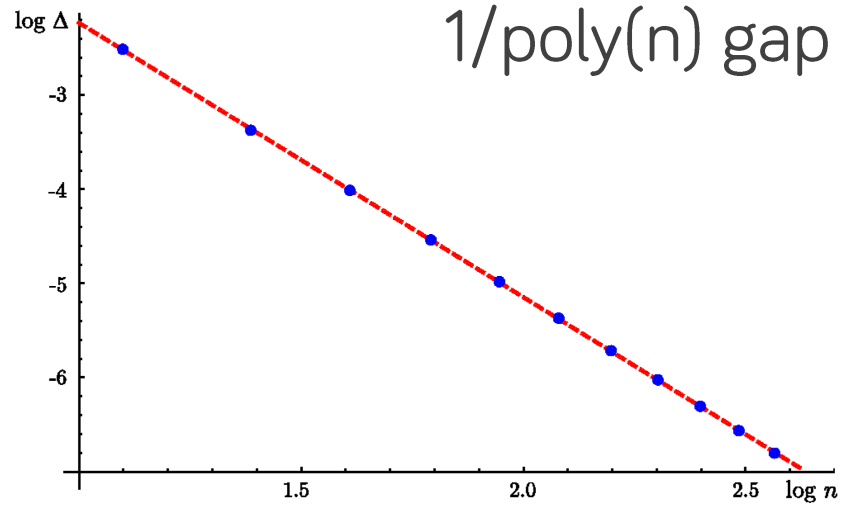
a unique, frustration-free,
entangled ground state



2 The bracket model: degeneracy, frustration & gap?

- add endpoints: the well-bracketed subspace wins

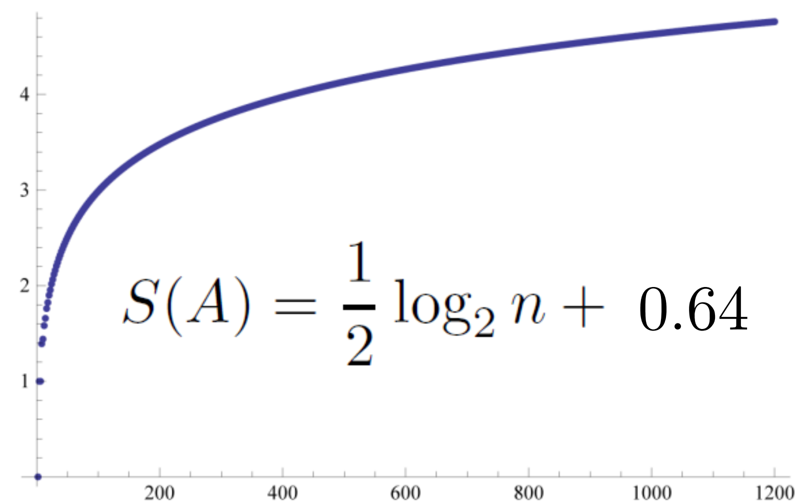
1/poly(n) gap



- a unique ground state frustration-free

- logarithmic entanglement entropy

$$S(A) = -\operatorname{Tr} \rho_A \log_2 \rho_A$$



$$S(A) = \frac{1}{2} \log_2 n + 0.64$$

2 The gap is small

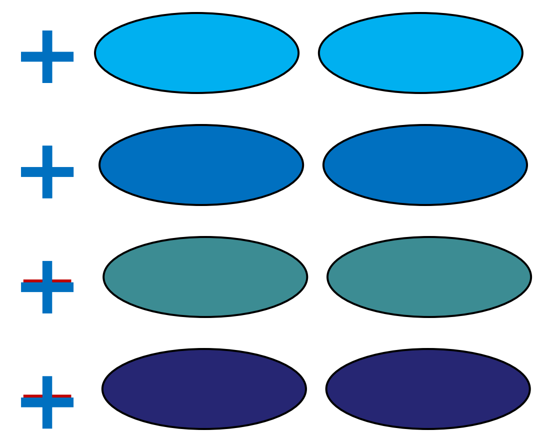
Test: twist the ground state.

- ground state: uniform

$$\sum_{m=0}^{n/2} \sqrt{p_m}\, |\hat{C}_{0,m}\rangle_A \otimes |\hat{C}_{m,0}\rangle_B$$

- an almost orthogonal state

  from some *k* on use $-\sqrt{p_m}$



- caught weakly by a few terms

  an upper bound on the gap

$$\Delta \leq O\left(\frac{1}{\sqrt{n}}\right)$$

2 Lower bounding the good subspace gap

-[]-[[]]- Hamiltonian
on Motzkin paths

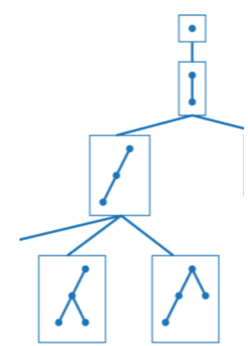
effective 
Hamiltonian
on Dyck paths

$P_{x,y}$ Markov chain
on Dyck paths

graph congestion
bounds the gap Δ

congestion from
canonical paths

canonical paths
from fractional matchings



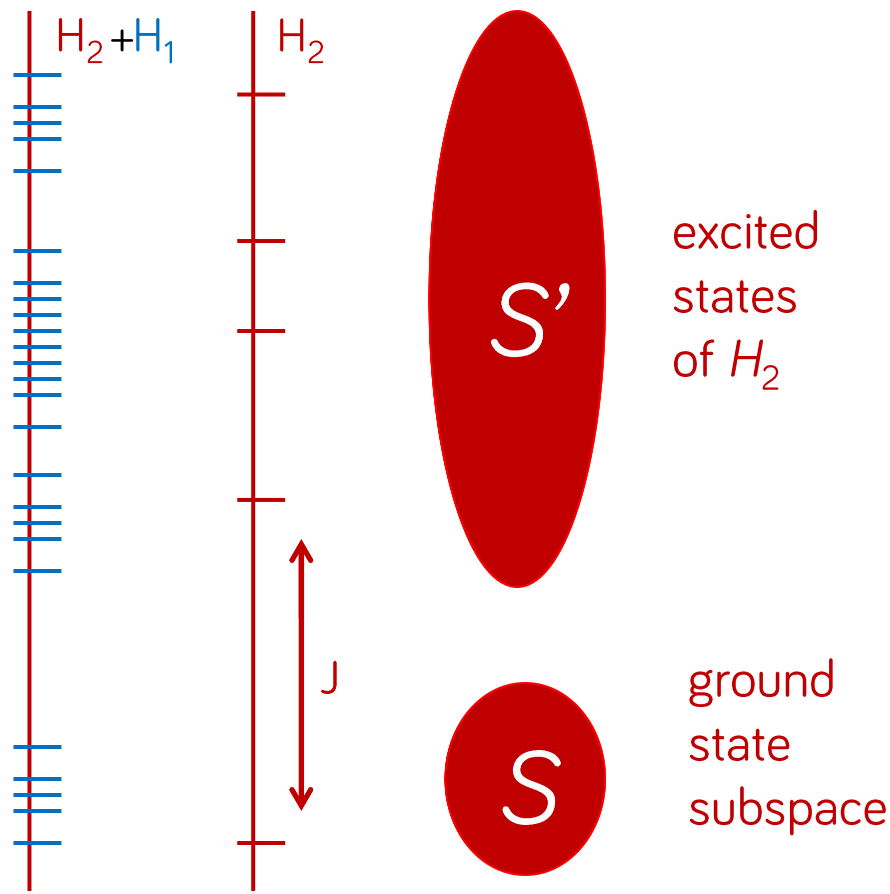
2 The projection lemma

Combining HUGE and tiny Hamiltonians.

$$H_2 + H_1$$

- estimate the small eigenvalues of $H_2 + H_1$ by an effective Hamiltonian

$$H_1|_S$$



2 Projection lemma, "good" subspace

- full Hamiltonian

$$\texttt{--} \leftrightarrow \texttt{[]}$$

$$H = H_{move} + H_{create} + H_{end}$$

$$\texttt{[-} \leftrightarrow \texttt{-[}$$

$$\texttt{-]} \leftrightarrow \texttt{]-}$$

2 Projection lemma, “good” subspace

- full Hamiltonian

-- ↔ []

$$H|_{good} = H_{move} + H_{create}$$

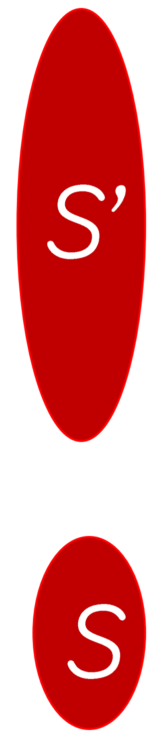
- pretend the “create” part is small

$$H_\epsilon = H_{move} + \epsilon H_{create}$$

Hsnbrg, gap $O(n^{-2})$

- low spectrum

$$H_{create}|S$$



excited states of H_{move}

well-bracketed words uniformly spread

ground states of H_{move}

2 From Motzkin paths to Dyck Paths (no spaces)

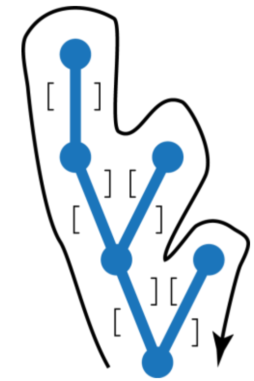
- a new basis

/_ + /‾\ + _/\

[]- [-] -[]

superpositions of Motzkin paths
with the same Dyck path

[[[]][]][]



labeled by
Dyck paths

- low spectrum

$H_{create}|S$

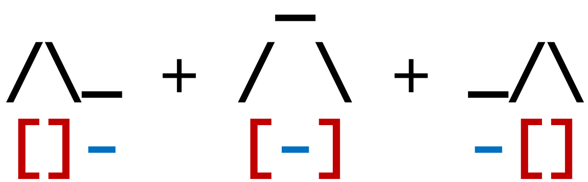
well-bracketed words
uniformly spread

S

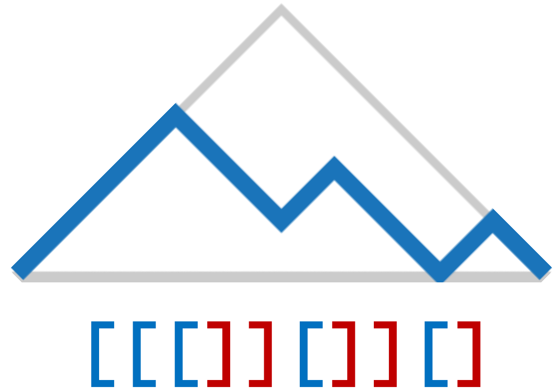
ground
states
of H_{move}

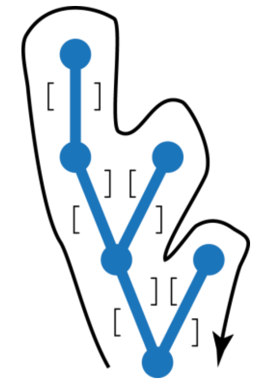
2 From Motzkin paths to Dyck Paths (no spaces)

- a new basis



superpositions of Motzkin paths
with the same Dyck path

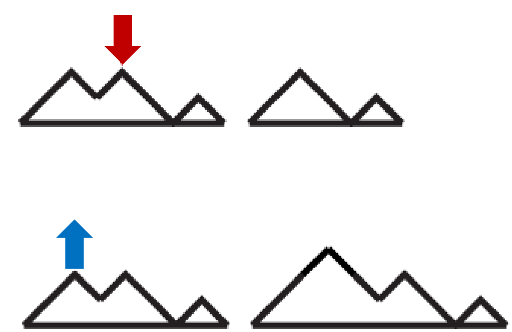




labeled by
Dyck paths

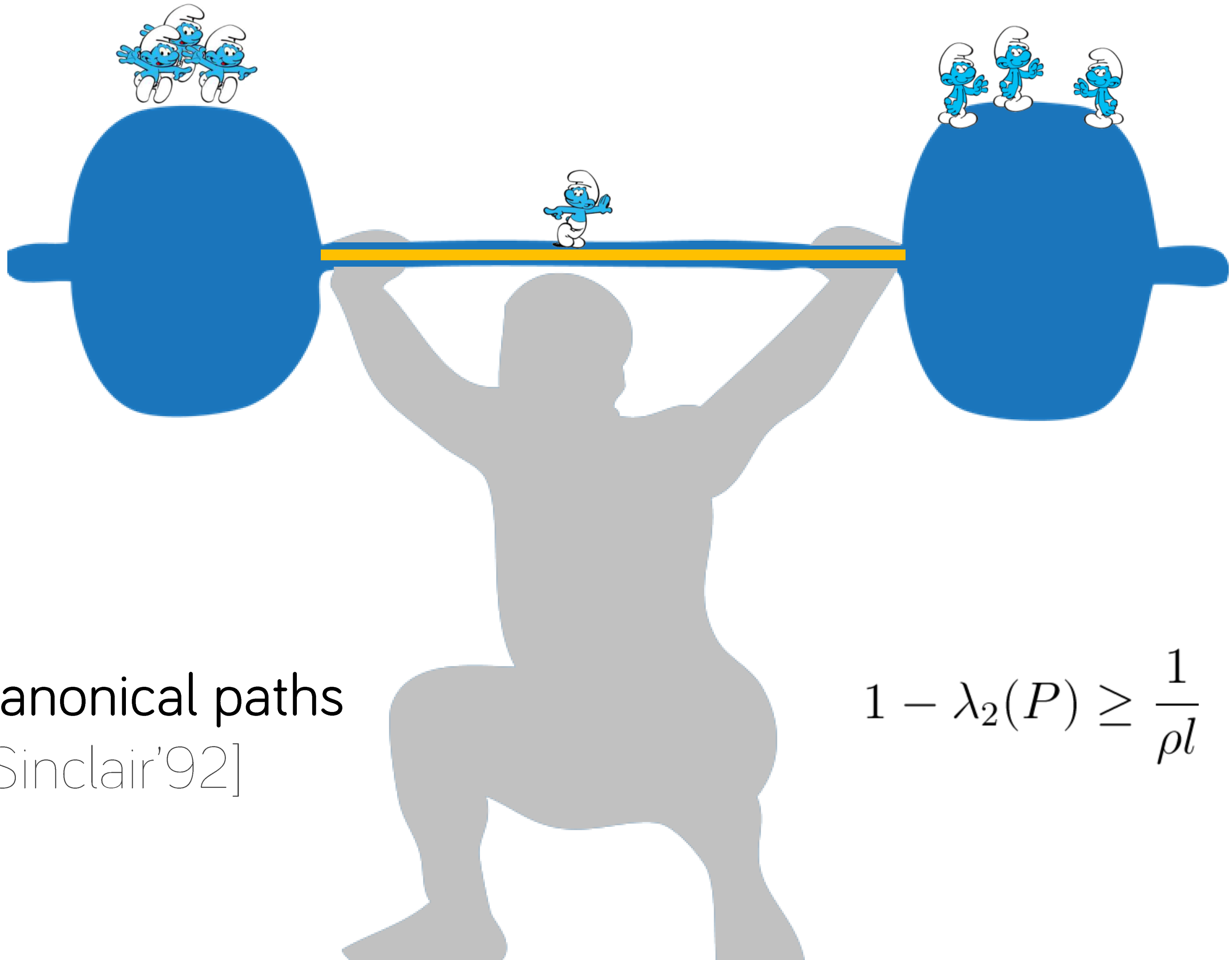
- effective Hamiltonian: a random walk on Dyck paths

connections: erosion/eruption



2 Congestion in a graph

Is any edge overworked?



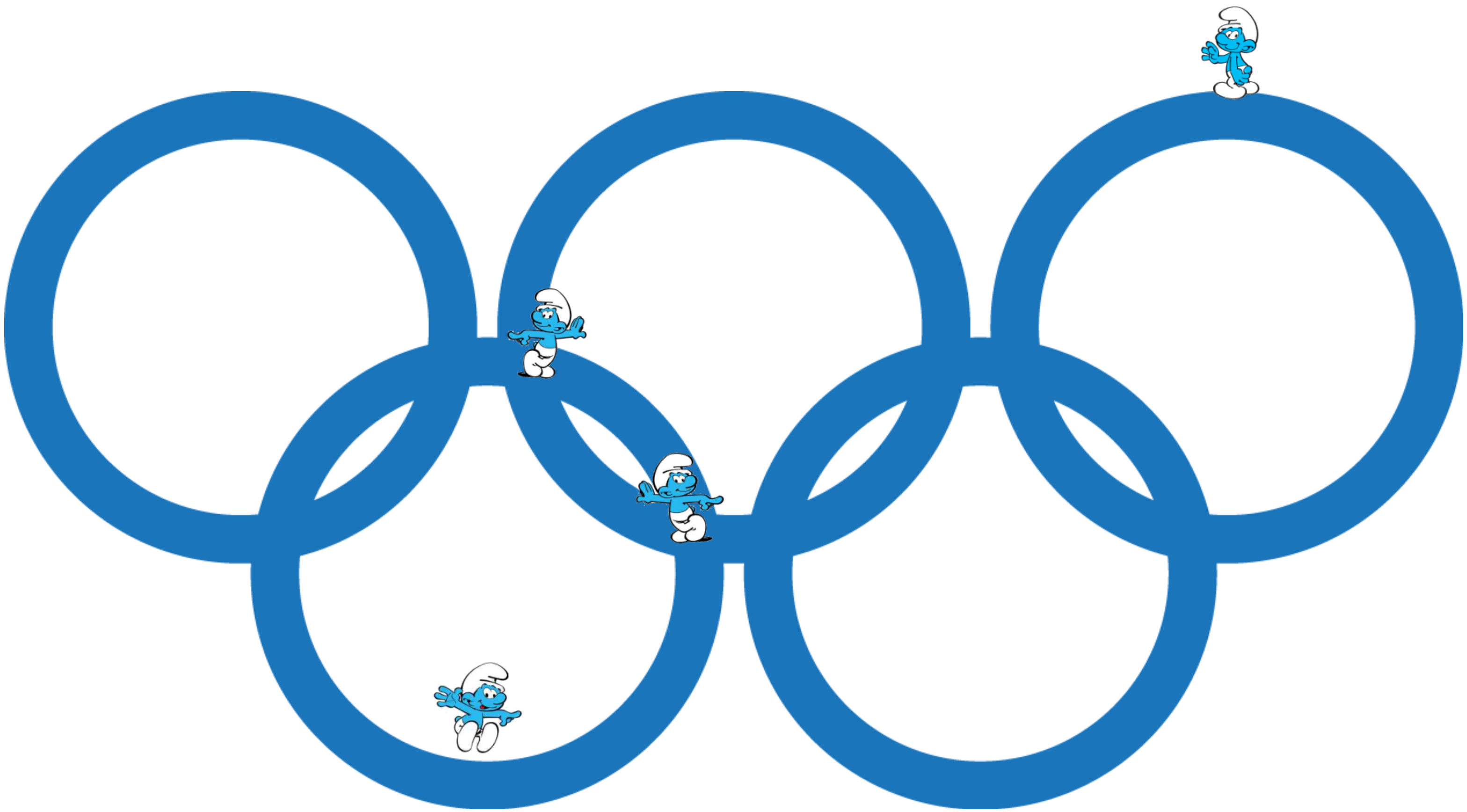
canonical paths

[Sinclair'92]

$$1 - \lambda_2(P) \geq \frac{1}{\rho l}$$

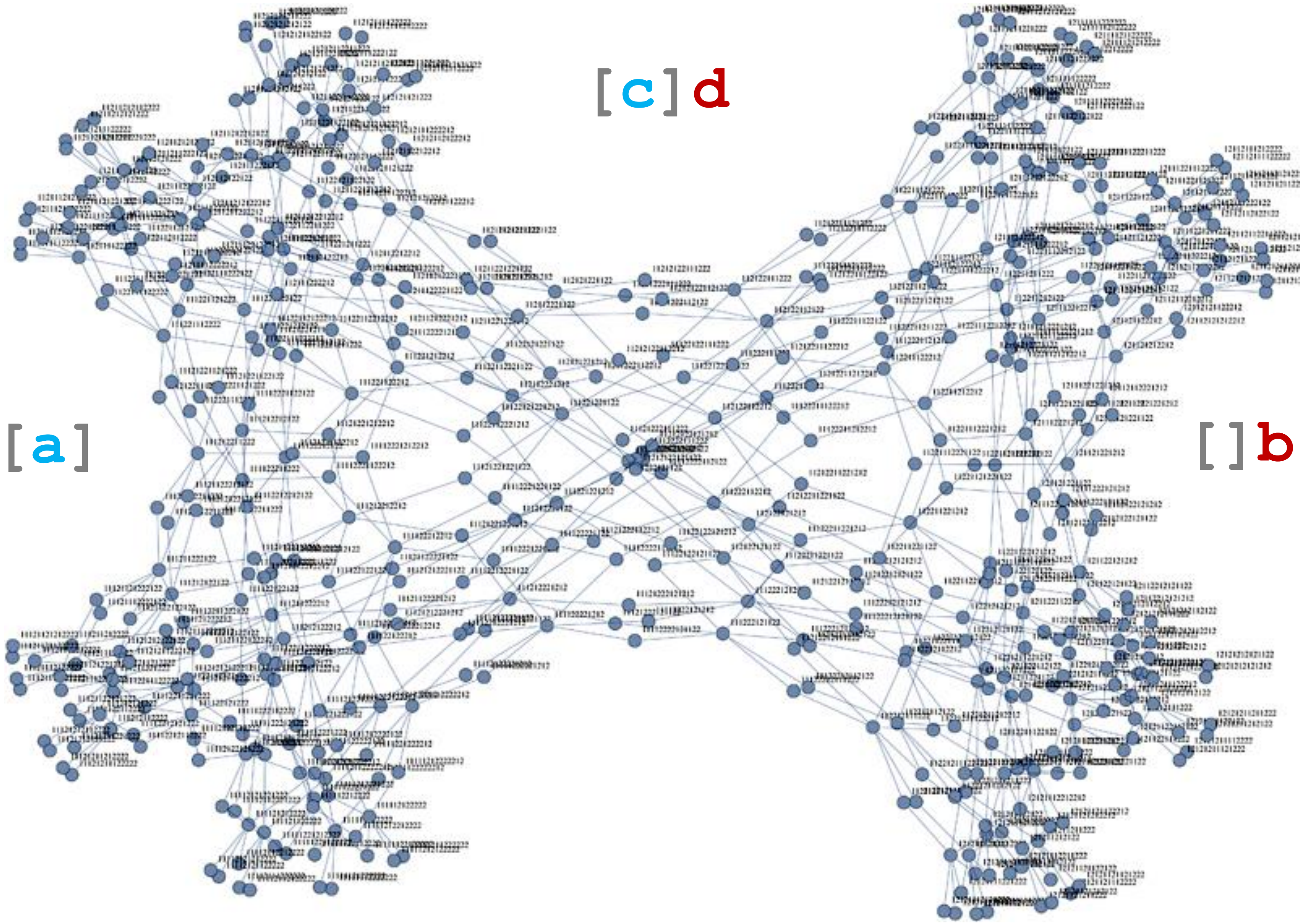
2 Congestion in a graph

Is any edge overworked?



2 Connecting Dyck paths

Looks good. Proof: fract. matchings.

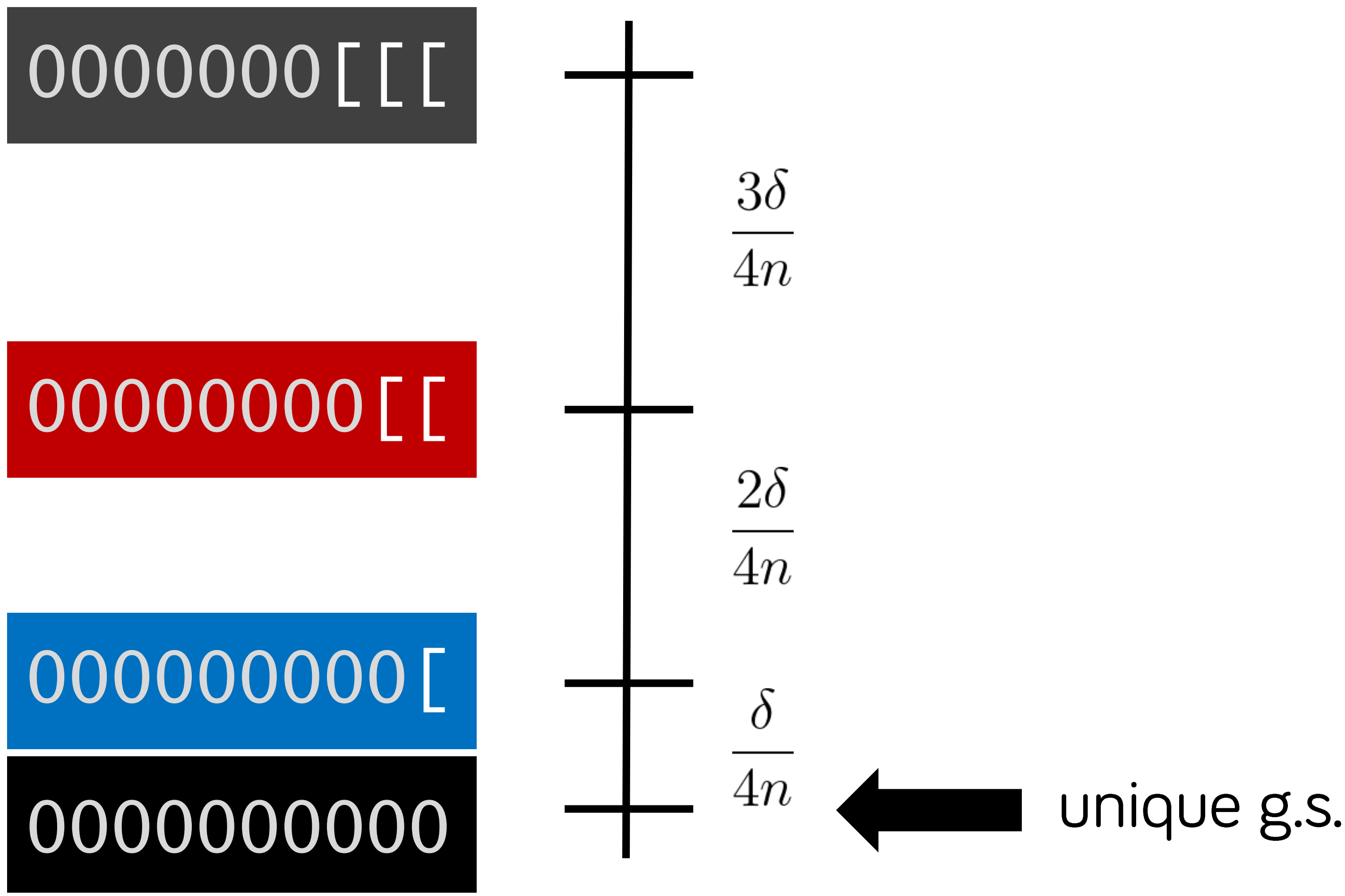


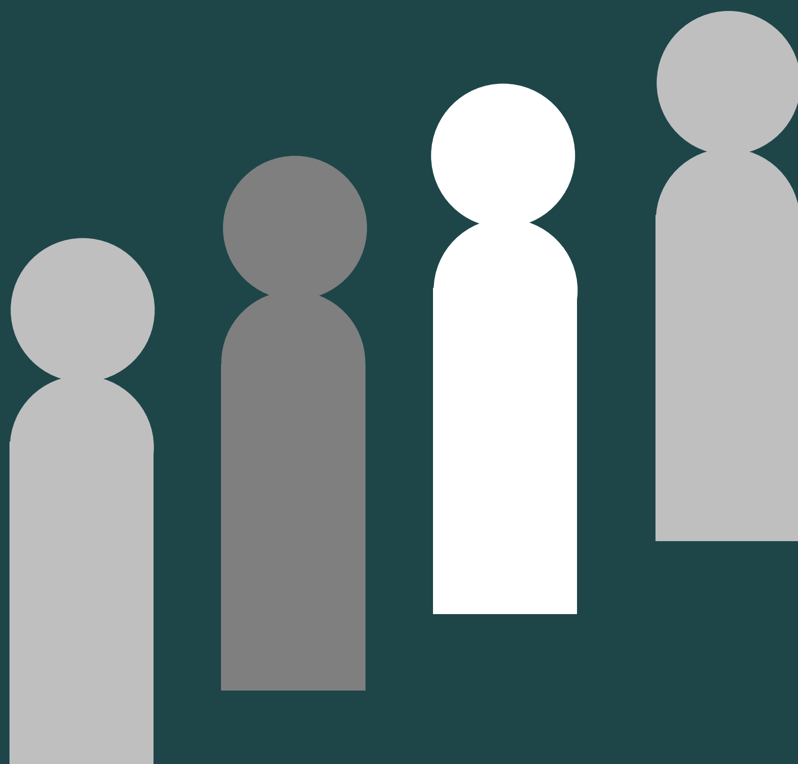
a polynomially small gap

$$\Delta = O(n^{-c})$$

2 Getting rid of endpoints using an external field

- a little perturbation… cost δ per particle (bracket)
  average # of particles in the uniform superpos.?





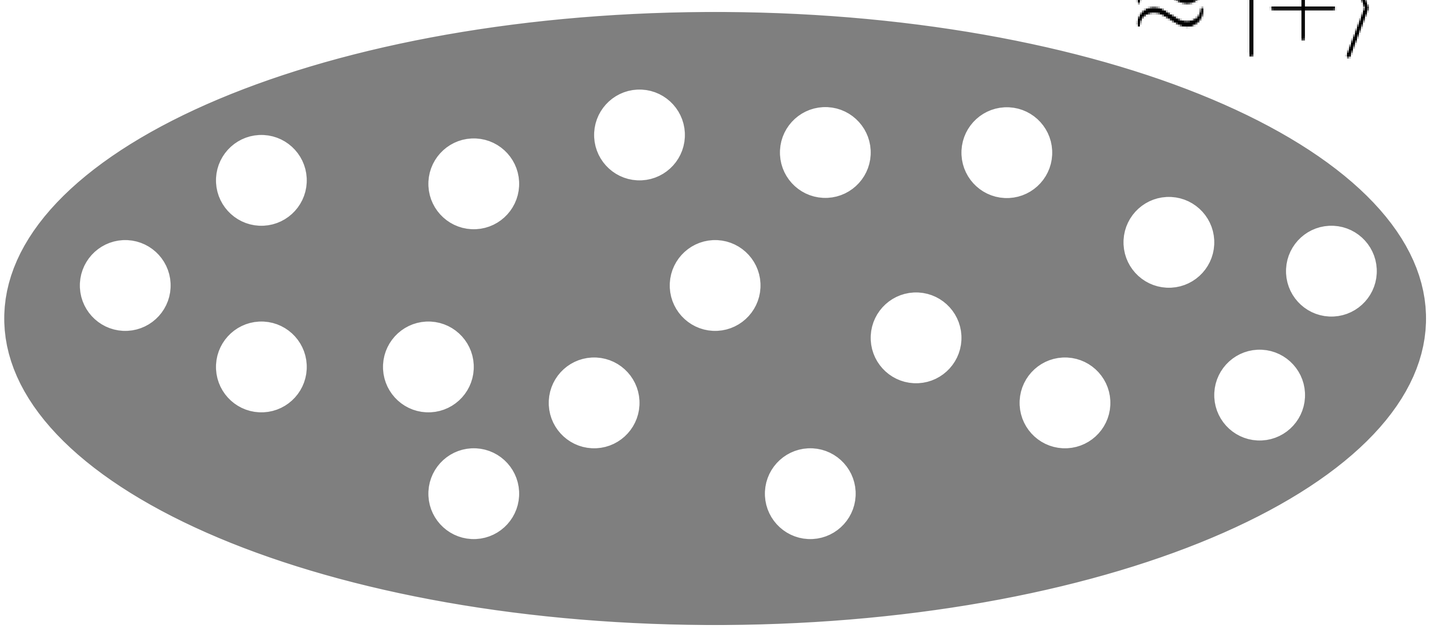
a gallery of models

3 Making models with unique entangled ground states

- a state with a lot of entanglement?

  a large (but not too large) superposition
  many (significant) Schmidt coeficients

$$\approx |+\rangle^{\otimes n}$$



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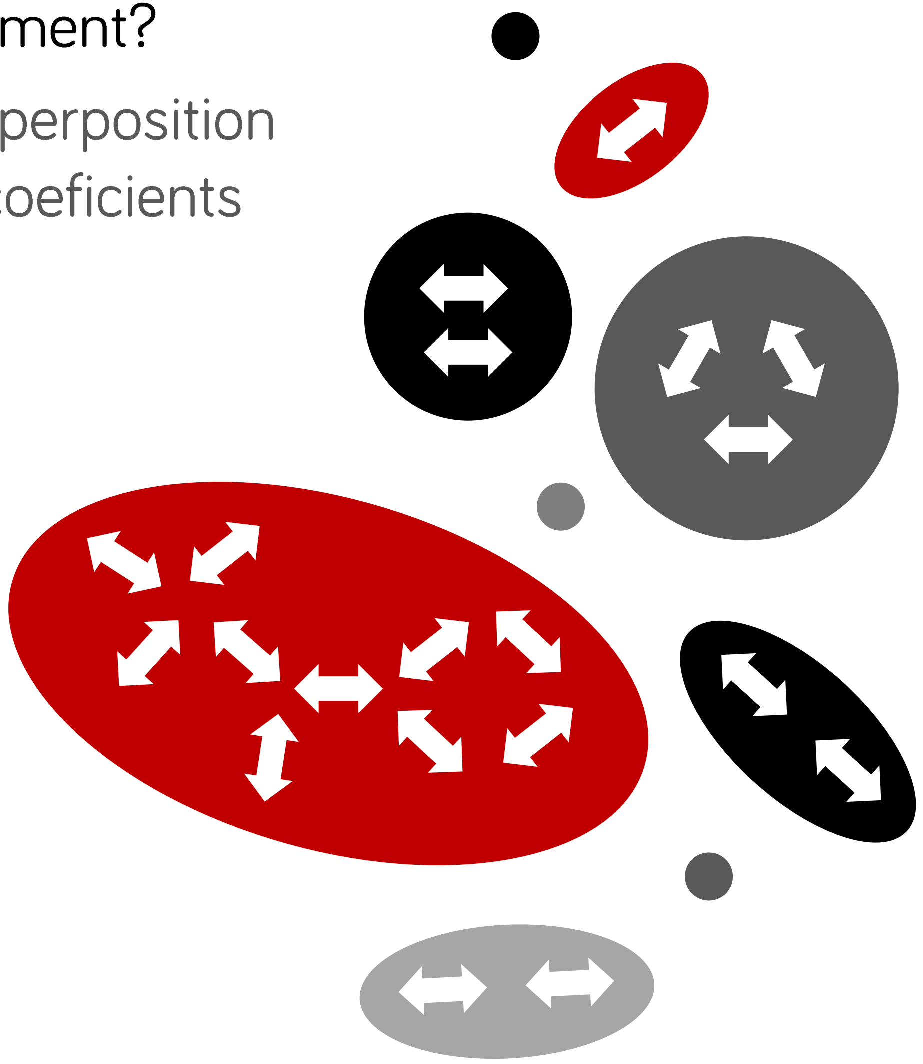
- building a superposition?

  transition rules, projectors
  invariant subspaces
  connected components



3 Making models with unique entangled ground states

- a state with a lot of entanglement?

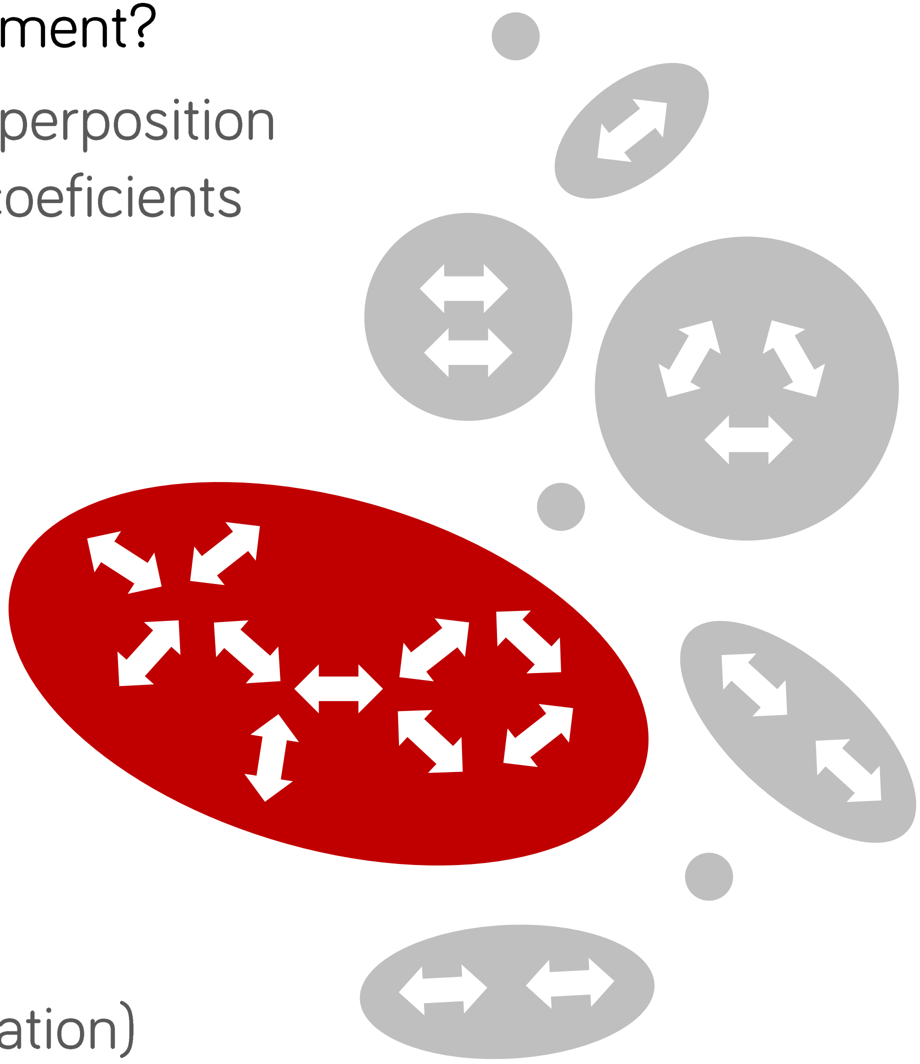
  a large (but not too large) superposition
  many (significant) Schmidt coeficients

- building a superposition?

  transition rules, projectors
  invariant subspaces
  connected components

- making it unique

  1- and 2-local rules,
  endpoint projectors,
  cost per particle (adds frustration)



3 The models so far

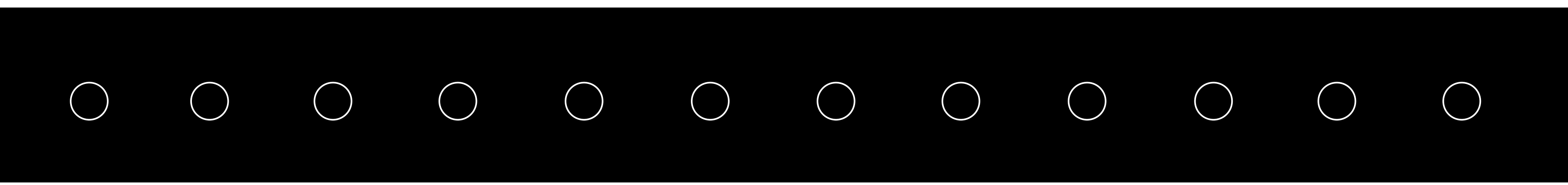
- **d=2** Heisenberg 0000000000000
  Hsnbrg + pair creation 0000**11**00**1**0**1**0
- **d=3** surfers 1111112000000
  AKLT $(-1)^p 2^a$ AB00A000B000
  brackets 00[]000[[]]0

$\Delta \propto n^{-\beta}$ $S \propto \log n$

3 The pair creation (pc-)model

- particle movement $\quad 0v \leftrightarrow v0$

  pair creation/annihilation $\quad 00 \leftrightarrow vv$



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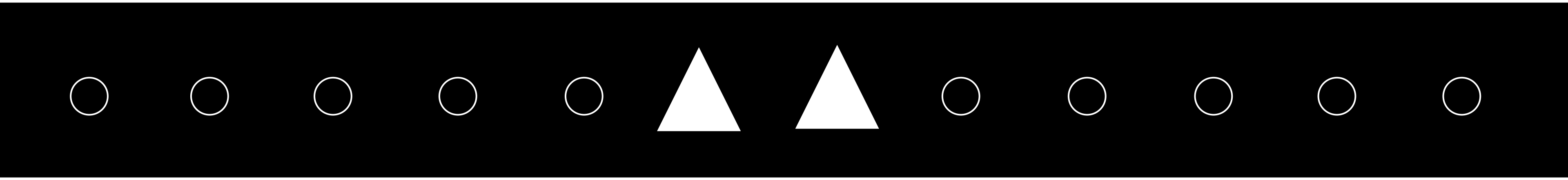

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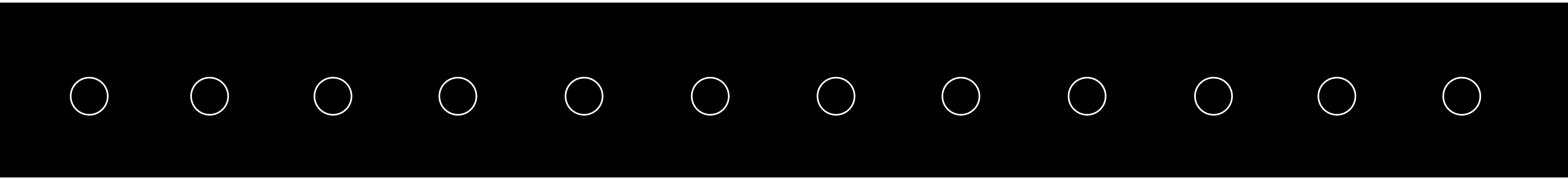

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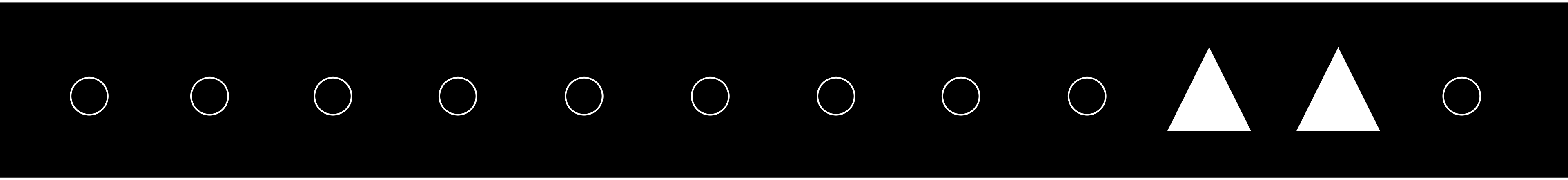

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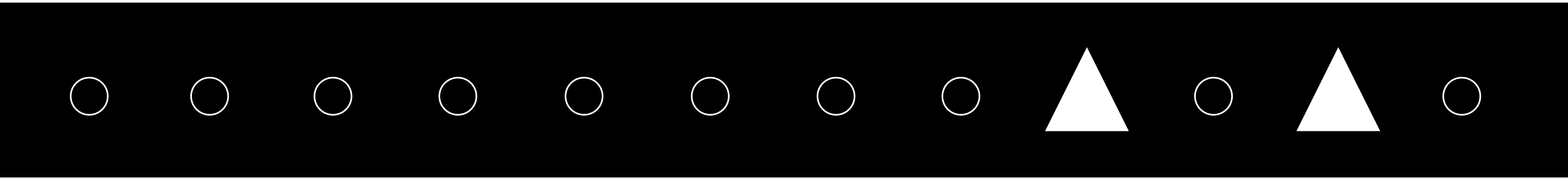

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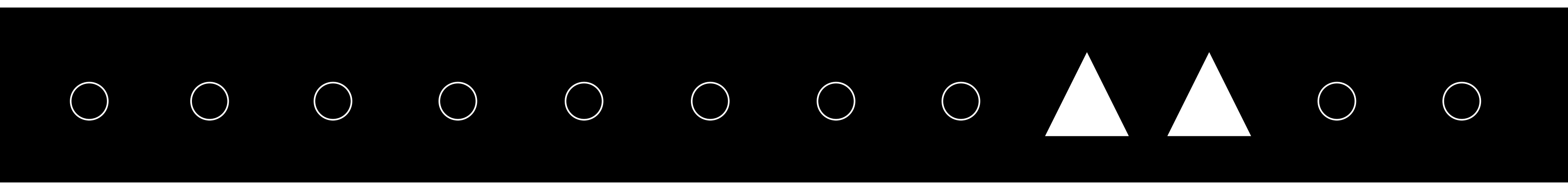

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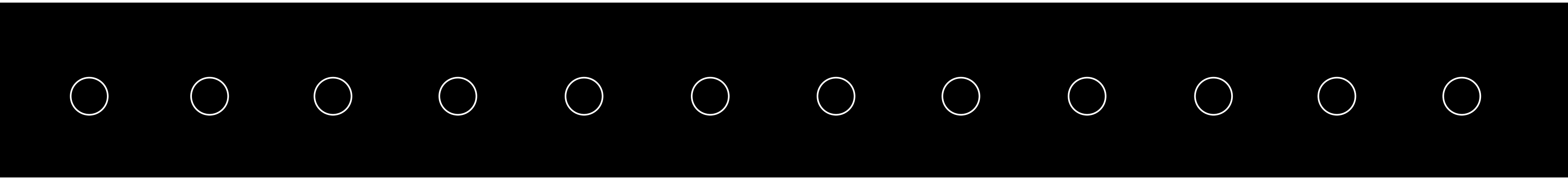


3 The pair creation (pc-)model

- *two* species of particles

  move freely

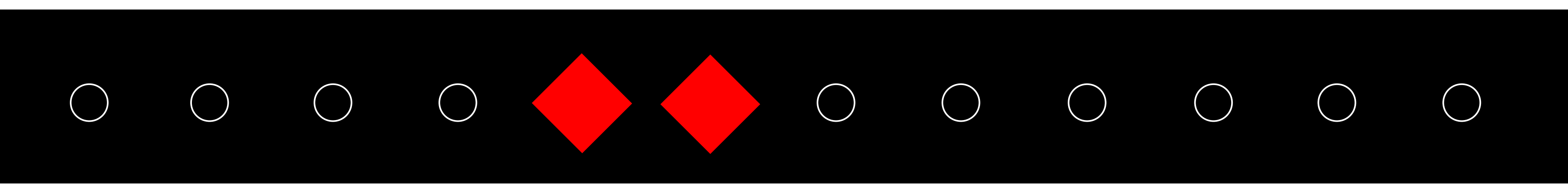
  each species: pair creation/annihilation

0v ↔ v0

0x ↔ x0

00 ↔ xx

00 ↔ vv



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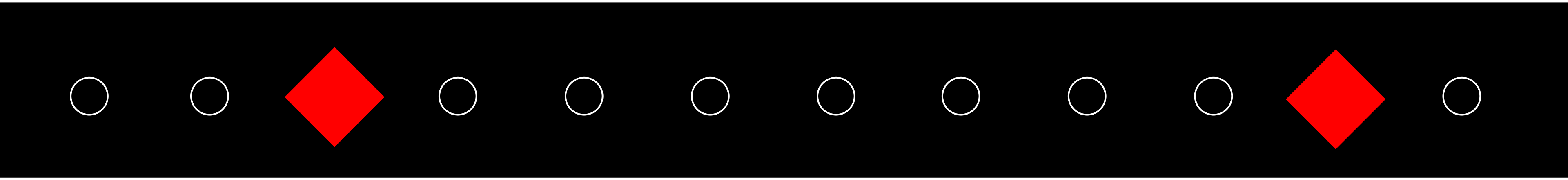

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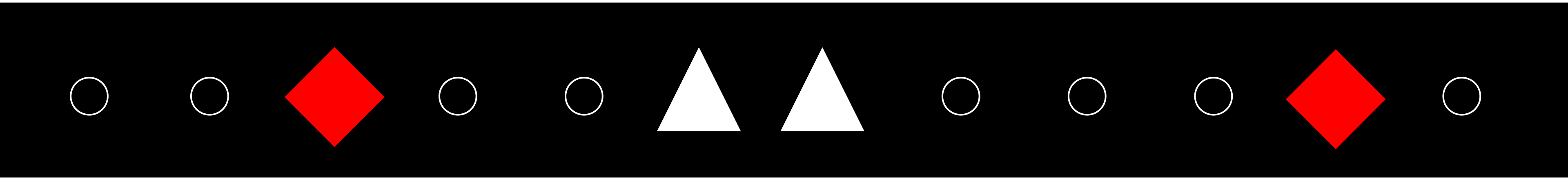

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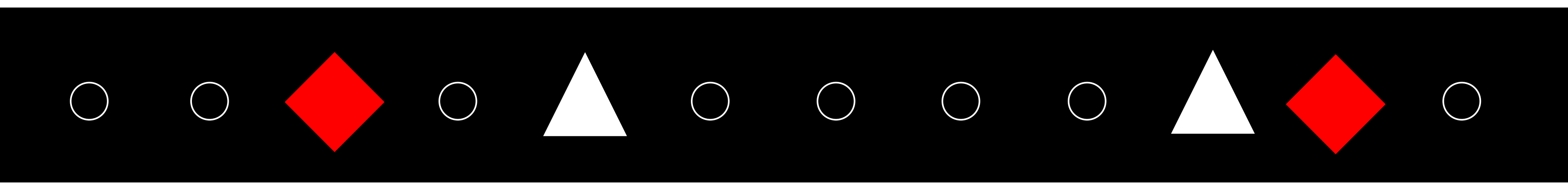

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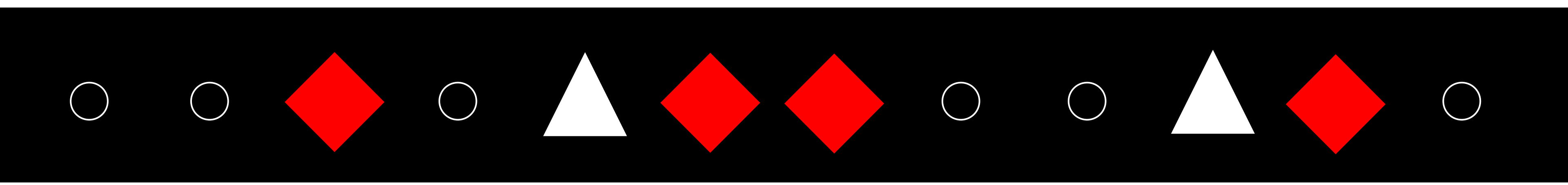

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  invariant subspaces: irreducible strings

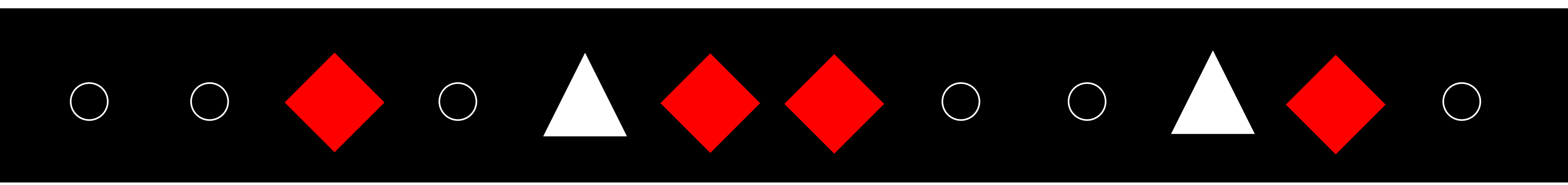
  Schmidt decomposition $\chi = 1 + 2\frac{n}{2}$

xvxv...xv

xvxv...xvx

vxvx...vx

vxvx...vxv



$$\Delta \propto n^{-\beta}$$

S : like brackets

Δ : more connected, better scaling

$$S \propto \log n$$

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  AKLT 0AB00A00B000
  brackets 00[]000[[]]0
  pair creation 00xx0vxxv000
  $\Delta \propto n^{-\beta} \quad S \propto \log n$
- **d=5** two kinds of brackets ([(0)]00[]0)
  $\Delta \propto n^{-\beta} \quad S \propto \sqrt{n}$

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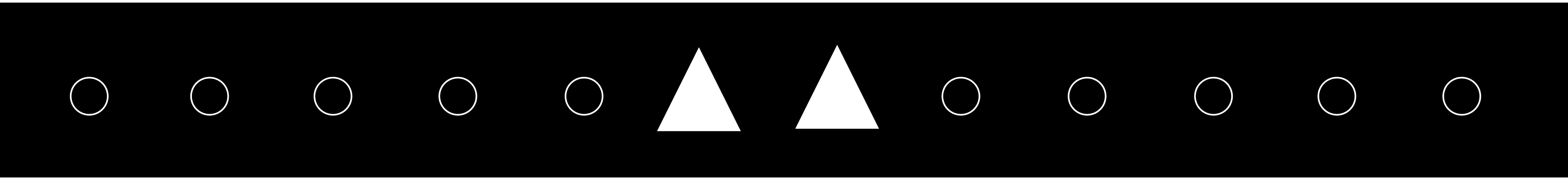

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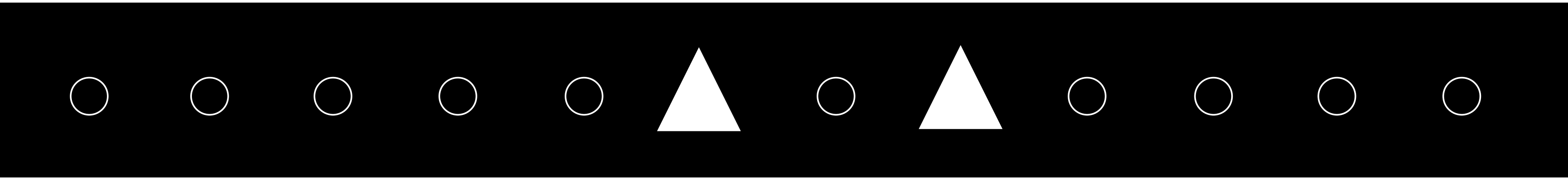

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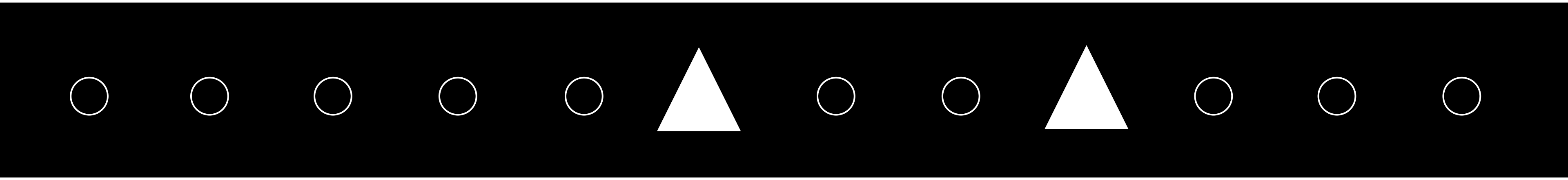

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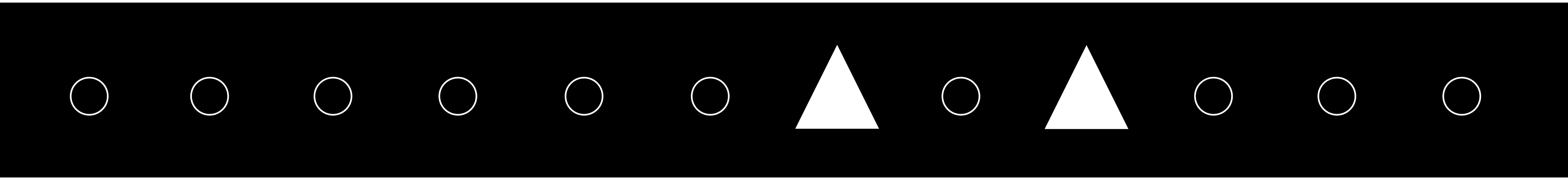

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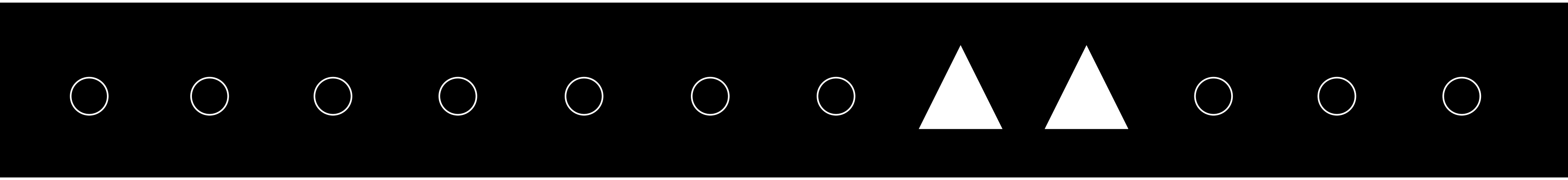

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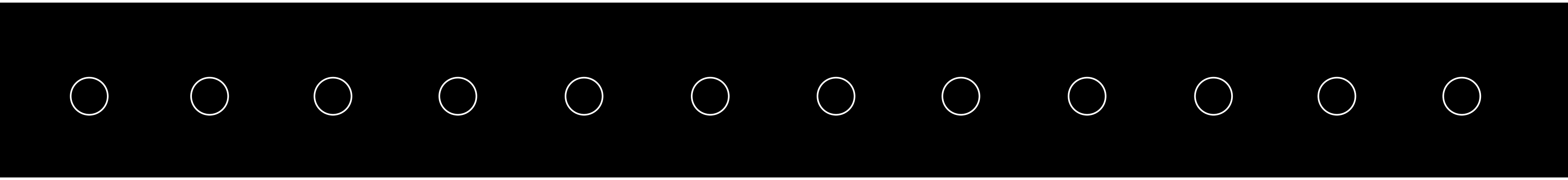

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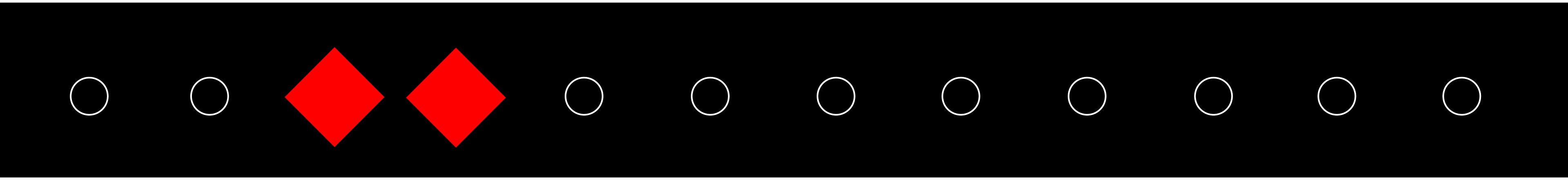

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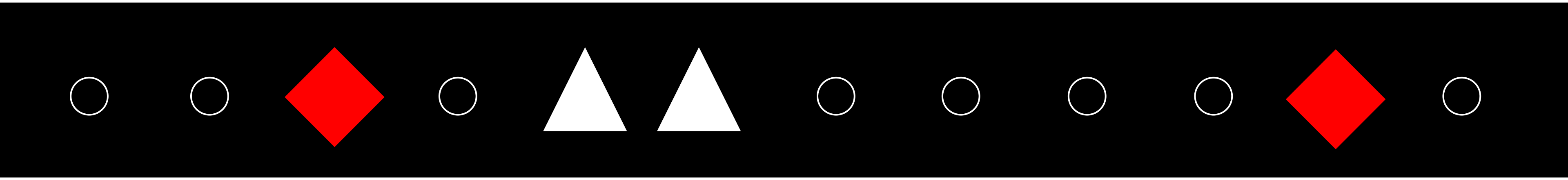

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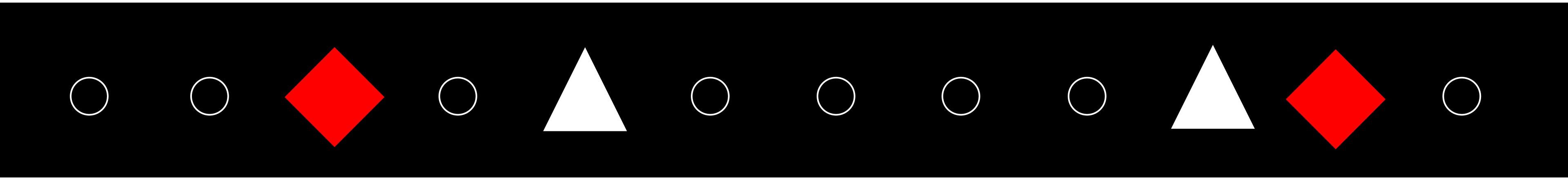

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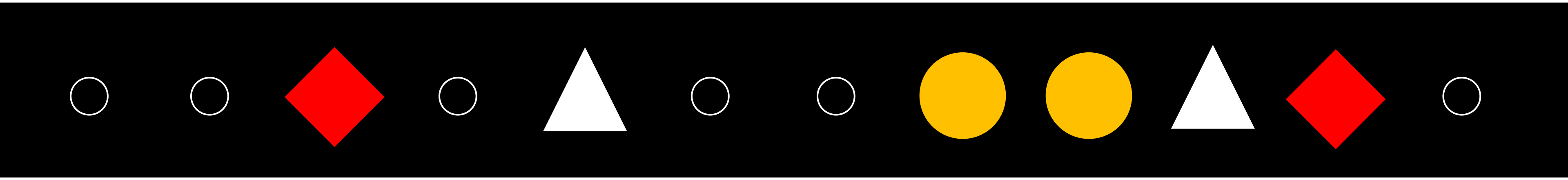

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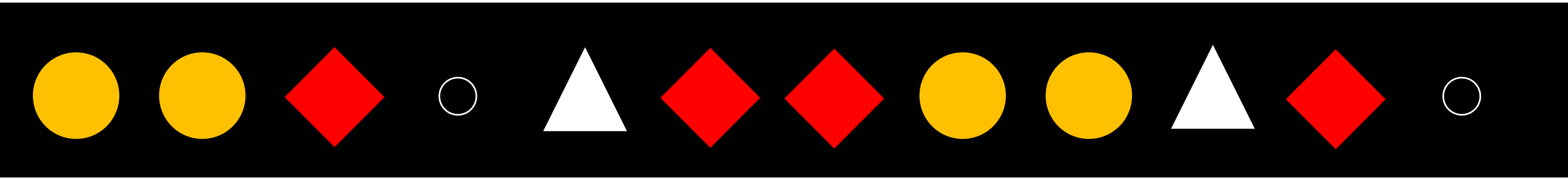

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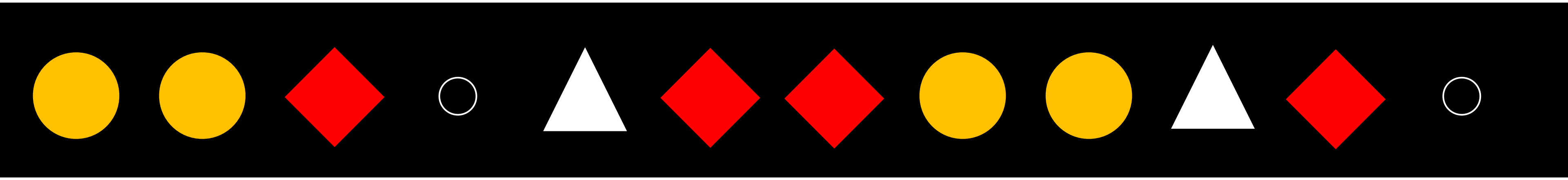
  Schmidt decomposition $\chi = 1 + s(s-1)^{\frac{n}{2}-1}$



$$\Delta \propto n^{-\beta}$$

S : like [([])] brackets
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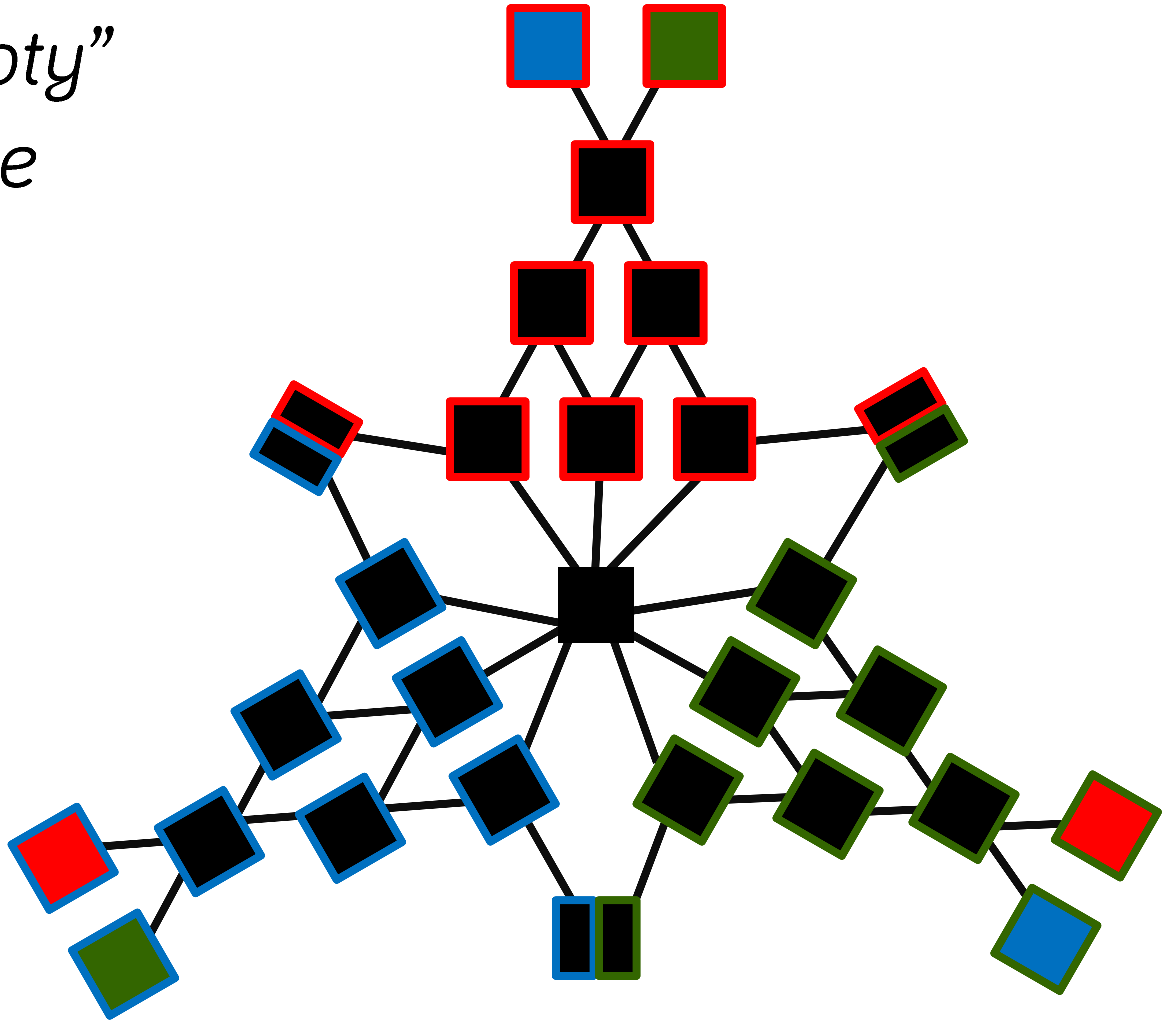
$$S \approx \sqrt{n}$$

$00 \leftrightarrow a_i a_i \quad 0a_i \leftrightarrow a_i 0$ 3-species pair creation

the "empty" subspace



an open
door

3 Different bracket types from parity

- brackets that change color as they move a...b...c...a
  create/annihilate brackets ab

00000000	00	–	ab	bracket creation
0ab00000				
0ab00ab0	a0	–	0b	movement changes color
0ab0c0b0	b0	–	0c	
0a0cc0b0	c0	–	0a	
0a0c0ab0				
0a00aab0				

- effectively: 3 types of brackets with d=4
  ab0000 0ab000 00ab00

$$\Delta \propto n^{-\beta}$$

$$S \approx \sqrt{n}$$

4 Low qudit chains, TI, FF chains, gap & entropy?

- can qutrits hide something hard?

  a class of H's and a promise problem?

  pushdown automaton state recognition?

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- pair/bracket creation with spin (singlets)?

$$|12-21\rangle$$

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- pair/bracket creation with spin (singlets)/general states?

$$\longleftarrow \quad |12-21\rangle \quad \longrightarrow$$

$$\rightrightarrows \quad 2|00\rangle - |AB-CD\rangle \quad \leftleftarrows$$

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- tuning the cost & the gap?

  DMRG [Feiguin], simple/interesting states?

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$$S \propto \Delta^{-\frac{1}{4}}$$

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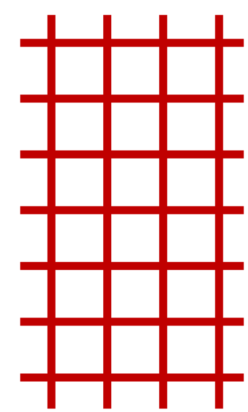
  DMRG [Feiguin], simple/interesting states?

- the best entanglement/gap relationship?
- use the combinatorics from/in

  qchem spin eig-functions, pert. gadgets

- connections to CFT's, cond matter models, 2D?



very
entangled
spin
chains

danielnagaj

slovakacademyofsciences

cahashormovassaghbravyi