local tests of global entanglement

and a counterexample to the generalized area law

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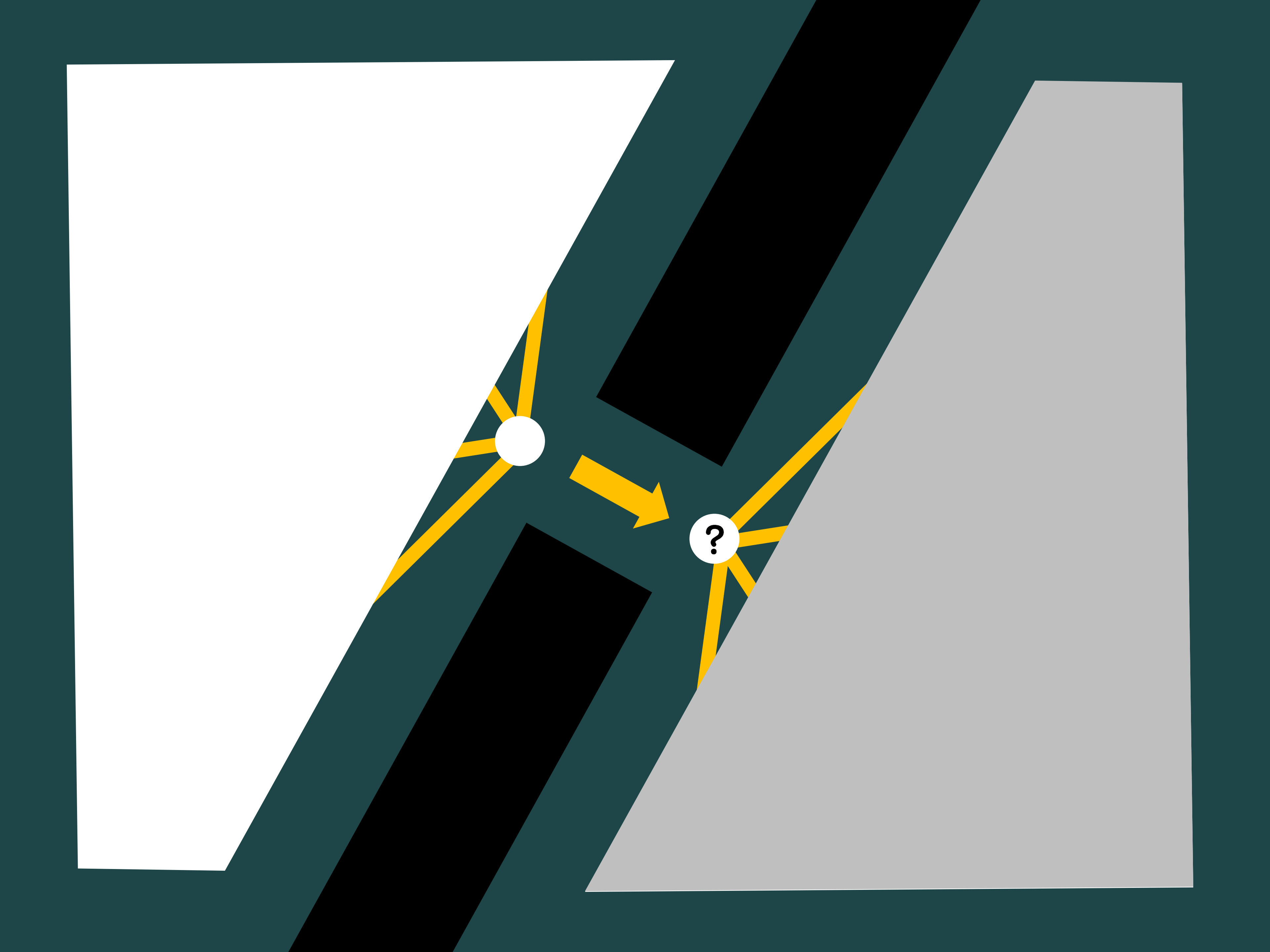
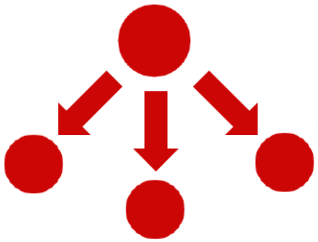
universität wien



1 **q. expanders**

maximally entangled states

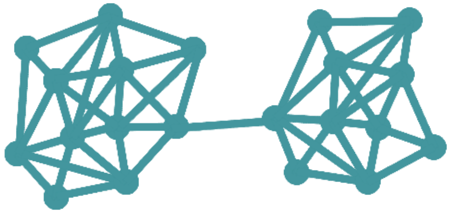


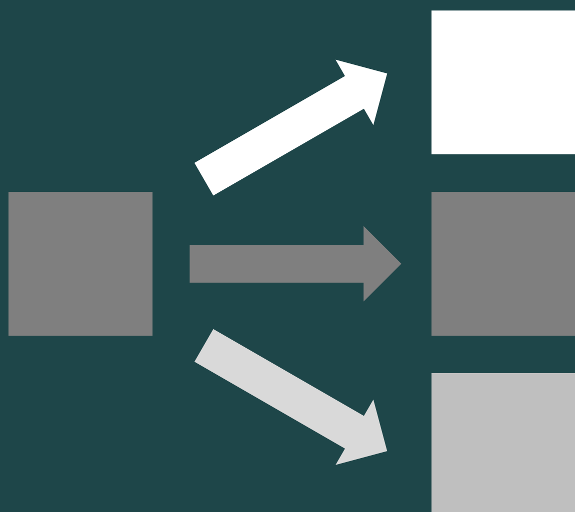
2 **entanglement**

testing and communication

3 **area law**

gaps, connections, correlations

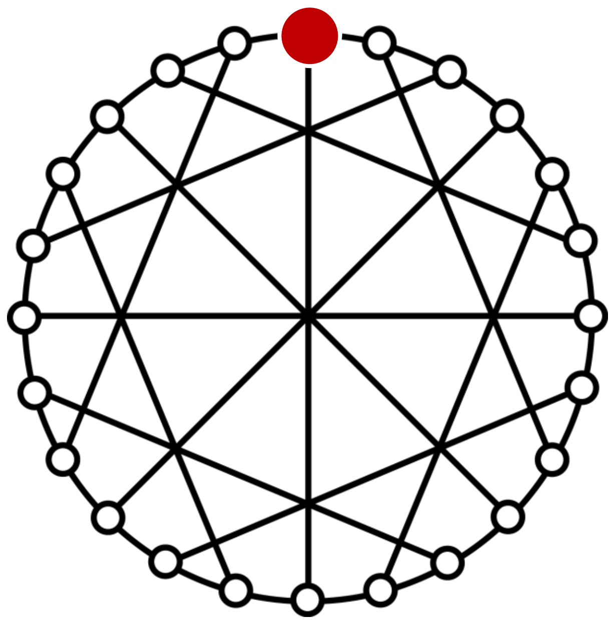




Quantum Expanders

1 Classical expanders

- walk on these graphs? mix fast!



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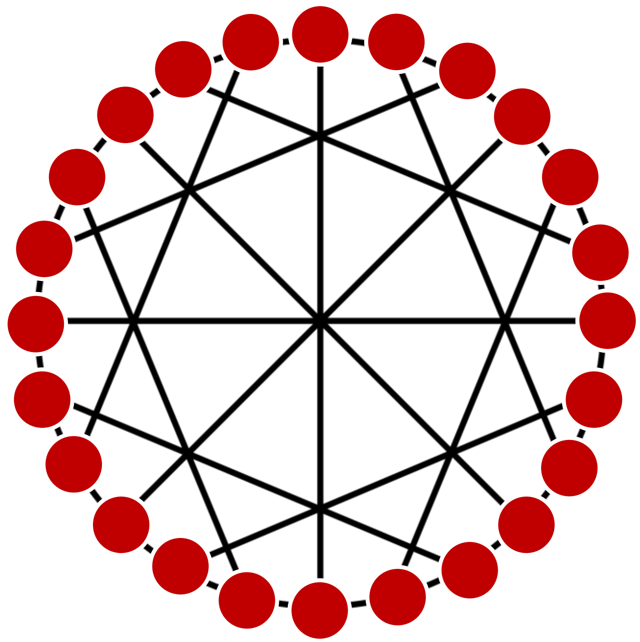

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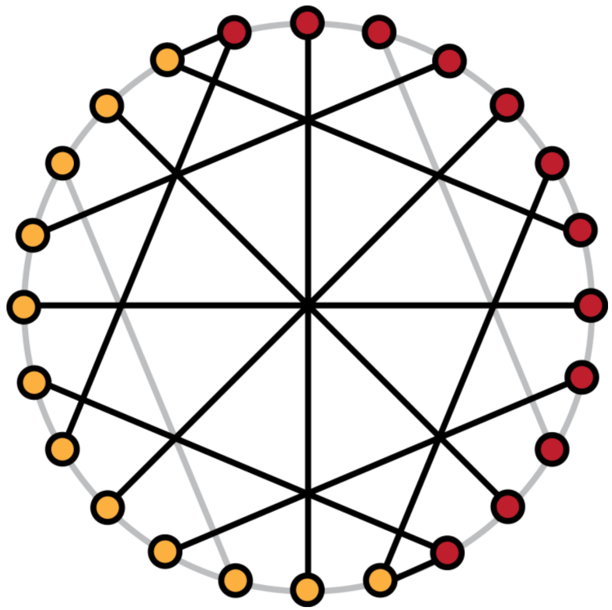
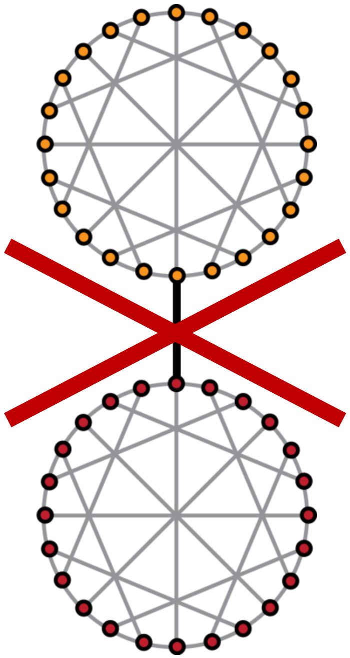
  divide in two? cut a lot (fraction) of edges!





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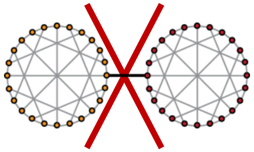
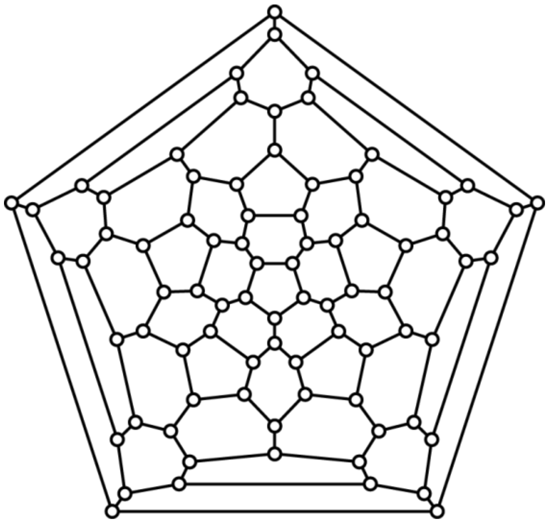
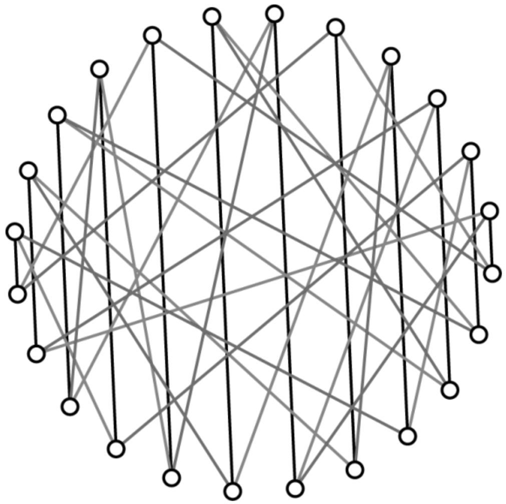


examples: Ramanujan, Cayley





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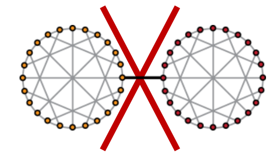
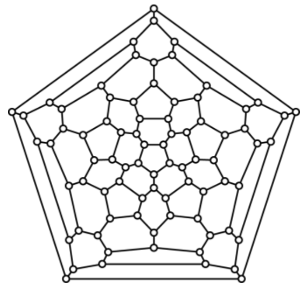
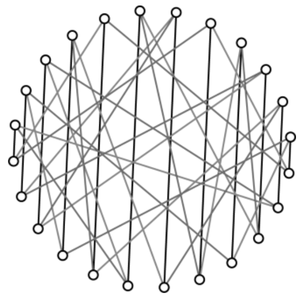




- explicit, constant-degree
  approximations to the full graph

  normalized adjacency matrix
  second largest eigenvalue $1-\lambda$

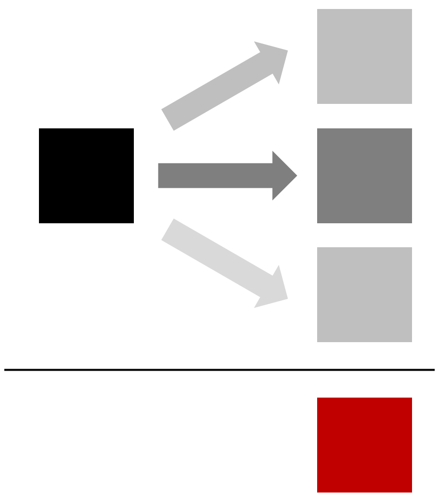


- a review [Hoory Linial Wigderson]
  a talk [Harrow quantum expanders youtube]

1 Mixing up something quantum

- applying random unitaries

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$



- classical expanders:
  explicit, constant-degree approximations to the full graph
  fast-mixing

1 Quantum expanders

- applying random unitaries from a small set
  a discrete approx. to Haar

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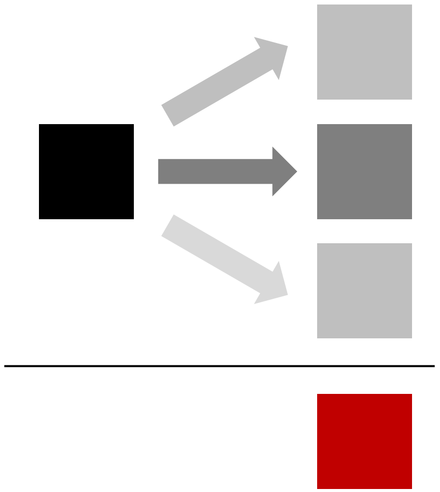

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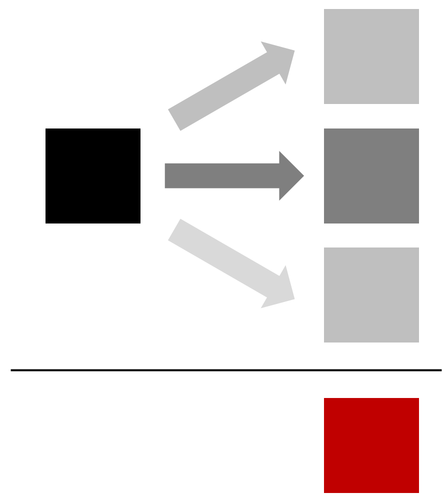
1 Quantum expanders

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$

- applying random unitaries from a small set
  a discrete approx. to Haar
- transform $N \times N$ matrices
  $\mathbb{I}$ stays, everything else changes



$$\|\mathcal{E}(X)\|_2 \leq \lambda \|X\|_2$$

small 2nd largest sing. value

- QE constructions for fixed k (..., 8, 4, 3) $\lambda \approx k^{-c}$
  [Ben-Arroya+ 07, Hastings '07 , Gross & Eisert '08, Hast. & Harr. '09, Gross '15]

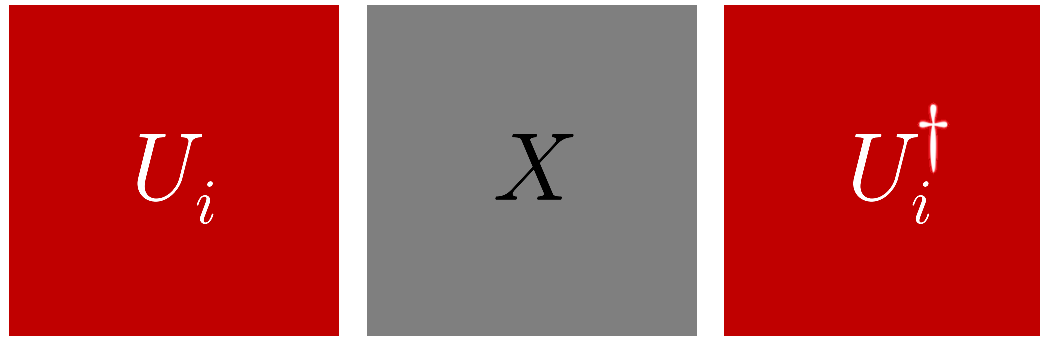
1 Quantum expanders

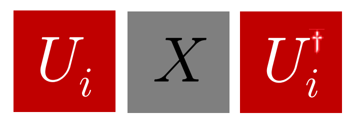
- transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^\dagger$$

a matrix that doesn't change?

$$X = \mathbb{I}$$





$$U_i X = X U_i$$

1 Quantum expanders

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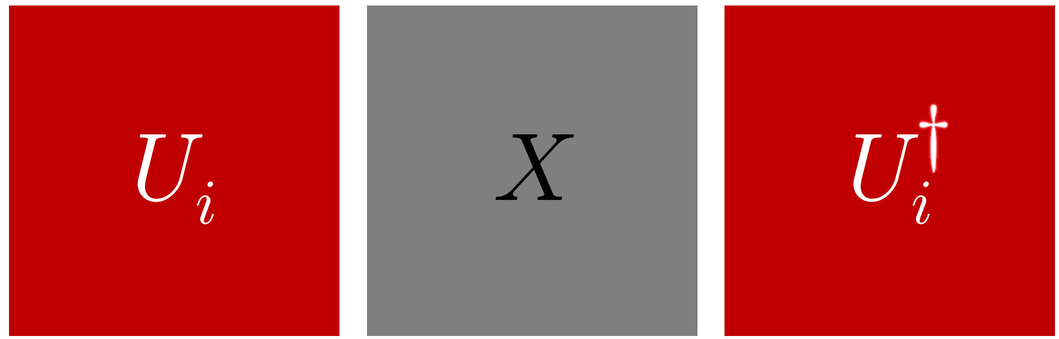

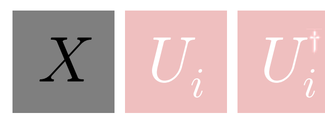

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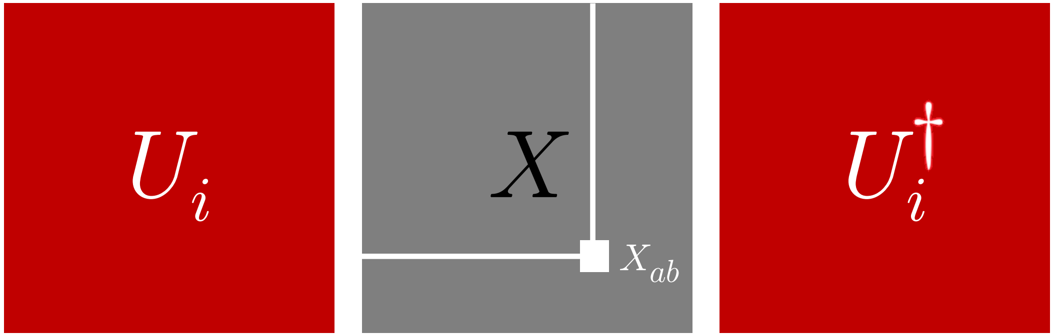


- interpreting matrices as 2-register states

$$X = \sum_{a,b} X_{ab} |a\rangle\langle b|$$

density matrix

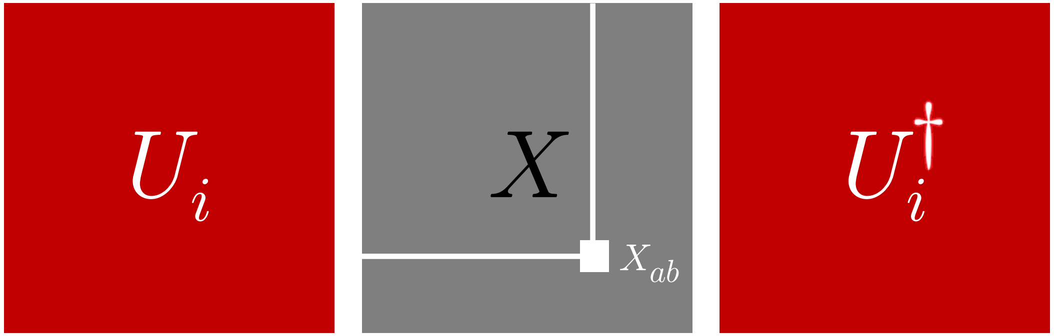
1 Quantum expanders

- transform *N*×*N* matrices

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$

a matrix that doesn't change?

$$X = \mathbb{I}$$



- interpreting matrices as 2-register states

$$|X\rangle = \sum_{a,b} X_{ab}|a\rangle|b\rangle$$

pure state

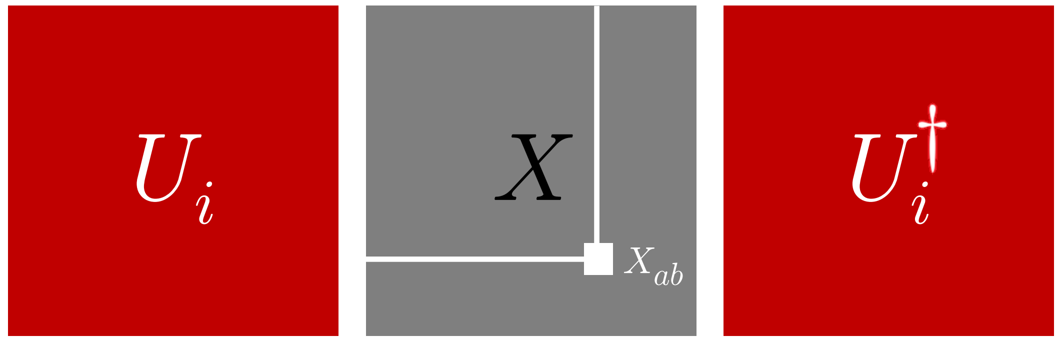
1 Quantum expanders

- transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^\dagger$$

a matrix that doesn't change?

$X = \mathbb{I}$



- interpreting matrices as 2-register states

$$\frac{1}{k} \sum_{i=1}^{k} (U_i \otimes U_i^*) \sum_{a,b} X_{ab} |a\rangle |b\rangle$$

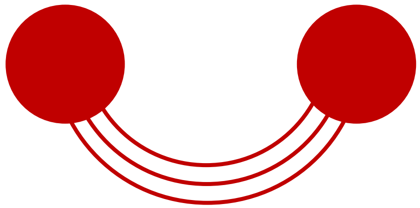
applying an expander distributively

a stationary state?

max. entangled! $\frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle |x\rangle$

1 Quantum expanders & 2 registers

- transform $N \otimes N$ states
  close to the depolarizing channel



$$\|\tilde{\mathcal{E}} - |\phi_N\rangle\langle\phi_N|\| = \lambda$$

- interpreting matrices as 2-register states

$$\frac{1}{k}\sum_{i=1}^{k}(U_i \otimes U_i^*)\sum_{a,b} X_{ab}|a\rangle|b\rangle$$

applying an expander distributively

a stationary state? max. entangled! $\frac{1}{\sqrt{N}}\sum_{x=1}^{N}|x\rangle|x\rangle$

local tests of global entanglement

2 EPR testing

- how costly is it to certify that we share a maximally entangled state?

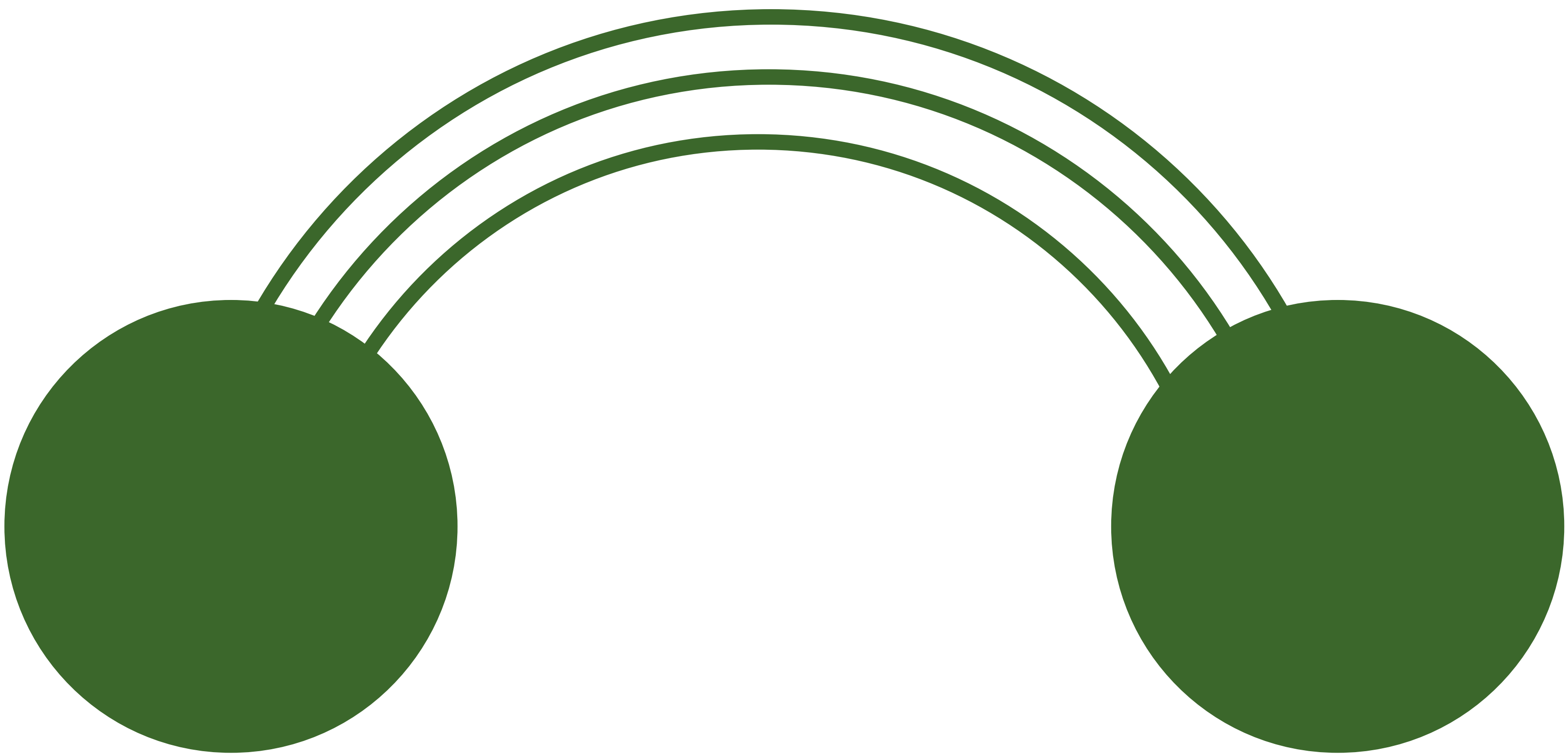
$$\frac{1}{\sqrt{N}}\sum_{x=1}^{N}|x\rangle|x\rangle$$

$O(\log\log(N) + \log(1/\lambda))$ qubits. [BDSW '96, BCGST '02]

2 EPR testing

- how costly is it to certify that we share a maximally entangled state?

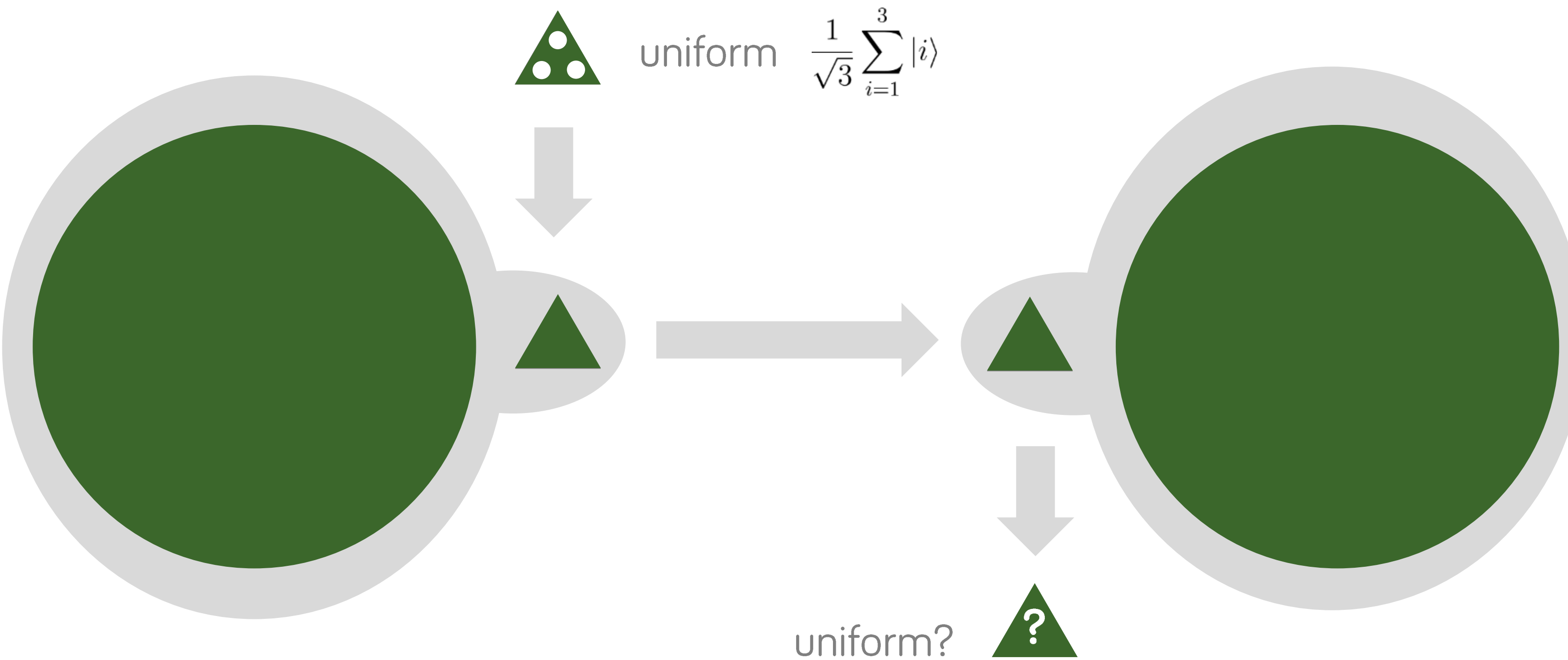
$$\frac{1}{\sqrt{N}}\sum_{x=1}^{N}|x\rangle|x\rangle$$



$O(\log(1/\lambda))$ qubits for error λ.

2 EPR testing

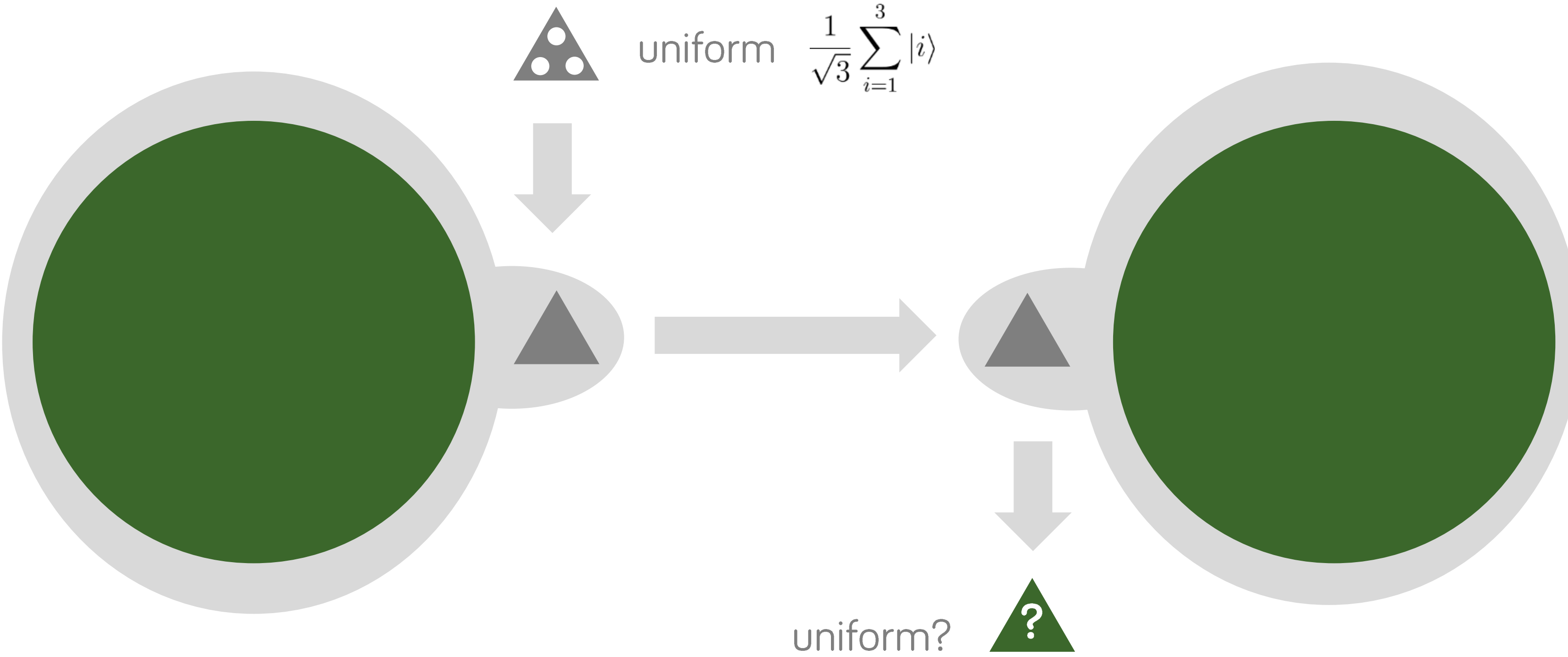
- how costly is it to certify that we share a maximally entangled state? $\frac{1}{\sqrt{N}}\sum_{x=1}^{N}|x\rangle|x\rangle$
- apply a quantum expander distributively $U_i \otimes U_i^*$



2 EPR testing

- when does the qutrit remain uniform?

$$\frac{1}{\sqrt{3}}\sum_{i=1}^{3}|i\rangle\left(U_i\otimes U_i^*\right)\sum_{a,b}X_{ab}|a\rangle|b\rangle$$

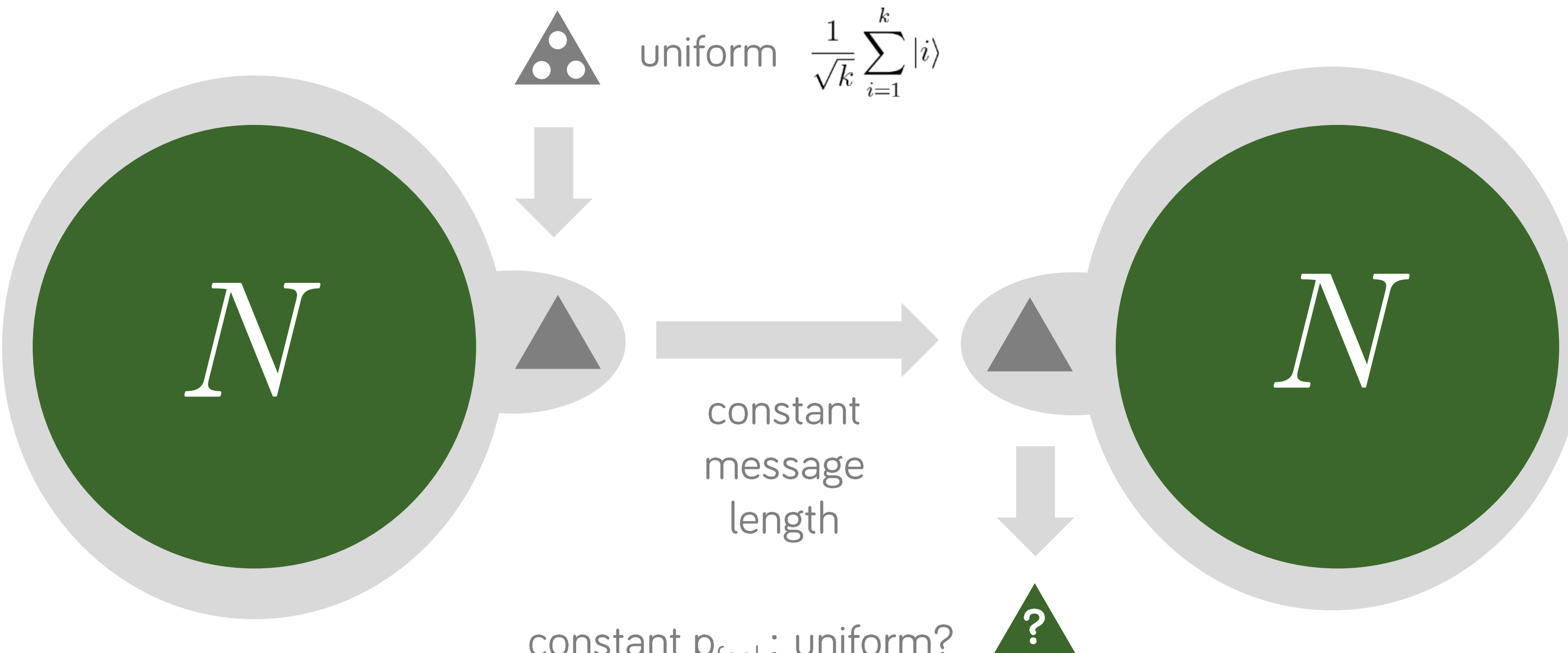


2 EPR testing

- when does the qutrit remain uniform?
  for max. entangled X

$$\frac{1}{\sqrt{3}}\sum_{i=1}^{3}|i\rangle\,(U_i \otimes U_i^*)\,\frac{1}{\sqrt{N}}\sum_{x=1}^{N}|x\rangle|x\rangle$$

- quantum expander property … soundness



little communication

local tests

constant error

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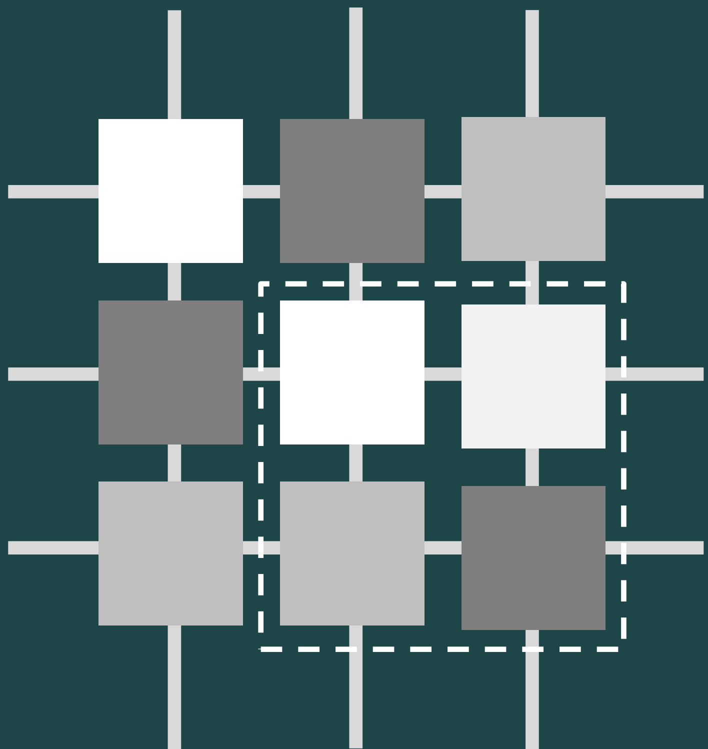
global correlations



a counterexample to the

generalized

area law

few connections

local interactions

constant gap

---

local correlations

few connections

local interactions

constant gap

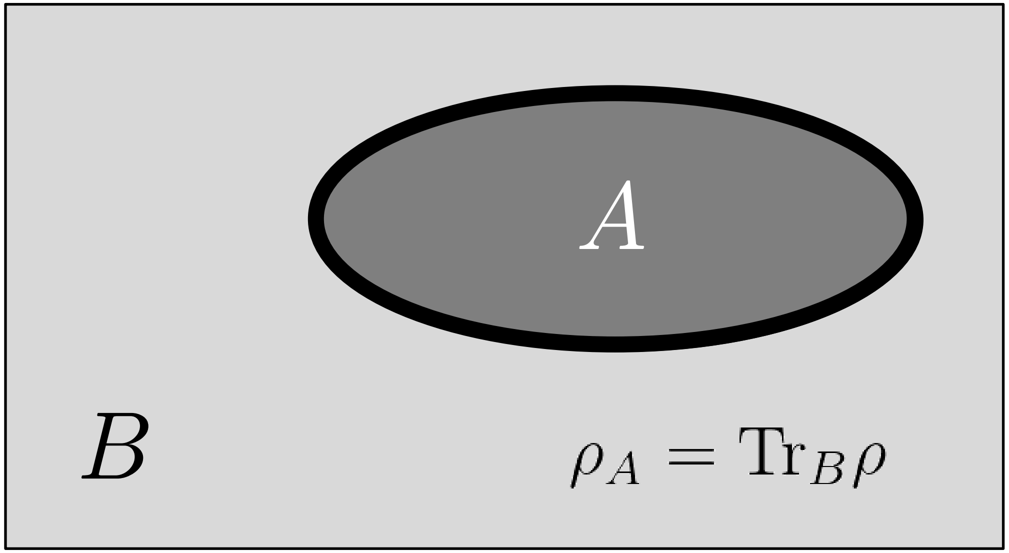
---

global correlations

3 Ground states of gapped quantum spin systems

- entanglement entropy

$$S = -\mathrm{Tr}(\rho_A \ln \rho_A)$$ ~ ~~volume~~
surface area

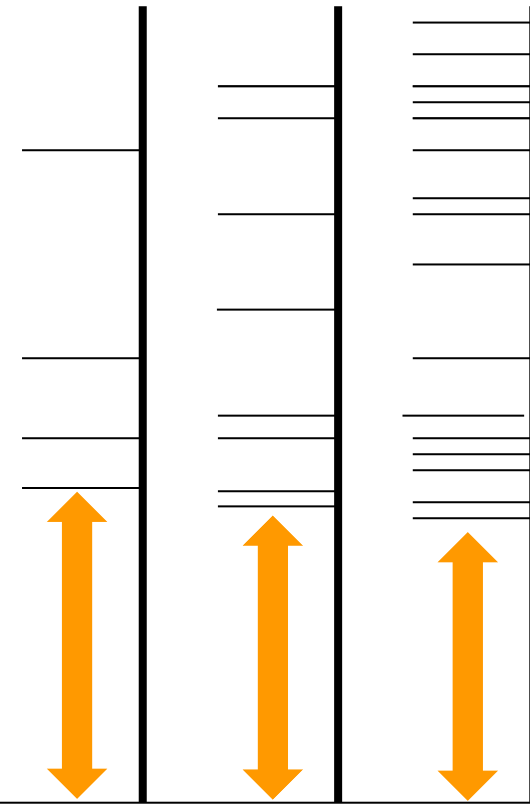


area law → a simple ground state?

3 Gapped 1D Hamiltonians

- Nothing closer than Δ to the ground state.

$N \to \infty$



the AKLT (spin-1) chain

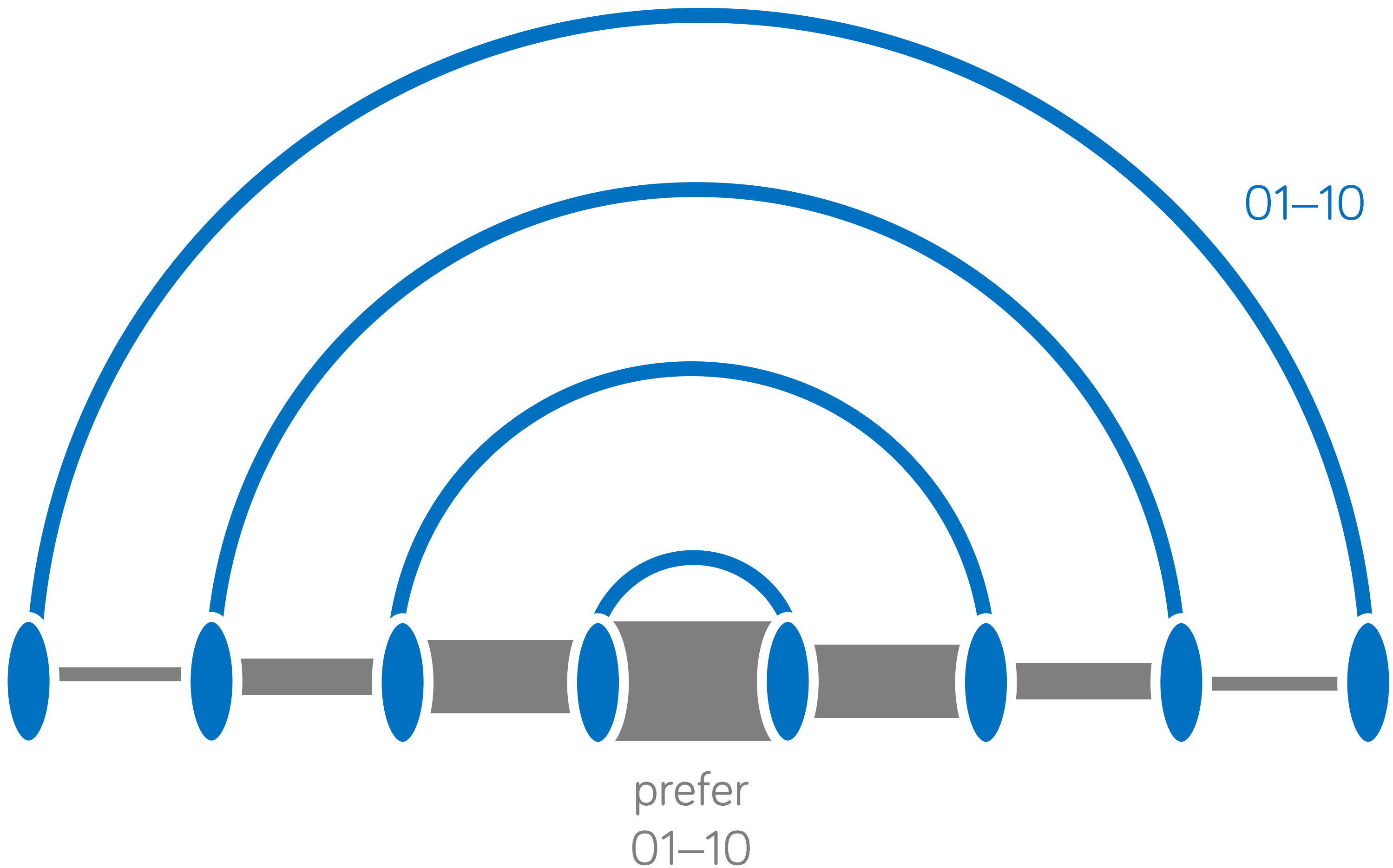
$$\sum_{j=1}^{N-1} \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3}\left(\vec{S}_j \cdot \vec{S}_{j+1}\right)^2$$

a biased walk in 1D

$$\sum_{j=1}^{N-1} \left(|j\rangle - B|j+1\rangle\right)\left(\langle j| - B\langle j+1|\right)$$

exp. falloff of corelations, MPS/DMRG work well

Without a gap, the entropy can be large. [Verstraete, Latorre+]



3 Ground states of gapped H's & the area law

- entanglement entropy

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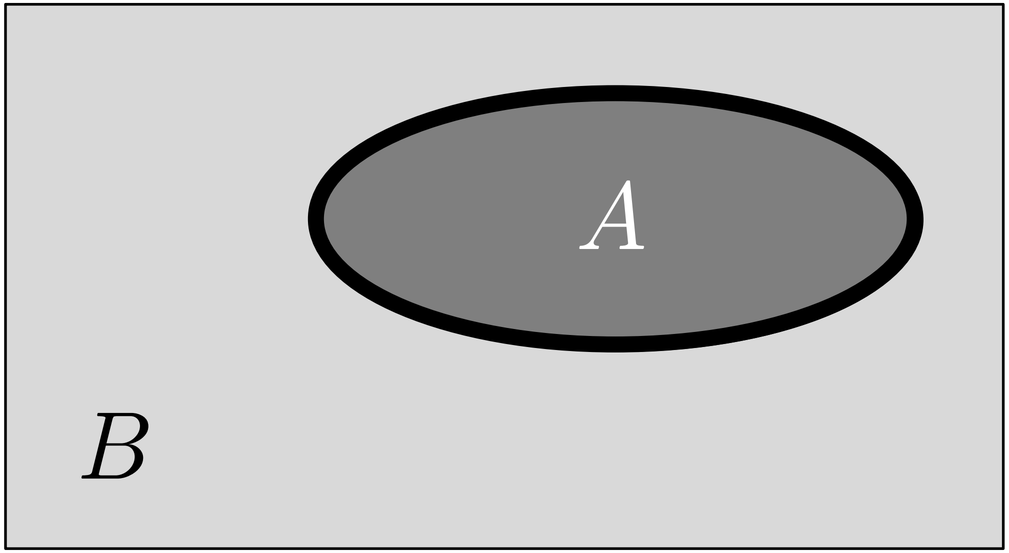

a gapped system ➡ an area law

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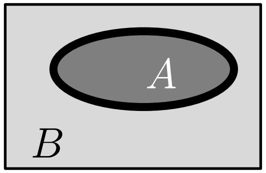


1D … theorems [Hastings 07, Arad+ 13]

algorithms [White 92, Vidal 03, Landau+ 13]

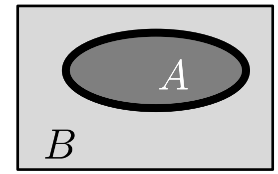
2D … we're close

a gapped system → an area law

3 Ground states of gapped H's & the area law

- entanglement entropy

$$S = -\mathrm{Tr}(\rho_A \ln \rho_A) \sim$$ ~~volume~~ surface area



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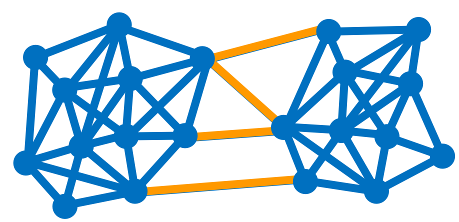
- generalized area conjecture

entropy ~ cut size



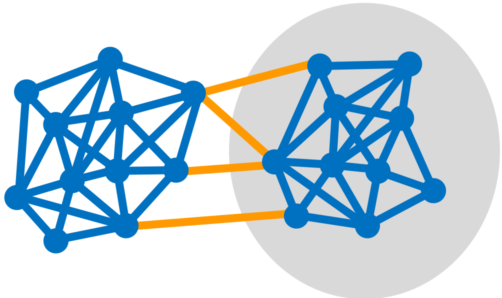
NO

a gap
a few links
O(1) terms

→

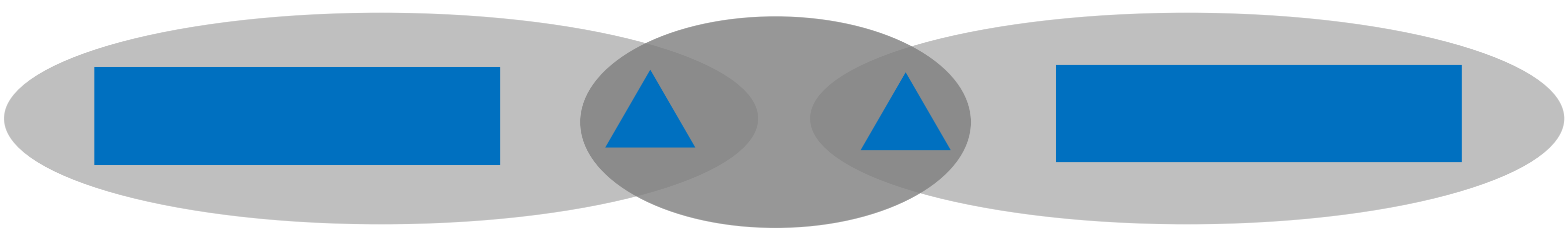
not much entanglement
(a "simple" ground state)

- generalized area conjecture
  entropy ~ cut size



Generalized area conjecture: the counterexample

- an $N \times 3 \times 3 \times N$ dimensional system
- a gapped, frustration-free Hamiltonian
  an $O(1)$ interaction between the qutrits



3 Generalized area conjecture: the counterexample

- an $N \times 3 \times 3 \times N$ dimensional system
- a gapped, frustration-free Hamiltonian
  an $O(1)$ interaction between the qutrits
- a unique, very entangled ground state
  $O(N)$ entanglement entropy across the cut

3 The 4-particle ($N \times 3 \times 3 \times N$) Hamiltonian

- a projector P_L with ground states $\frac{1}{\sqrt{3}}\left(|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle\right)$

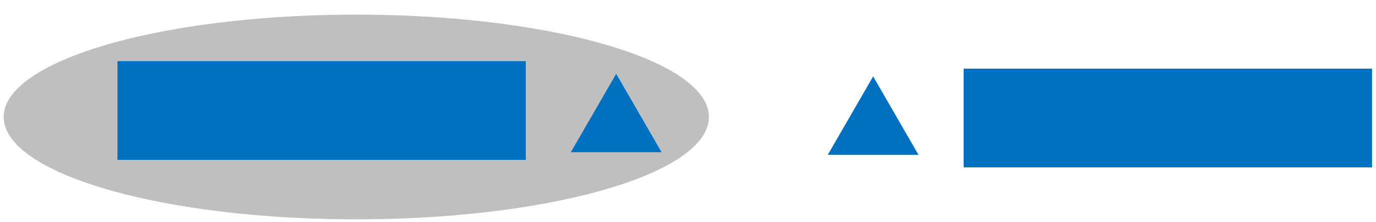
- as a vector

$$\begin{pmatrix} x \\ Ax \\ Bx \end{pmatrix} \otimes |j\rangle \otimes |y\rangle$$

as a matrix

X_1	X_2	X_3
AX_1	AX_2	AX_3
BX_1	BX_2	BX_3

P_L



3 The 4-particle ($N \times 3 \times 3 \times N$) Hamiltonian

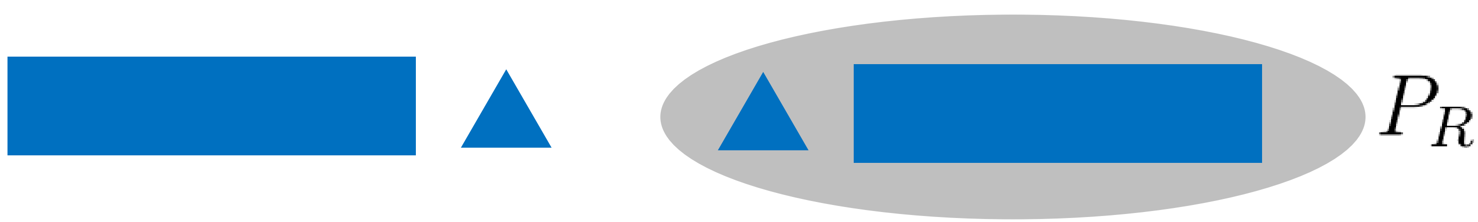
- a projector P_R
  with ground states $$\frac{1}{\sqrt{3}}\left(|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle\right)$$

- as a vector $|i\rangle \otimes |x\rangle \otimes$

y	yA	yB

  as a matrix

Y_1	$Y_1 A$	$Y_1 B$
Y_2	$Y_2 A$	$Y_2 B$
Y_3	$Y_3 A$	$Y_3 B$



3 The 4-particle ($N \times 3 \times 3 \times N$) Hamiltonian

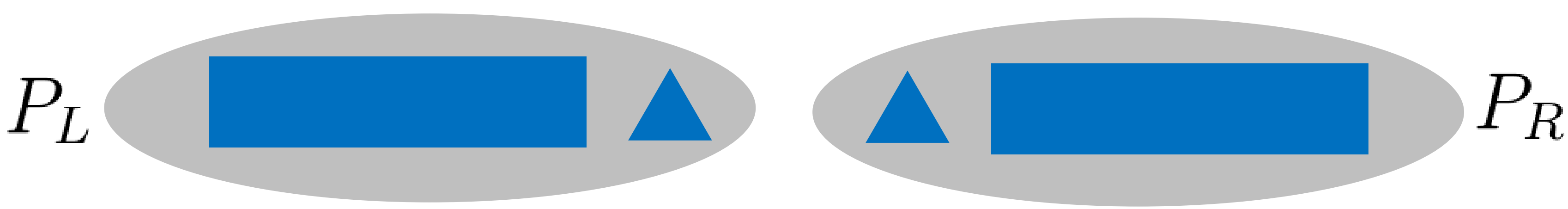
- a projector P_L $\quad \left(|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle\right)/\sqrt{3}$

  a projector P_R $\quad \left(|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle\right)/\sqrt{3}$

X_1	X_2	X_3
AX_1	AX_2	AX_3
BX_1	BX_2	BX_3

X	XA	XB
AX	AXA	AXB
BX	BXA	BXB

Y_1	Y_1A	Y_1B
Y_2	Y_2A	Y_2B
Y_3	Y_3A	Y_3B



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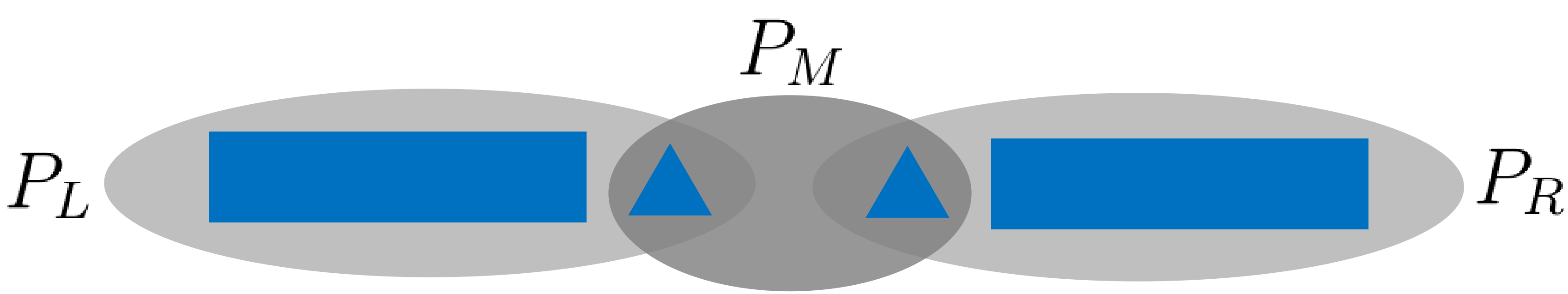
  a projector P_R $\quad (|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)\,/\sqrt{3}$
  a projector P_M

X	XA	XB
AX	AXA	AXB
BX	BXA	BXB



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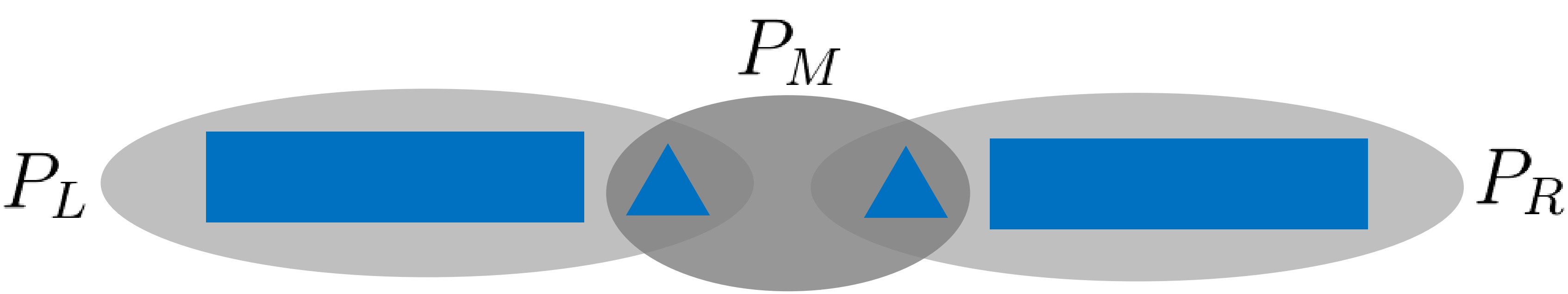
- a projector P_L $\quad (|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle)\,/\sqrt{3}$
  a projector P_R $\quad (|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)\,/\sqrt{3}$
  a projector P_M $\quad$ enforce symmetry: 12 & 21
  13 & 31

- who commutes with A and B?

  only the identity, as $[I, A, B]$ is a q. expander

X	XA	XB
AX	AXA	AXB
BX	BXA	BXB



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  a projector P_R $\quad (|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)\,/\sqrt{3}$

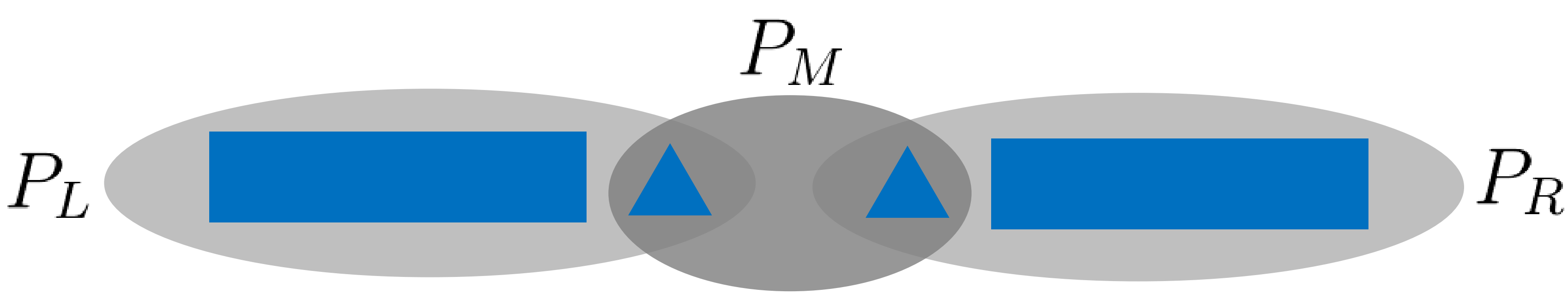
  a projector P_M $\quad$ enforce symmetry: 12 & 21, 13 & 31

- who commutes with A and B?

  only the identity, as $[I, A, B]$ is a q. expander

I	A	B
A	AA	AB
B	BA	BB

$$\frac{1}{3\sqrt{N}}$$



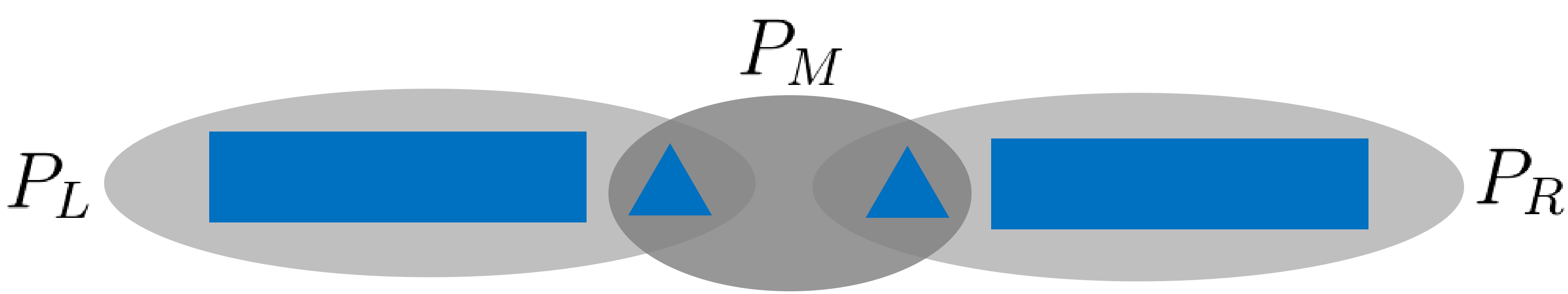
3 The 4-particle ($N \times 3 \times 3 \times N$) Hamiltonian

a unique, very entangled ground state

a gapped Hamiltonian, O(1) terms, frust. free

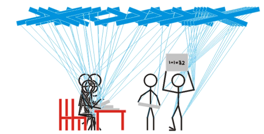
I	A	B
A	AA	AB
B	BA	BB

$$\frac{1}{3\sqrt{N}}$$



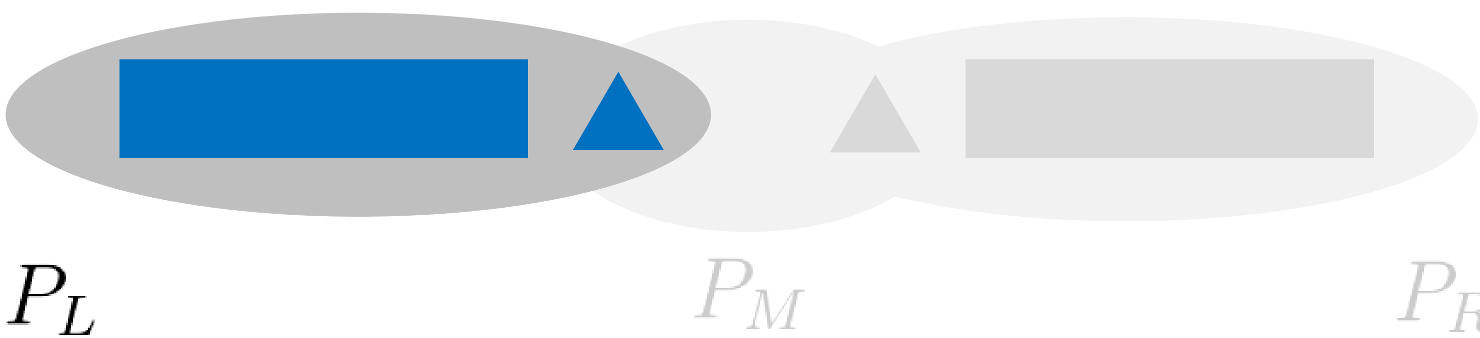
3 Making the counterexample local

- a quantum circuit, history state
  Kitaev's LH, 1D, qudits



prepare $$\frac{1}{\sqrt{3}}\left(|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle\right)$$

from $$\frac{1}{\sqrt{3}}\left(|1\rangle + |2\rangle + |3\rangle\right)|x\rangle$$



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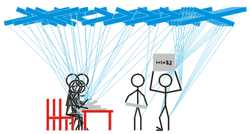
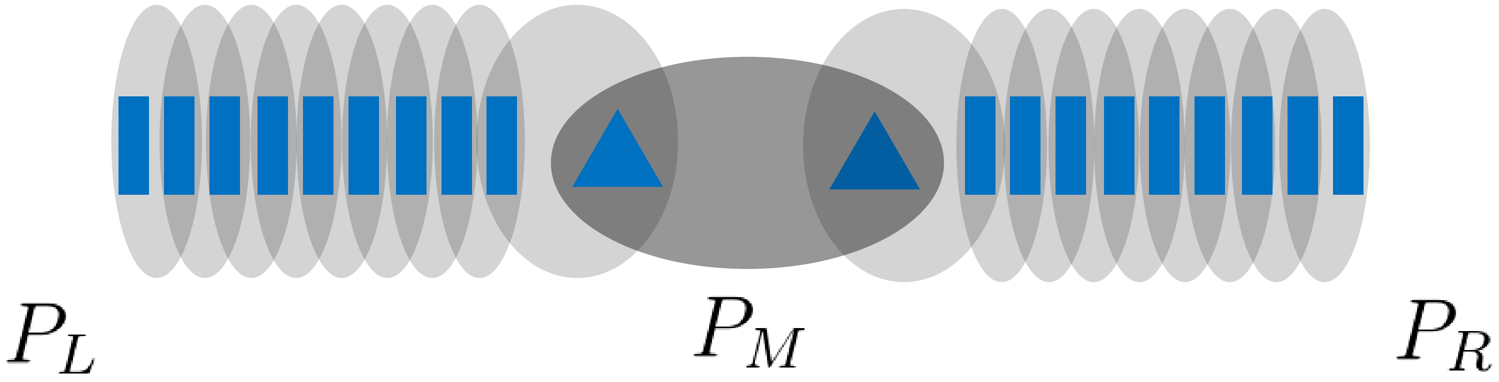


prepare $\frac{1}{\sqrt{3}}\left(|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle\right)$

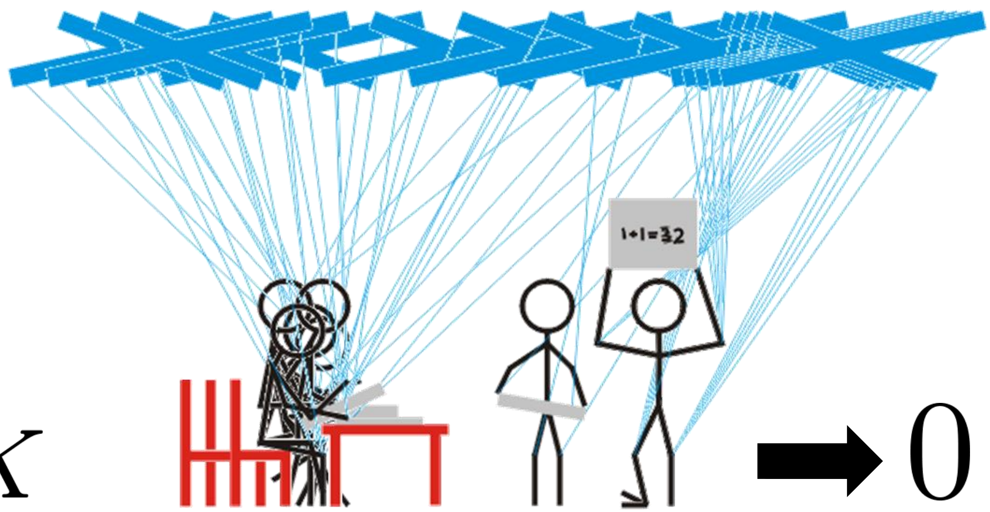
from $\frac{1}{\sqrt{3}}\left(|1\rangle + |2\rangle + |3\rangle\right)|x\rangle$



3 Making the counterexample local

- a quantum circuit, history state
  Kitaev's LH, 1D, qudits

H_K



$\rightarrow 0$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} \underbrace{|\varphi_t\rangle}_{U_t \cdots U_1 |\varphi_0\rangle} \otimes |t\rangle$$

3 The history state: a ground state of a qudit chain

2-local

c-o-n-d-i-t-i-o-n-s

d=13 [AGIK'08]

d=8 [HNN'13]

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

3 The history state: a ground state of a qudit chain

2-local

c-o-n-d-i-t-i-o-n-s

clock encoding

state progression

initialization

idling

$$|\cdots\rangle|0\rangle \otimes |0\rangle$$

$$|\varphi_t\rangle \otimes |t\rangle$$

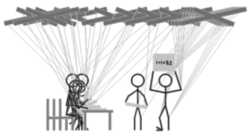
$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

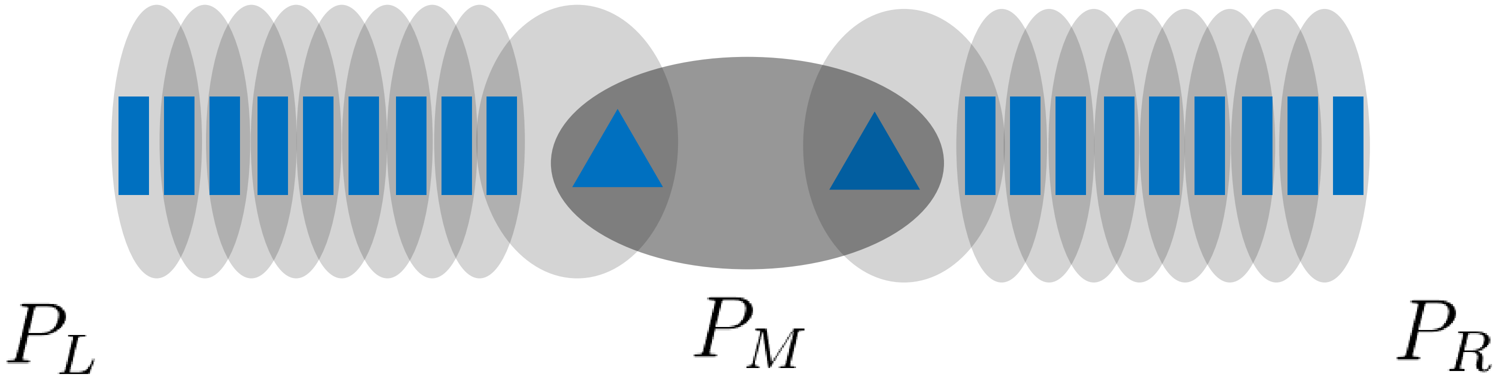
$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

most of the state has the result

3 Making the counterexample local

- a quantum circuit, history state
  Kitaev's LH, 1D, qudits
  an approx. groundstate, a small 1/poly(n) gap





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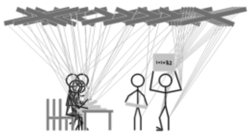
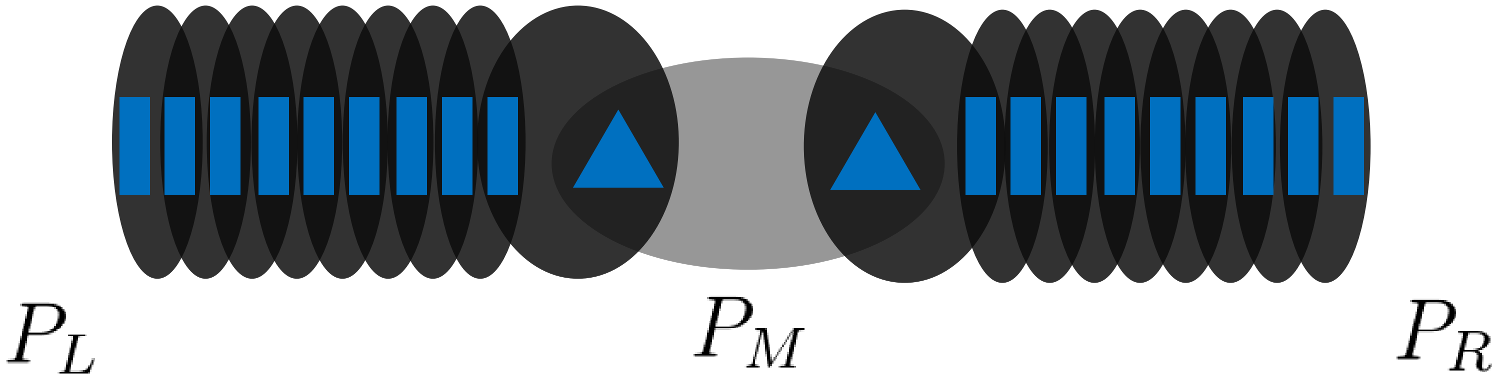


- rescale P_L, P_R (not the middle!)
  a constant gap, huge couplings



3 Making the counterexample local

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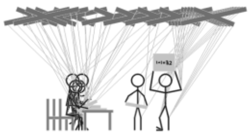

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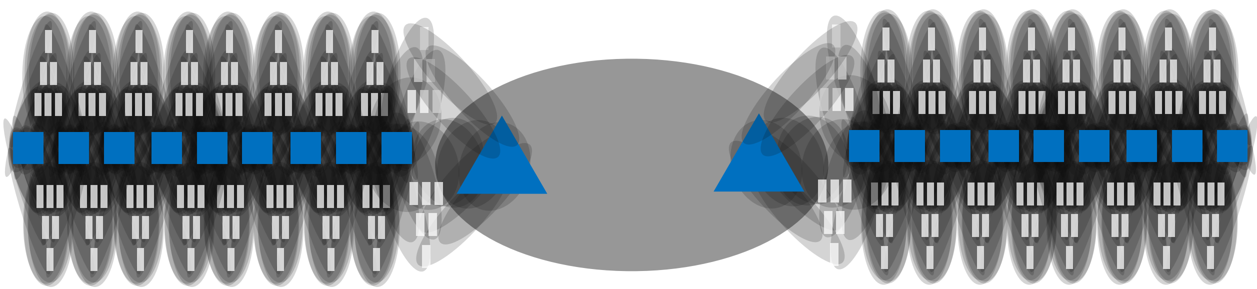
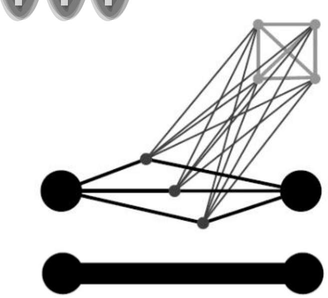


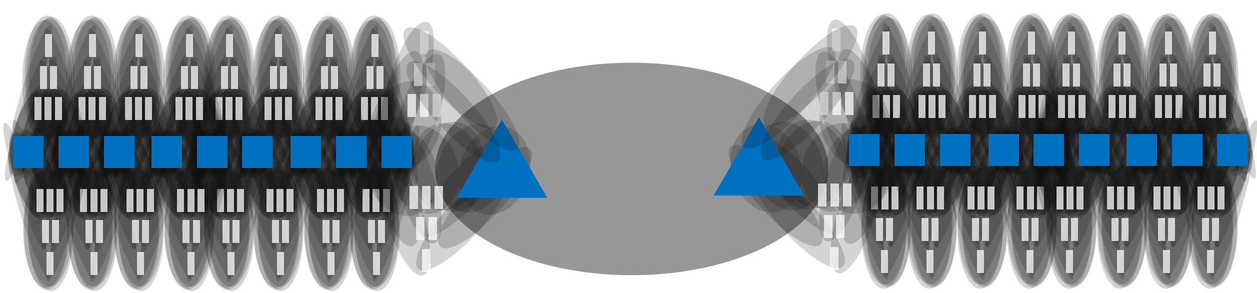
- decompose using gadgets [Cao, N.]
  ~~huge couplings~~ many spins, high degree



3 A local Hamiltonian, (N × M) × 3 × 3 × (N × M)

- a unique and very entangled ground state

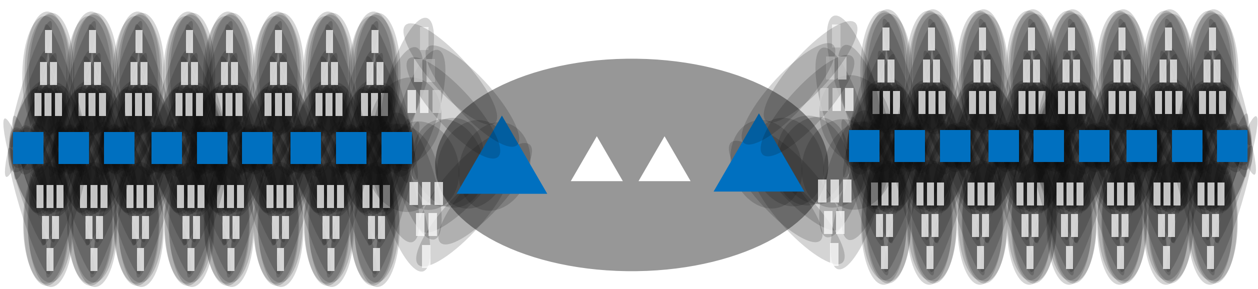
$$\approx |w\rangle \otimes \begin{bmatrix} I & A & B \\ A & AA & AB \\ B & BA & BB \end{bmatrix}$$



- a constant gap
  O(1) or smaller norm terms, frustration

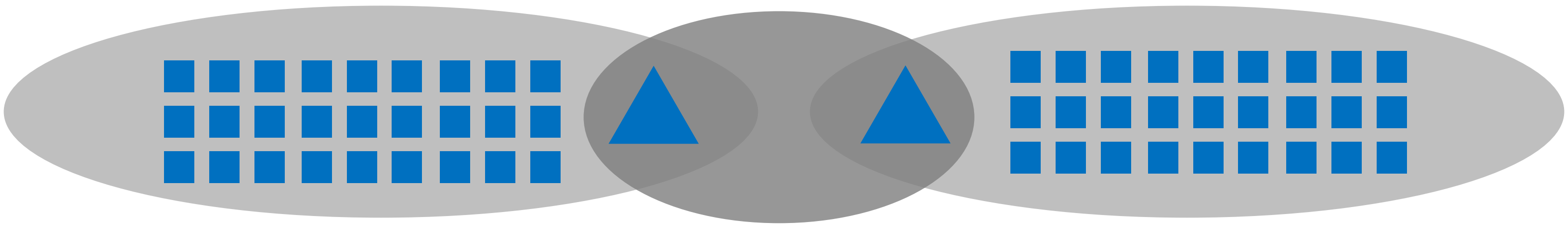
3 What next?

- a longer middle chain?



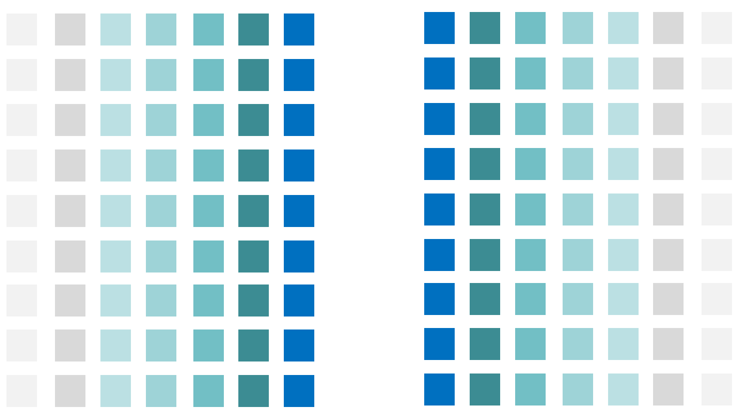
3 What next?

- a longer middle chain?
- a nice lattice on the sides?



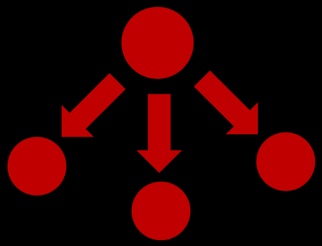
3 What next?

- a longer middle chain?
- a nice lattice on the sides?
- a new area conjecture: count the cut & things nearby?



1 **q. expanders**

maximally entangled states

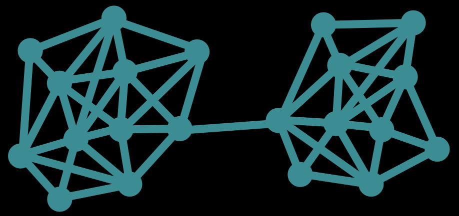


2 **entanglement**

testing and communication

3 **area law**

gaps, connections, correlations



local tests of global entanglement

and a counterexample to the generalized area law

D. Aharonov, A. Harow, Z. Landau, D. Nagaj, M. Szegedy, U. Vazirani

universität wien