local tests of global entanglement

and a counterexample to the generalized area law

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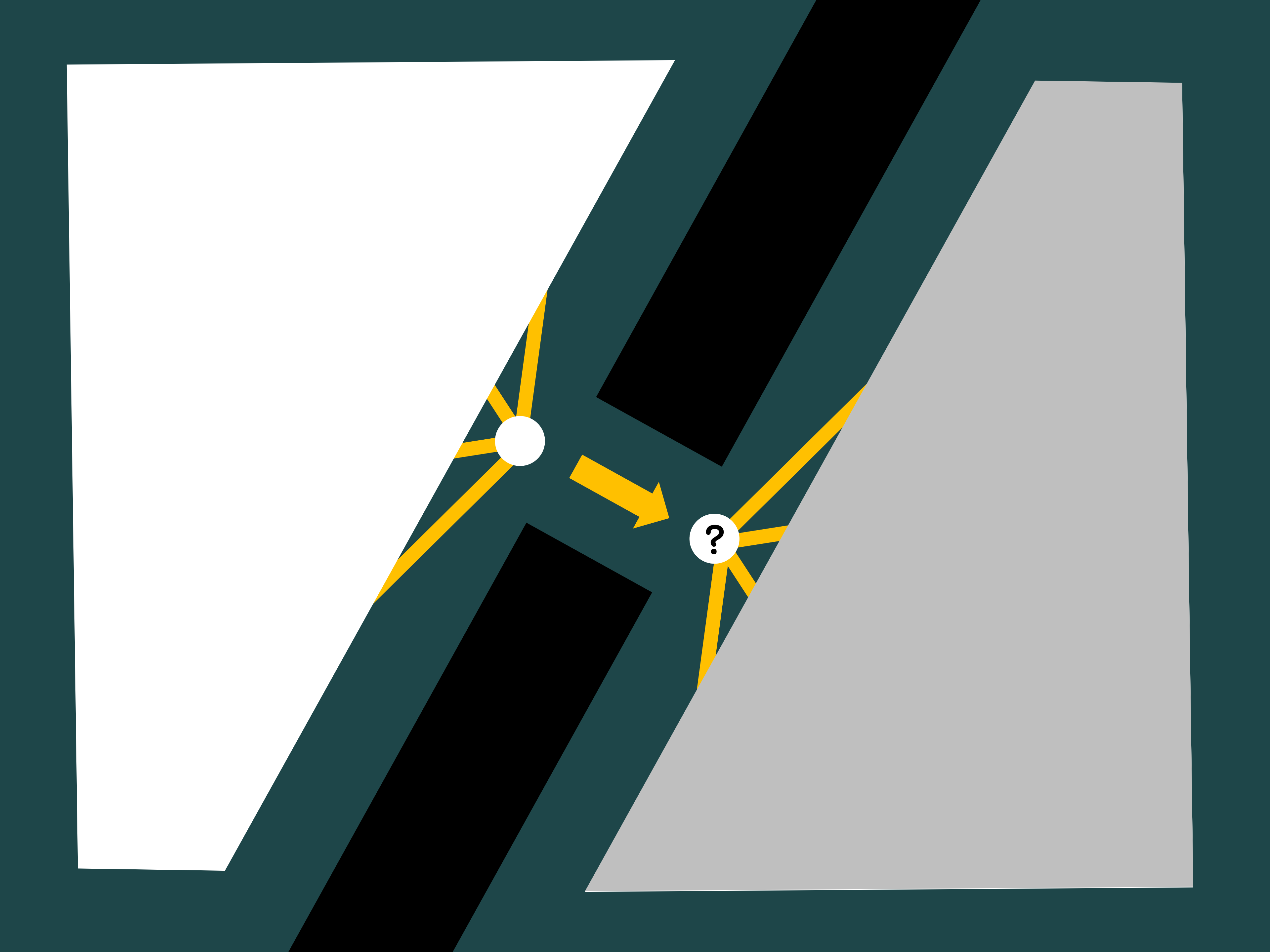
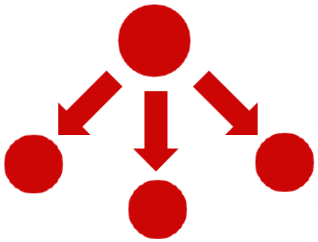
RCQI



1

1 **q. expanders**

maximally entangled states

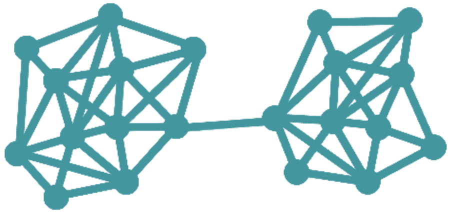


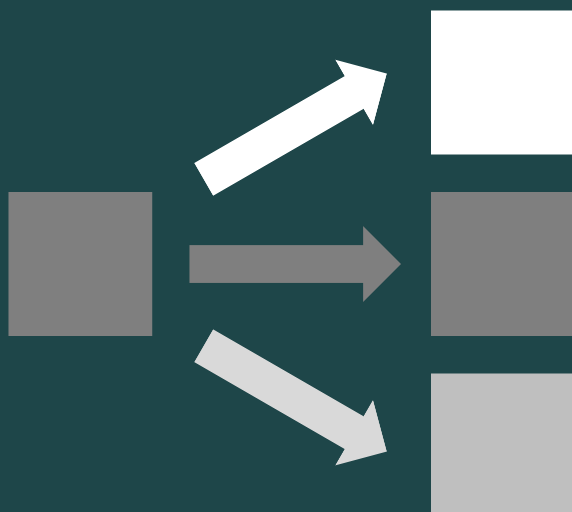
2 **entanglement**

testing and communication

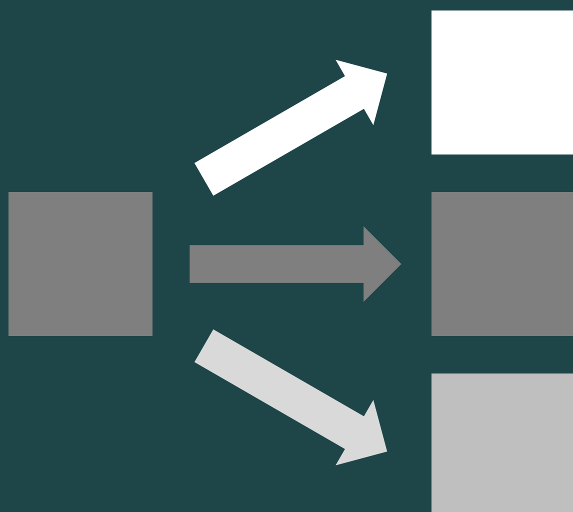
3 **area law**

gaps, connections, correlations





Quantum Expanders



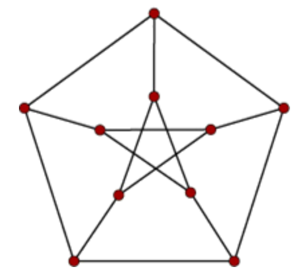
Expanders Everywhere!
1-5 December 2014
Neuchâtel
Ragnar Freij, Aalto University
Camilla Hollanti, Aalto University
Pierre-Nicolas Jolissaint, Uni. de Neuchâtel
Emmanuel Kowalski, ETH Zürich
Damian Osajda, IMPAN and Uniwersytet
Hervé Oyono-Oyono, Université de
Joachim Rosenthal, University of Zürich
Alina Vdovina, University of Newcastle
Organisers:
Ana Khukhro, Université de Neuchâtel
Alain Valette, Université de Neuchâtel
https://sites.google.com/site/expanderseverywhere/
(part of the Swiss Doctoral Program in Mathematics supported by CUSO)

1 Classical expanders

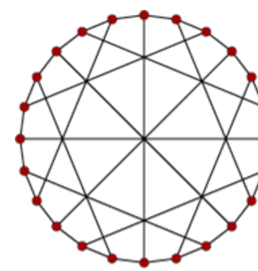
- Aram Harrow's talk, QHC workshop at the
  [youtube Harrow quantum expanders]

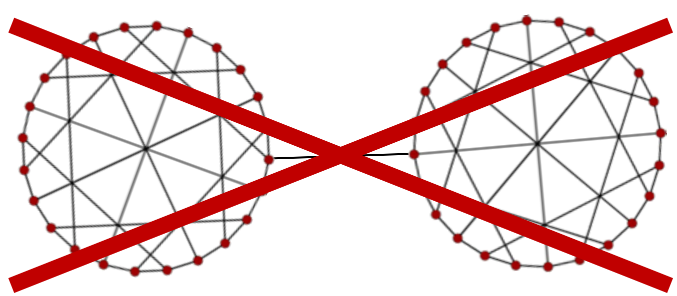
SIMONS INSTITUTE for the Theory of Computing

- graphs that "mix" well
  divide in two? cut a lot (fraction) of edges!



examples: Cayley graphs





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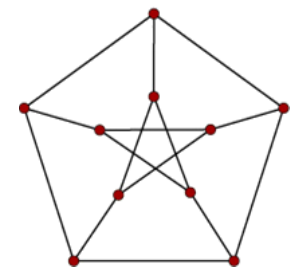
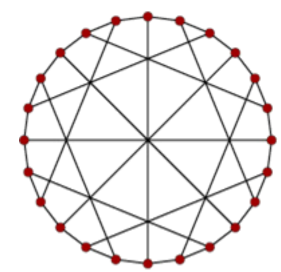


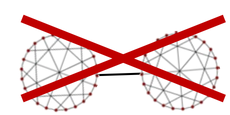
- graphs that "mix" well
  divide in two? cut a lot (fraction) of edges!

  examples: Cayley graphs



- normalized adjacency matrix
  second largest eigenvalue 1–λ

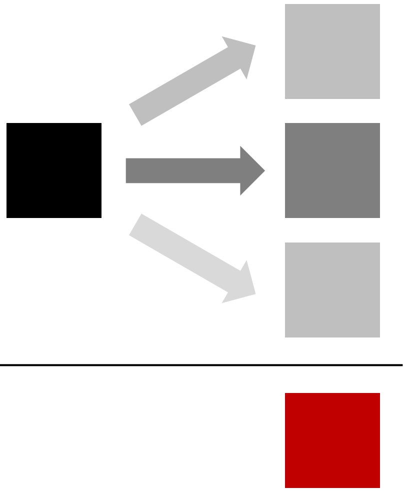
- classical expanders: constant-degree
  approximations to the full graph



1 Mixing up something quantum

- applying random unitaries

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$

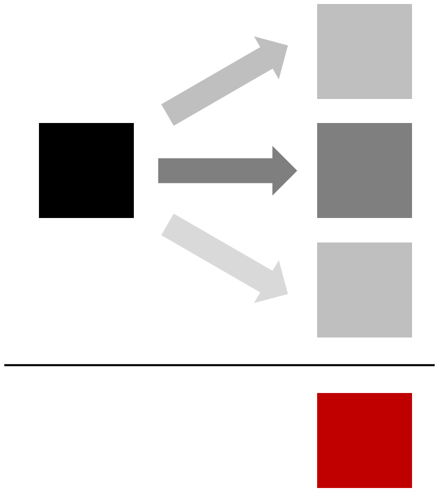


- classical expanders: constant-degree approximations to the full graph

1 Quantum expanders

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$

- applying random unitaries from a set
  a discrete approximation to the Haar measure



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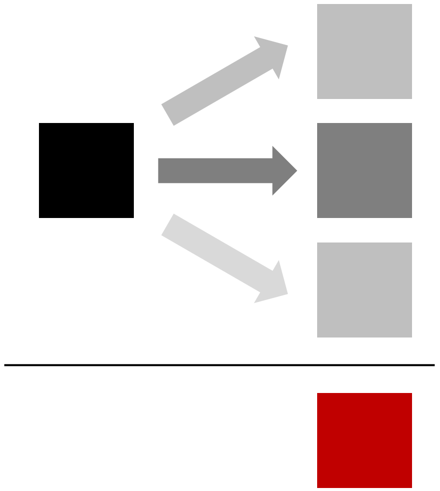

- transform $N \times N$ matrices

- a small second largest singular value λ
  not far from the depolarizing channel

$$\|\hat{\mathcal{E}} - |\phi_D\rangle\langle\phi_D|\| = \lambda$$

- q. expander constructions, also for fixed k (=8. =3?)
  [Ben-Arroya+ 07, Hastings '07 , Gross & Eisert '08, Hastings & Harrow '09]

1 Quantum expanders

- transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$

- a matrix that doesn't change?

$$X = \mathbb{I}$$

$$U_i \quad X \quad U_i^\dagger$$

$$U_i X = X U_i$$

$$X \quad U_i \quad U_i^\dagger$$

1 Quantum expanders

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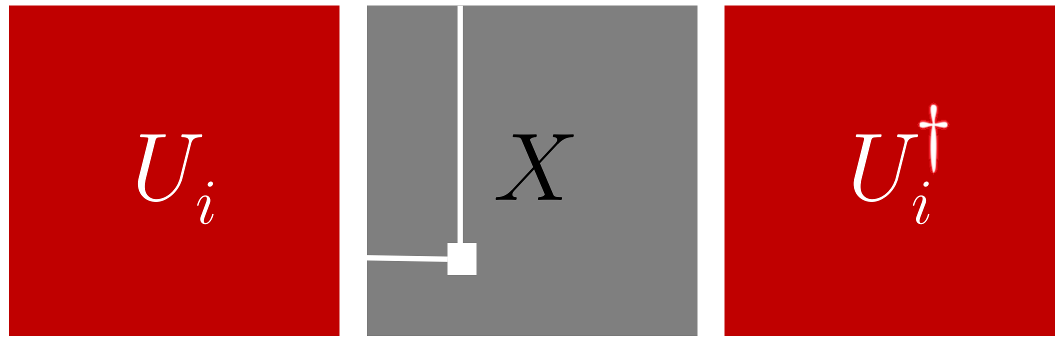

- interpreting matrices as 2-register states

$$\sum_{a,b} X_{ab} |a\rangle\langle b|$$

density matrix

1 Quantum expanders

- transform $N \times N$ matrices

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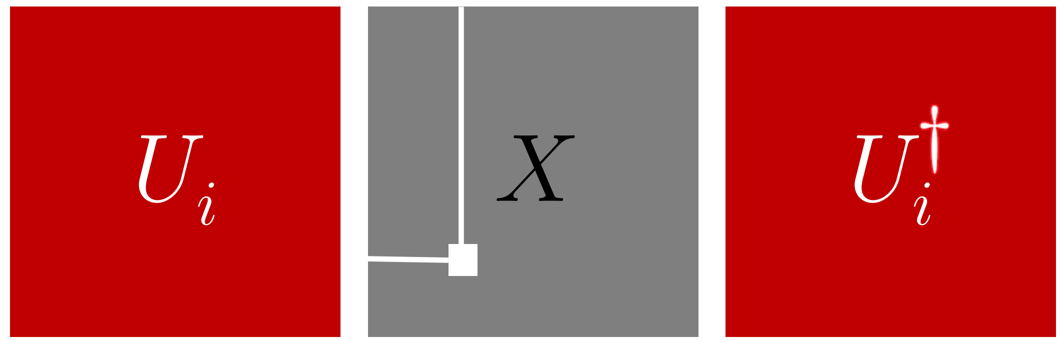

- interpreting matrices as 2-register states

$$\frac{1}{k}\sum_{i=1}^{k} (U_i \otimes U_i^*) \sum_{a,b} X_{ab}|a\rangle|b\rangle$$

$\Big\downarrow$ pure state

- distributively applying an expander

stationary?

$$\sum_{a,b} X_{ab}|a\rangle|b\rangle$$

1 Quantum expanders

- transform $N \times N$ matrices $$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$
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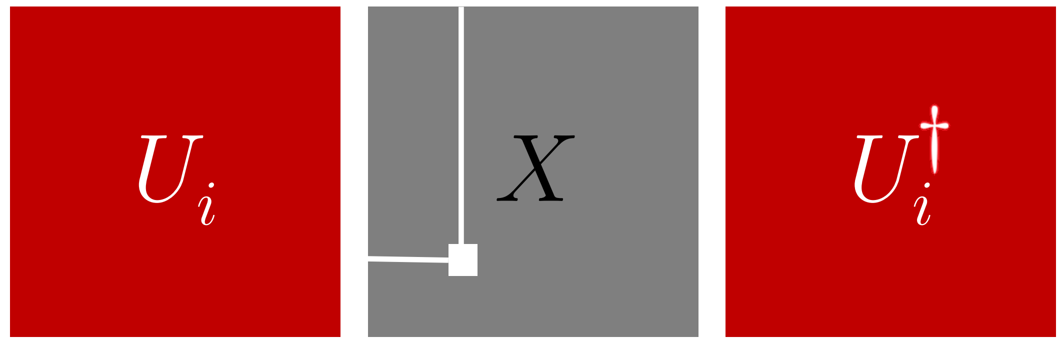


- interpreting matrices as 2-register states $$\frac{1}{k}\sum_{i=1}^{k} (U_i \otimes U_i^*) \sum_{a,b} X_{ab}|a\rangle|b\rangle$$
- distributively applying an expander

stationary? X=I … max. entangled! $$|\Phi_N\rangle = \frac{1}{\sqrt{N}}\sum_{x=1}^{N} |x\rangle|x\rangle$$

local tests of global entanglement

2 EPR testing

- how costly is it to certify that we share a maximally entangled state?

$$\frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle|x\rangle$$

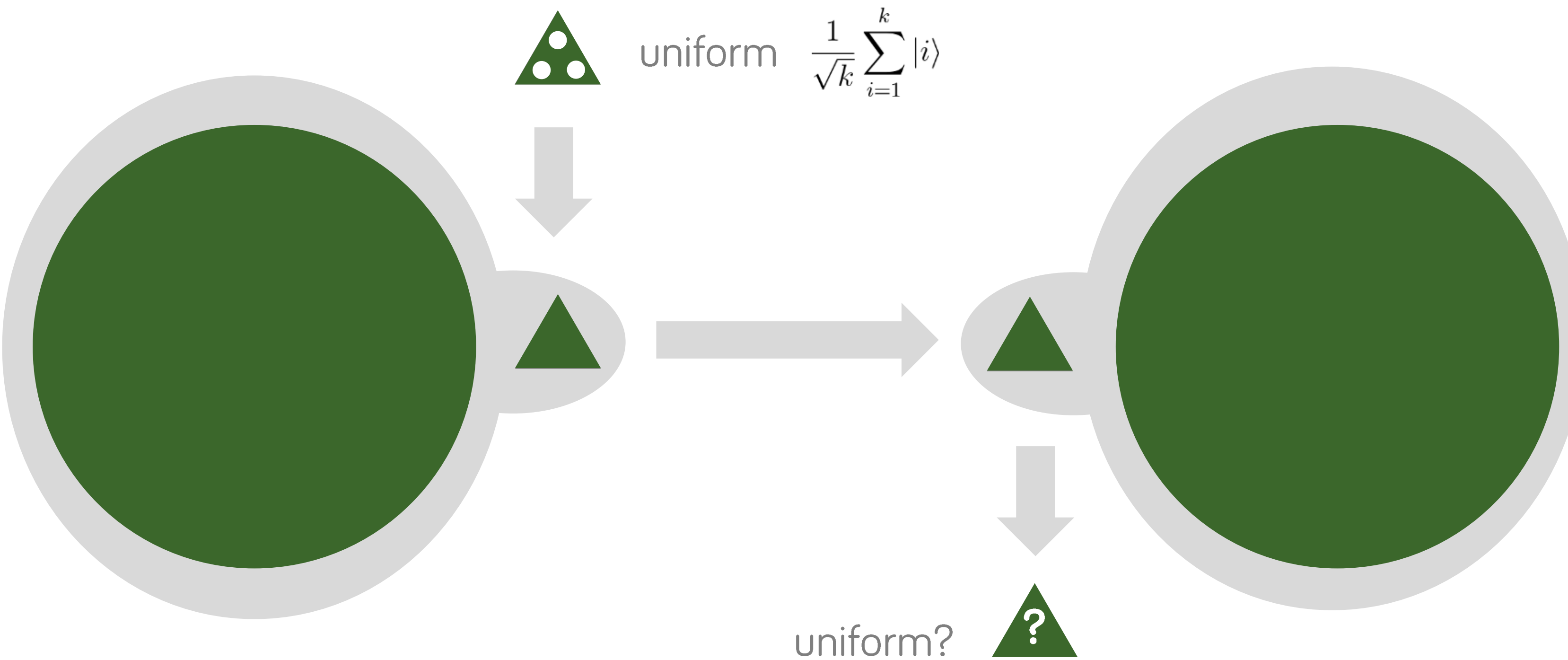
2 EPR testing

- how costly is it to certify that we share a maximally entangled state?

$$\frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle |x\rangle$$

- apply a quantum expander distributively

$$U_i \otimes U_i^*$$



2 EPR testing

- action on states

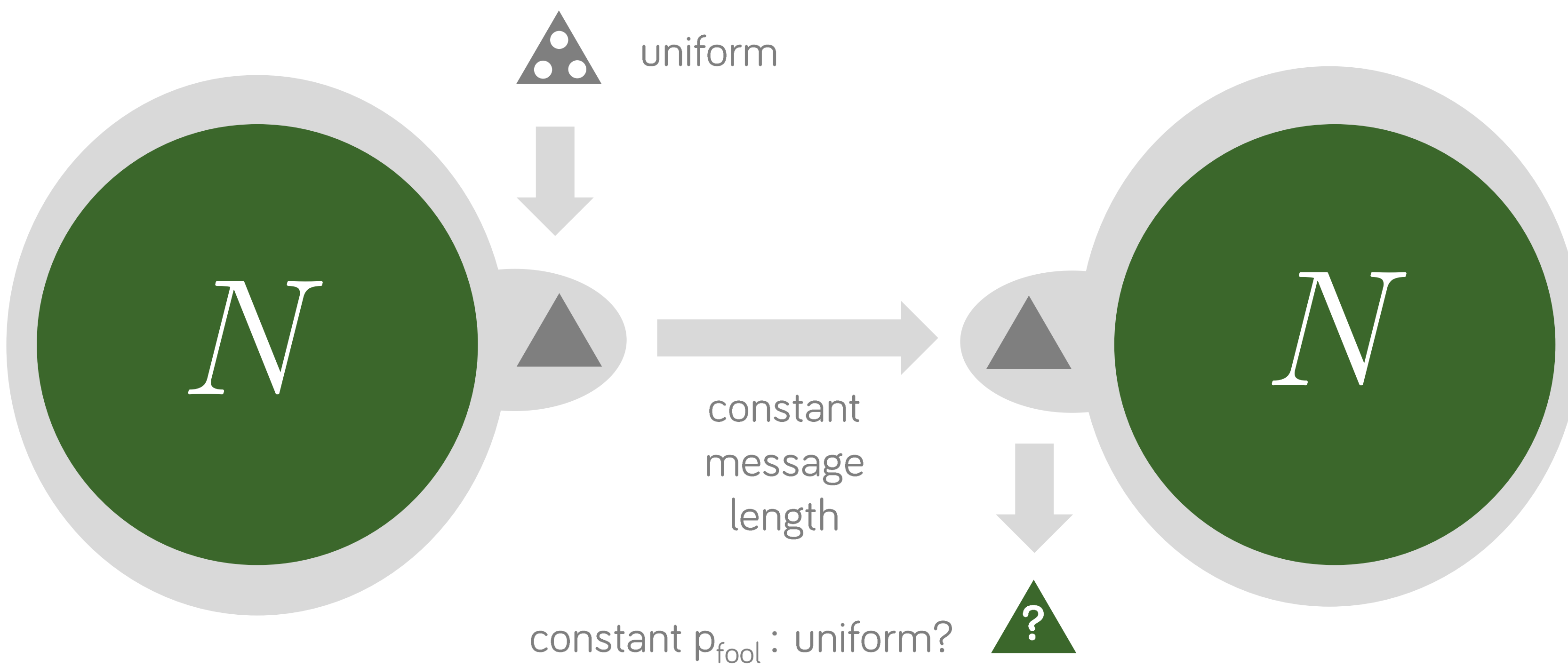
$$\frac{1}{\sqrt{k}}\sum_{i=1}^{k}|i\rangle\,(U_i\otimes U_i^*)\,|X\rangle$$

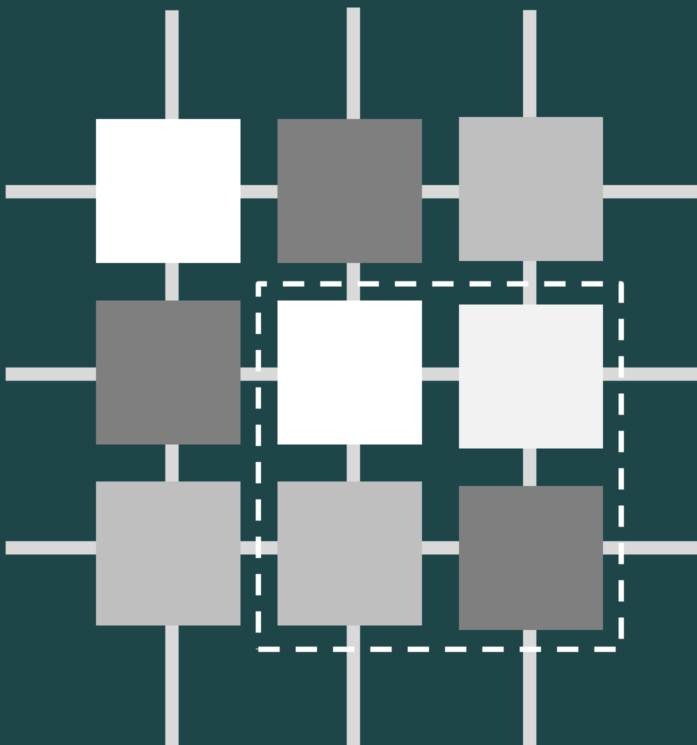
- does the qutrit remain uniform?
  when X commutes with all U_i

$$\sum_{a,b} X_{ab}|a\rangle|b\rangle$$

- quantum expander property ... soundness

$$\boxed{U_i}\;\boxed{X}\;\boxed{U_i^\dagger}$$





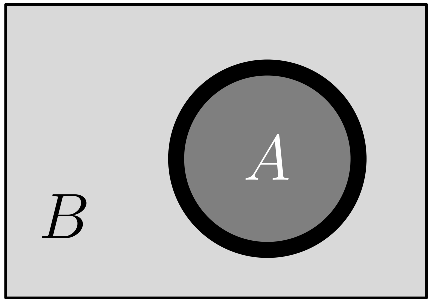
a counterexample to the generalized area law

3 Area law: ground states of (gapped) q. spin systems

- entanglement entropy

$$S = -\mathrm{Tr}(\rho_A \ln \rho_A) \sim \cancel{\text{volume}}$$

area



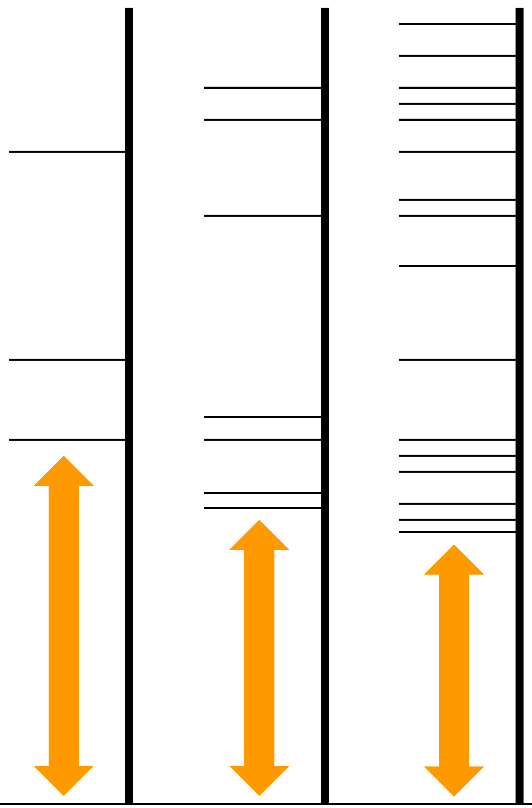
$$\rho_A = \mathrm{Tr}_B \rho$$

a gapped system ... a simple ground state?

3 Gapped Hamiltonians

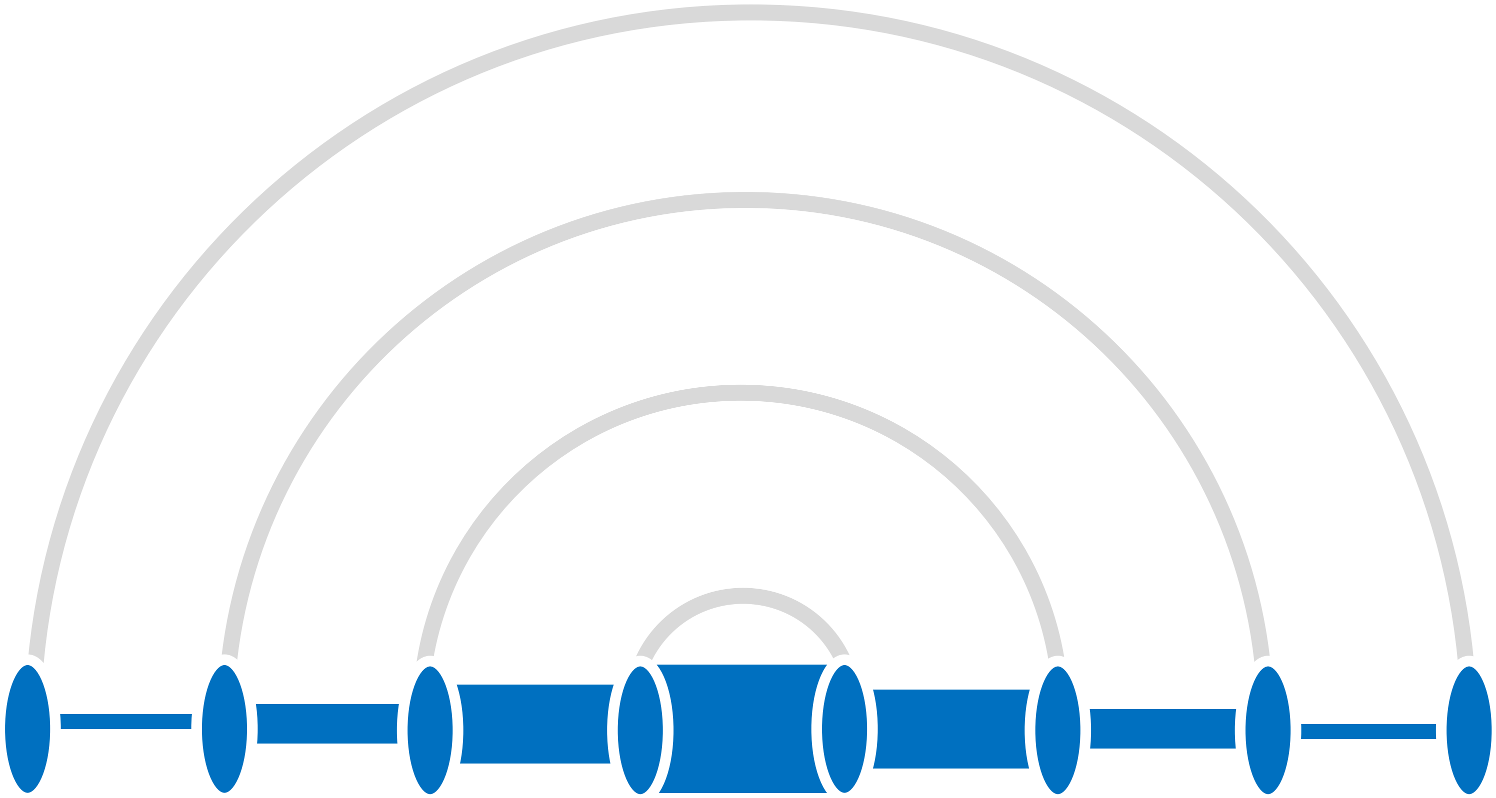
- Are there any states close to the ground state when we take the thermodynamic limit?

$N \to \infty$



- local, $O(1)$ norm terms

~~an inverse-poly gap?~~ $\cancel{\Delta = \frac{c}{N} \to 0}$



- without a gap, the entropy can be large

[Verstraete, Latorre+]

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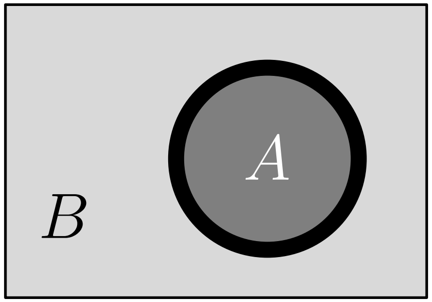
Schmidt coeff's fall off



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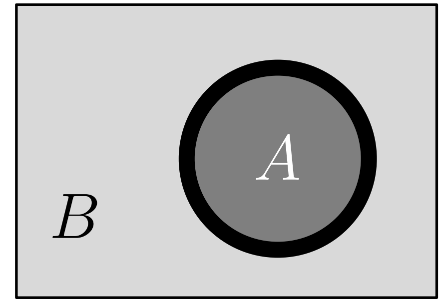
$$S = -\mathrm{Tr}(\rho_A \ln \rho_A) \sim$$ ~~volume~~ area

Schmidt coeff's fall off



$$\rho_A = \mathrm{Tr}_B \rho$$

- 1D … algorithms [White 92, Vidal 03, Landau+ 13]
  theorems [Hastings 07, Arad+ 13]

- 2D … we're close
  small gap? large local dimension?

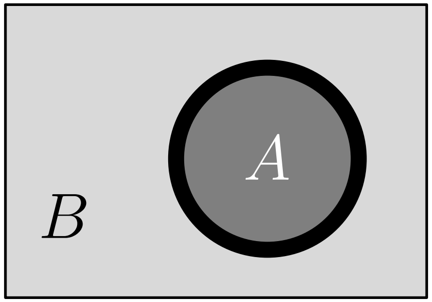
a gapped system … a simple ground state?

3 Area law: ground states of (gapped) q. spin systems

- entanglement entropy

  $S = -\mathrm{Tr}(\rho_A \ln \rho_A)$ ~ ~~volume~~
  area

  Schmidt coeff's fall off



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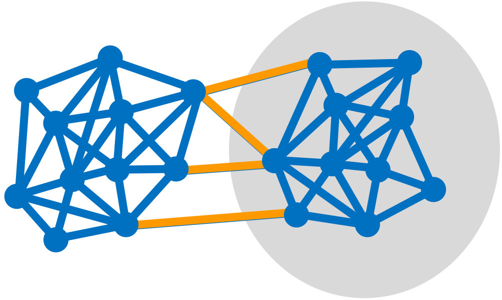
- generalized area conjecture
  entropy ~ cut size



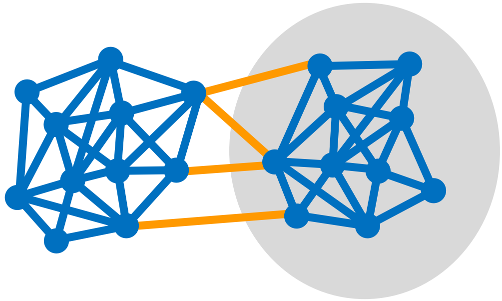
Not true.

a gap
a few links
O(1) terms

→

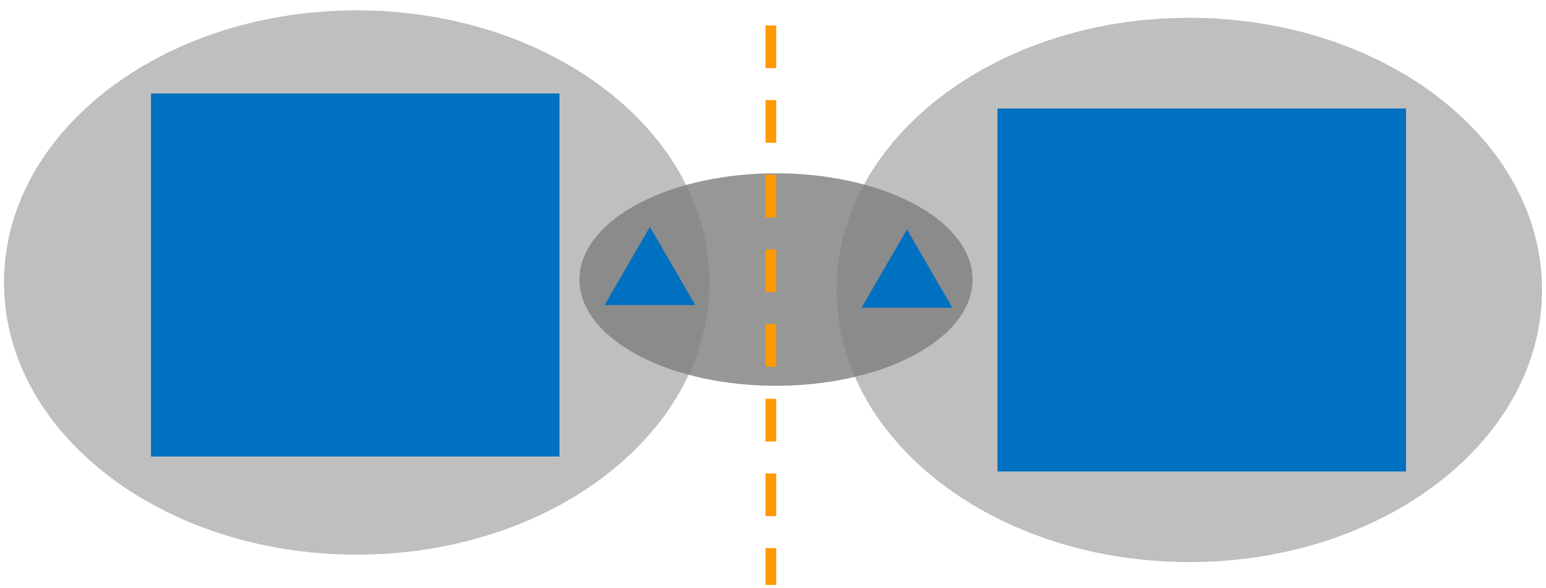
not much entanglement
(a “simple” ground state)

- generalized area conjecture
  entropy ~ cut size



3 Our counterexample to the generalized area conjecture

- an $N \times 3 \times 3 \times N$ dimensional system
- a frustration-free, gapped, Hamiltonian
- a single O(1) interaction of two 3×3 subsystems
- a unique, very entangled ground state with O(N) entanglement entropy across the cut



3 A nonlocal Hamiltonian, $N\times 3\times 3\times N$

- a projector P_L with ground states

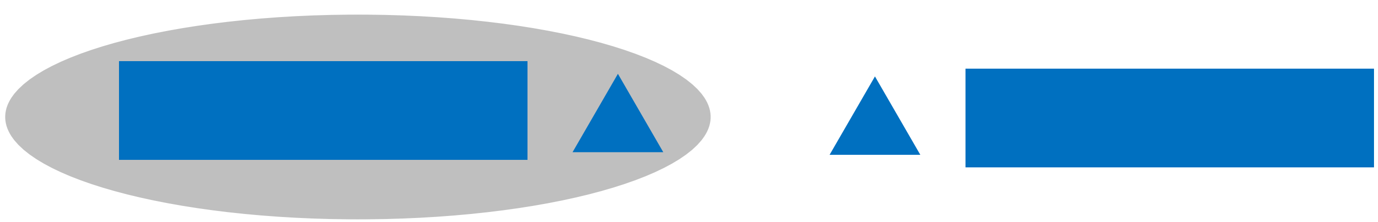
$$\frac{1}{\sqrt{3}}\left(|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle\right)$$

- as a vector

$$\begin{array}{|c|}\hline x \\ \hline Ax \\ \hline Bx \\ \hline \end{array} \otimes |j\rangle \otimes |y\rangle$$

as a matrix

$$\begin{array}{|c|c|c|}\hline X_1 & X_2 & X_3 \\ \hline AX_1 & AX_2 & AX_3 \\ \hline BX_1 & BX_2 & BX_3 \\ \hline \end{array}$$

P_L 

3 A nonlocal Hamiltonian, $N\times 3\times 3\times N$

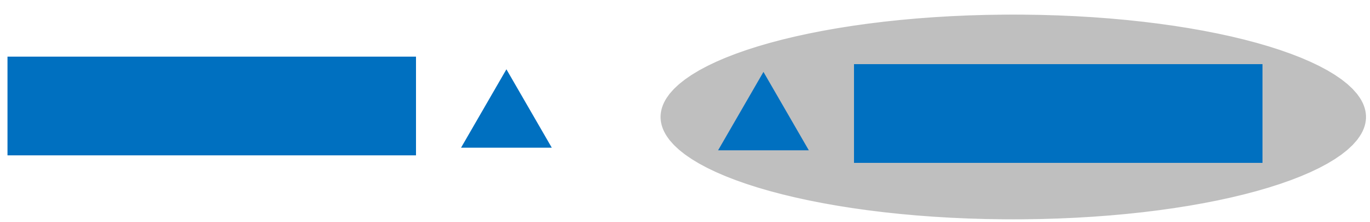
- a projector P_R with ground states $$\frac{1}{\sqrt{3}}\left(|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle\right)$$

- as a vector $|i\rangle \otimes |x\rangle \otimes$

y	yA	yB

as a matrix

Y_1	$Y_1 A$	$Y_1 B$
Y_2	$Y_2 A$	$Y_2 B$
Y_3	$Y_3 A$	$Y_3 B$



P_R

3 A nonlocal Hamiltonian, $N \times 3 \times 3 \times N$

- a projector P_L $\quad (|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle)/\sqrt{3}$

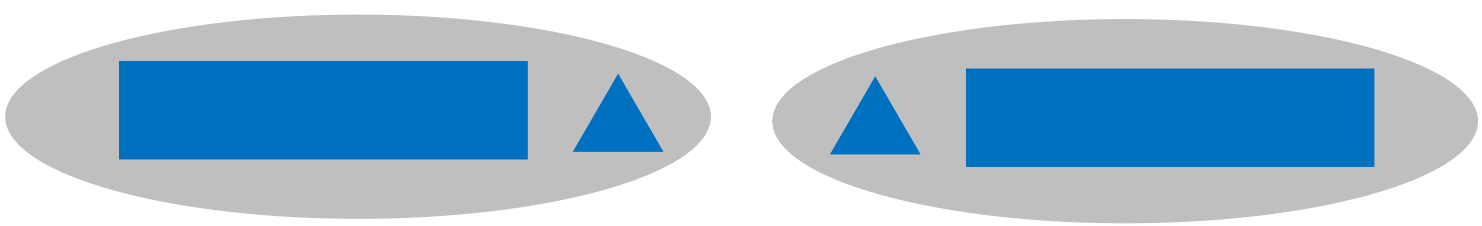
  a projector P_R $\quad (|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)/\sqrt{3}$

X_1	X_2	X_3
AX_1	AX_2	AX_3
BX_1	BX_2	BX_3

X	XA	XB
AX	AXA	AXB
BX	BXA	BXB

Y_1	Y_1A	Y_1B
Y_2	Y_2A	Y_2B
Y_3	Y_3A	Y_3B

P_L



P_R

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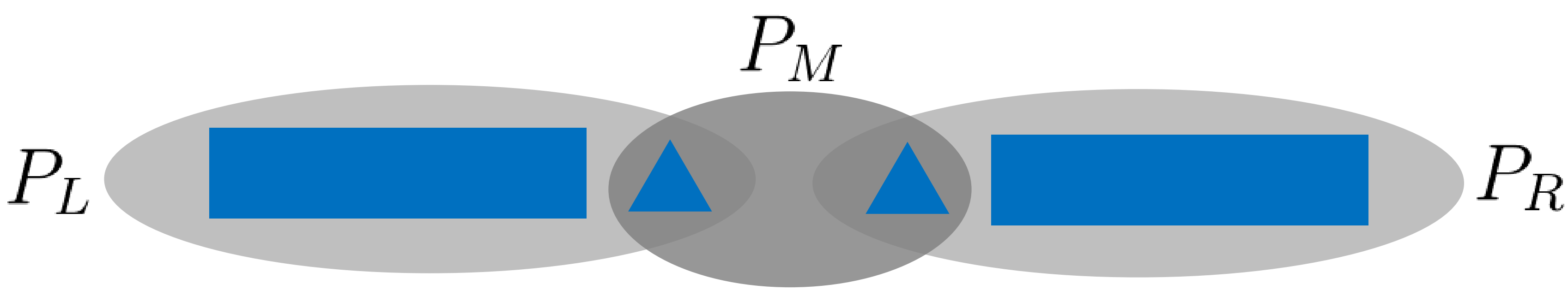
  a projector P_R $\quad (|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)/\sqrt{3}$

  a projector P_M

X	XA	XB
AX	AXA	AXB
BX	BXA	BXB



3 A nonlocal Hamiltonian, $N \times 3 \times 3 \times N$

- a projector P_L
  a projector P_R
  a projector P_M

$$\left(|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle\right)/\sqrt{3}$$

$$\left(|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle\right)/\sqrt{3}$$

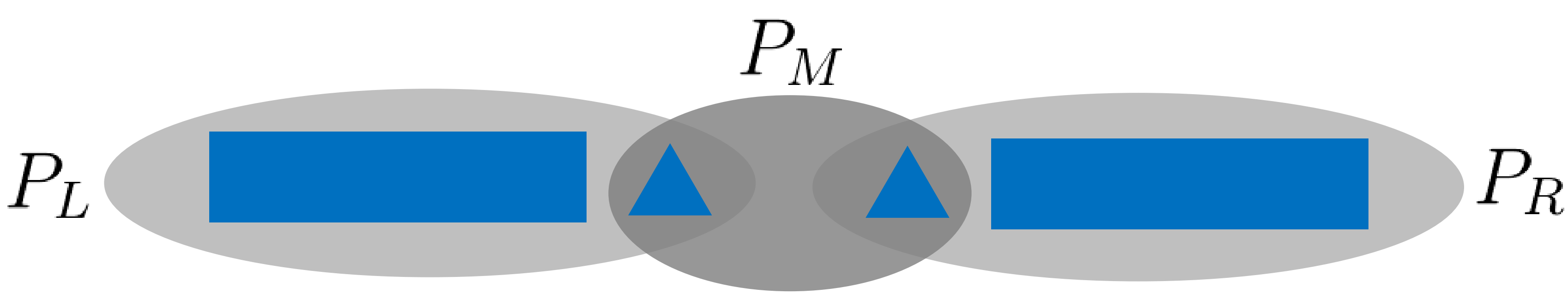
enforce symmetry

for 12 & 21
for 13 & 31

- who commutes with A and B?

only the identity, as $[I, A, B]$ are a q. expander

X	XA	XB
AX	AXA	AXB
BX	BXA	BXB



3 A nonlocal Hamiltonian, $N\times 3\times 3\times N$

- a projector P_L $\qquad$ $\left(|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle\right)/\sqrt{3}$

  a projector P_R $\qquad$ $\left(|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle\right)/\sqrt{3}$

  a projector P_M $\qquad$ enforce symmetry $\quad$ for 12 & 21

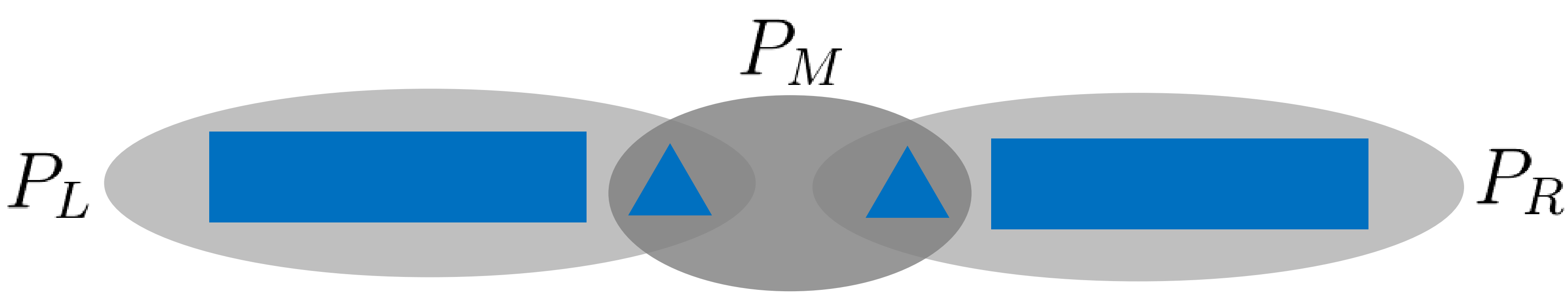
  for 13 & 31

- who commutes with A and B?

  only the identity, as $[I, A, B]$ are a q. expander

I	A	B
A	AA	AB
B	BA	BB

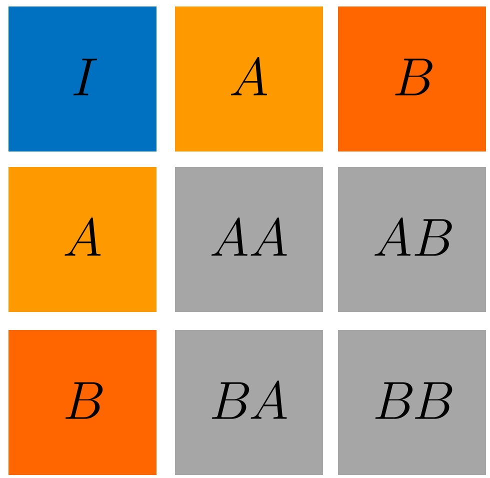
$$\frac{1}{3\sqrt{N}}$$



3 A nonlocal Hamiltonian, $N \times 3 \times 3 \times N$

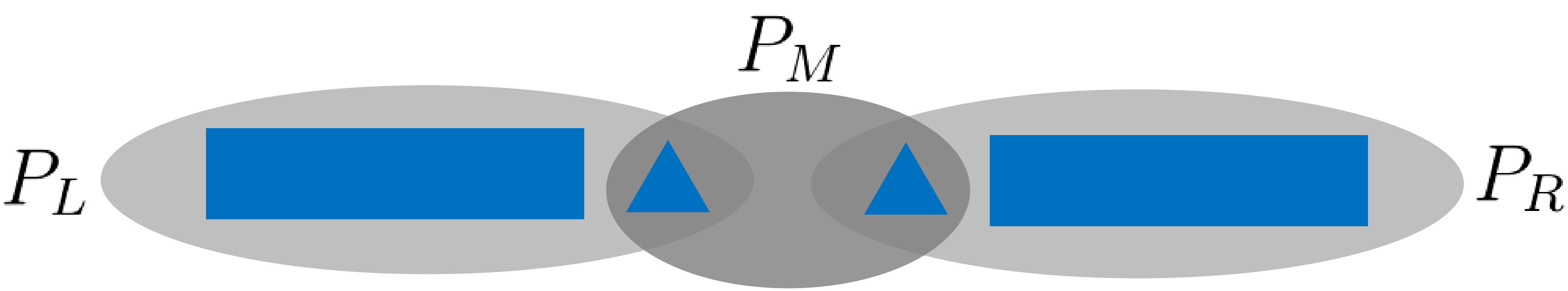
- ground state: unique
  very entangled

- Hamiltonian:
  frustration free
  gapped



$$\sum_{x=1}^{N} |x\rangle|x\rangle$$

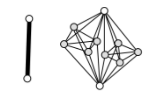
$$\frac{1}{3\sqrt{N}}$$

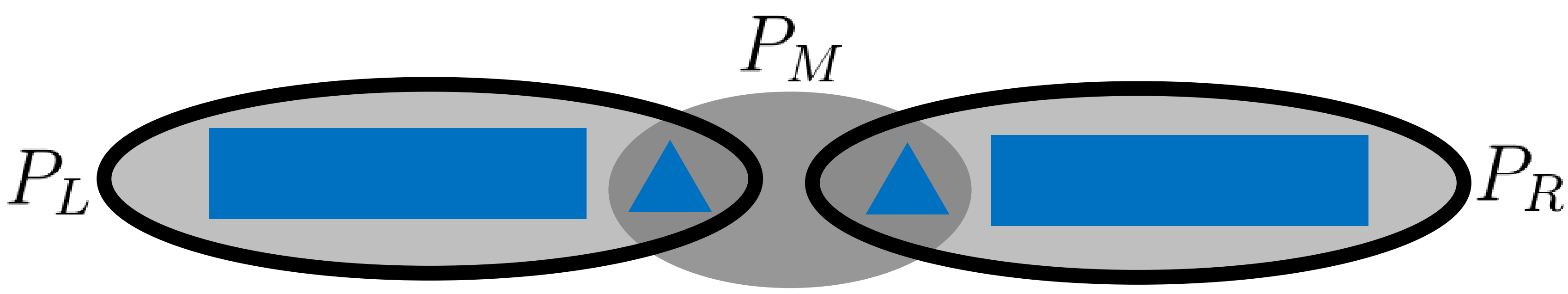


3 Making the counterexample local

- quantum expander $[I, A, B]$ … quantum circuits …
  ~~nonlocal projectors~~ … Kitaev's LH & history states 
  an approximate groundstate, a very small gap

- rescale P_L, P_R (not the middle!)
  huge, nonphysical couplings

- use new "strengthening gadgets" [N., Cao] 
  ~~large interaction strength~~ … extra particles, high degree

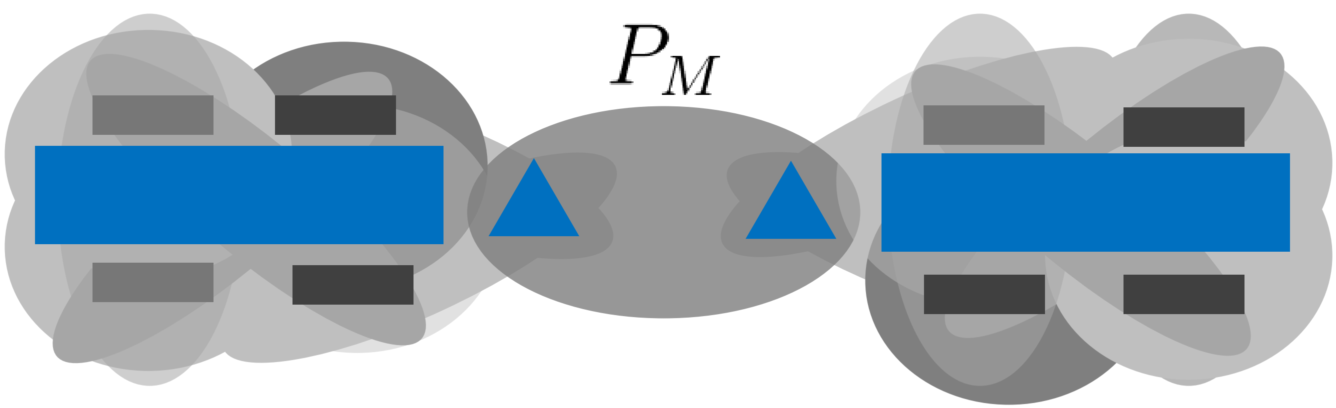


3 A local Hamiltonian, $(N+n)\times 3\times 3\times(N+n)$

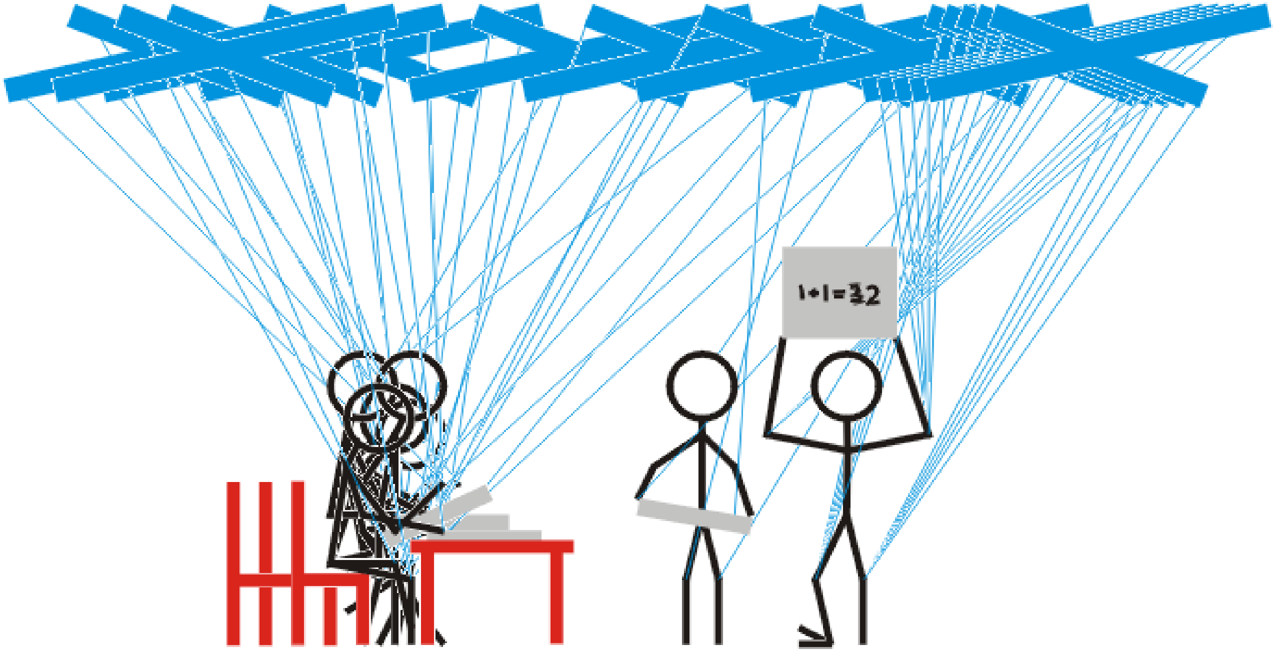
- frustrated, but still gapped
  O(1) norm terms

- a unique and
  still very entangled
  ground state

$$\approx |w\rangle \otimes \begin{array}{|c|c|c|} \hline I & A & B \\ \hline A & AA & AB \\ \hline B & BA & BB \\ \hline \end{array}$$

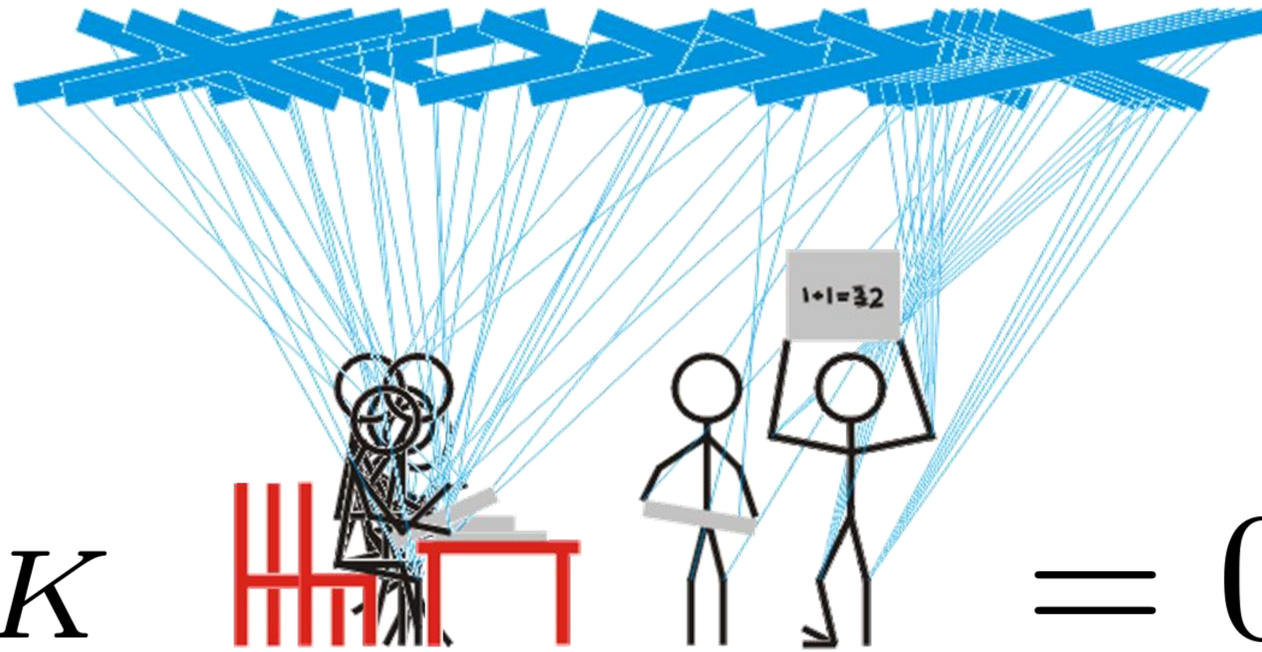


3 Implementing circuits locally: Feynman’s computer



3 The history state: a ground state

H_K



$= 0$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} \underbrace{|\varphi_t\rangle}_{U_t \cdots U_1 |\varphi_0\rangle} \otimes |t\rangle$$

3 The history state: a ground state

k-local

c-o-n-d-i-t-i-o-n-s

clock encoding

state progression

initialization

idling

$$|\cdots\rangle|u\rangle \otimes |0\rangle$$

$$|\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

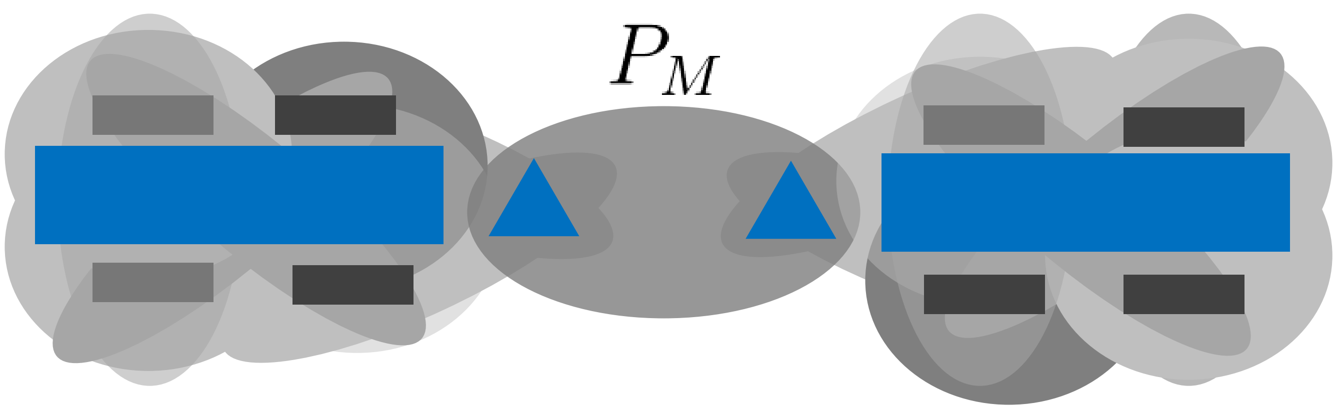
most of the state has the result

3 A local Hamiltonian, $(N+n)\times 3\times 3\times(N+n)$

- frustrated, but still gapped
  O(1) norm terms

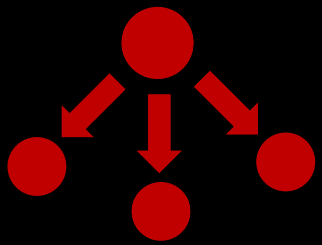
- a unique and
  still very entangled
  ground state

$$\approx |w\rangle \otimes \begin{matrix} I & A & B \\ A & AA & AB \\ B & BA & BB \end{matrix}$$



1 **q. expanders**

maximally entangled states

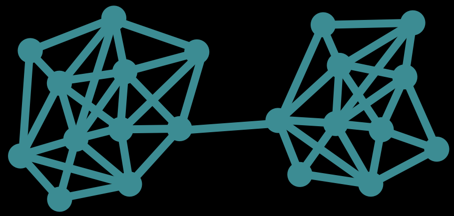


2 **entanglement**

testing and communication

3 **area law**

gaps, connections, correlations



local tests of global entanglement

and a counterexample to the generalized area law

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