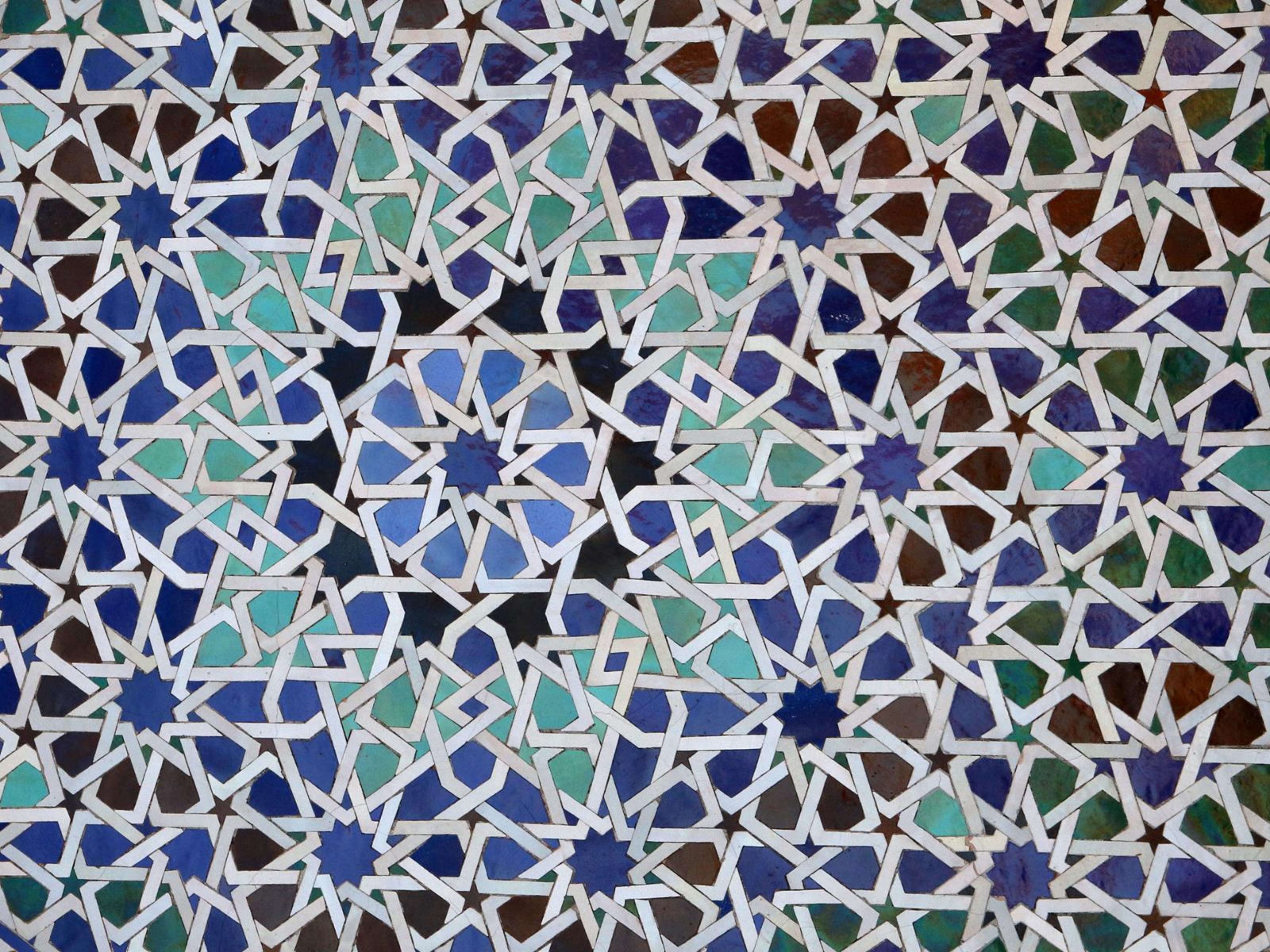

Local tests of global entanglement

and a counterexample to the generalized area law.

D. Aharonov, A. Harow, Z. Landau, D. Nagaj, M. Szegedy, U. Vazirani

universität wien

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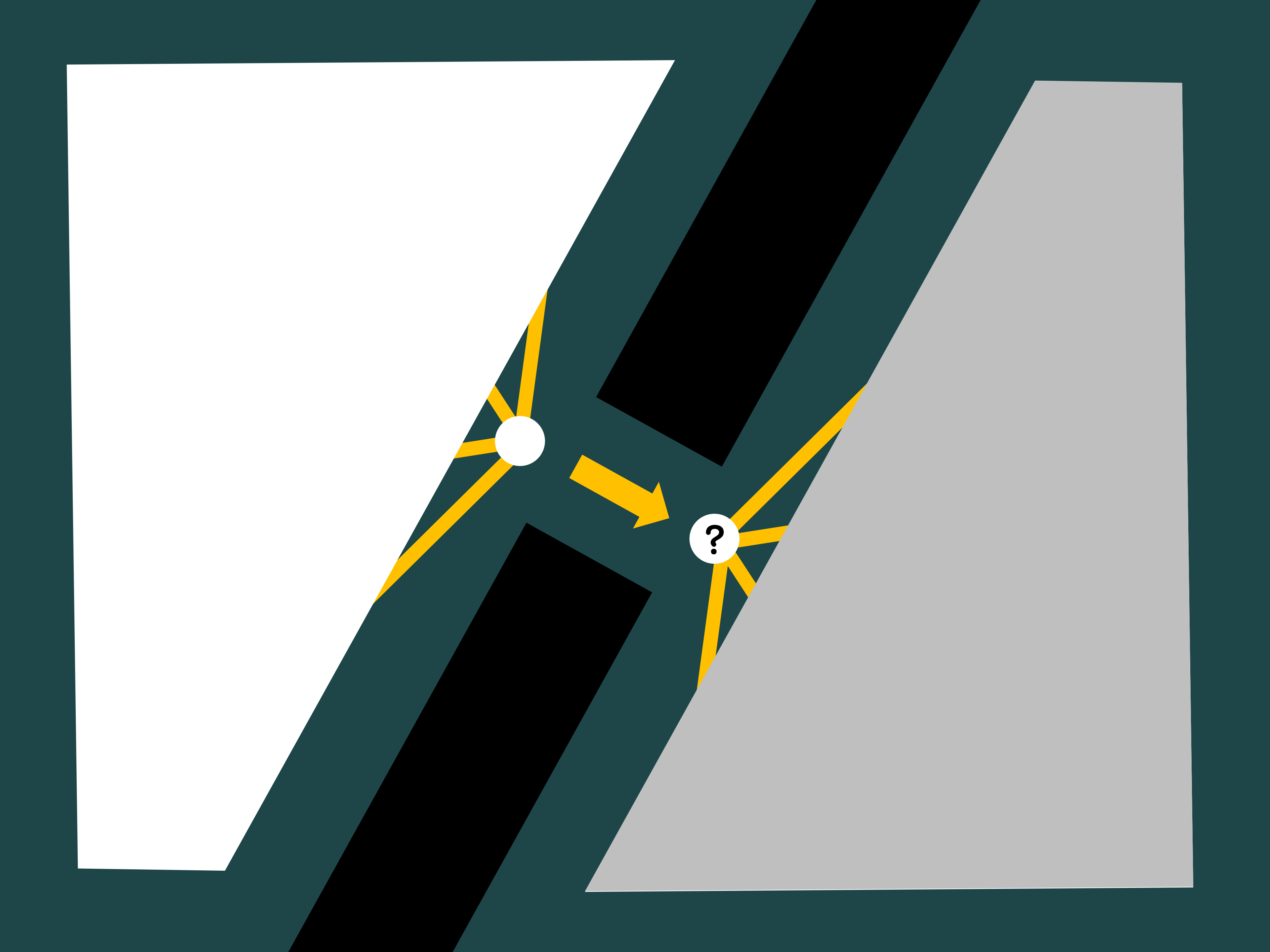
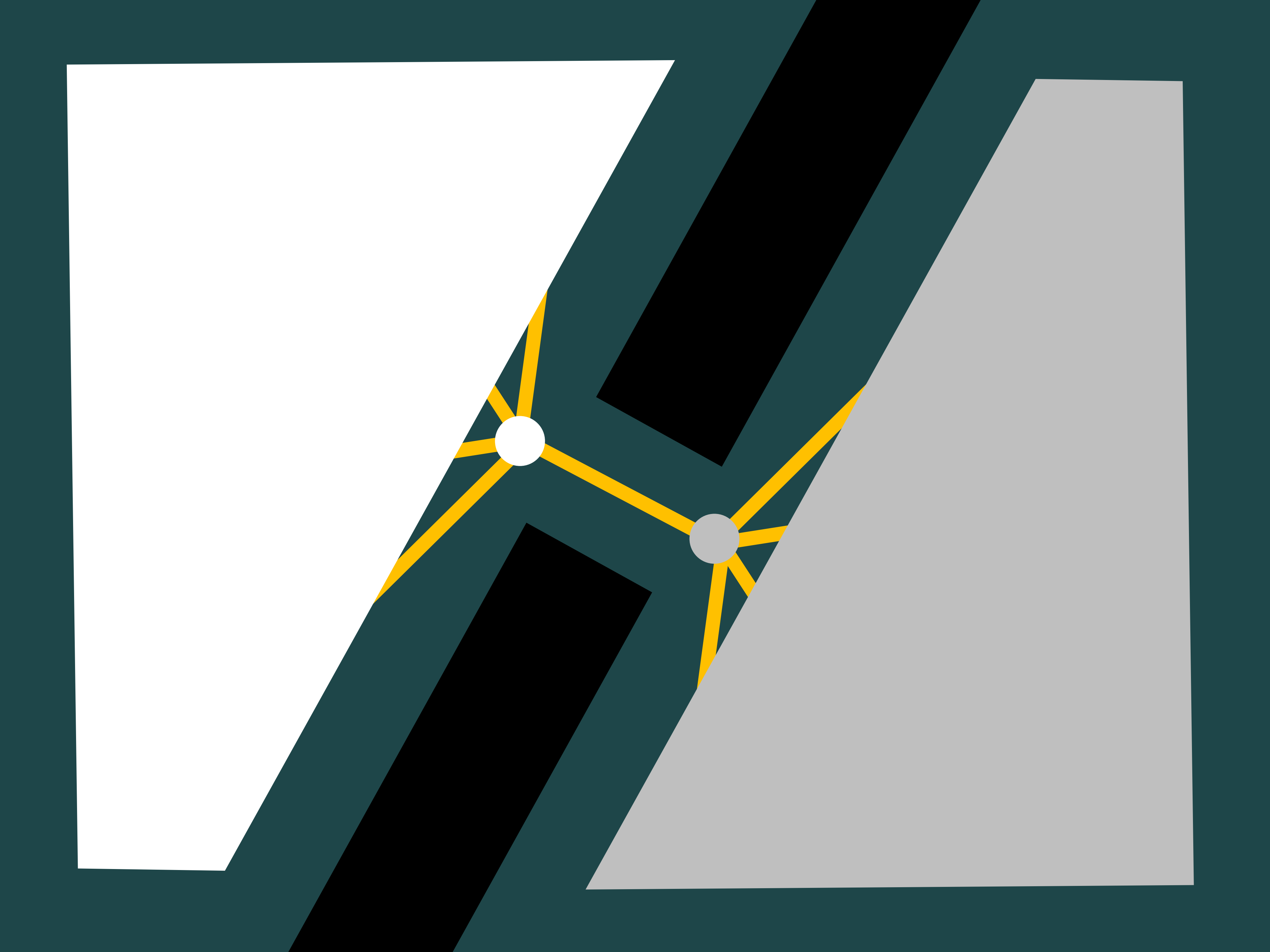
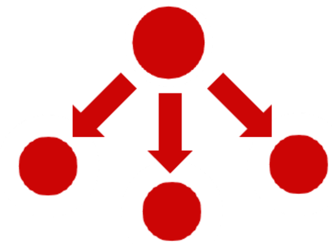




1 **q. expanders**

maximally entangled states

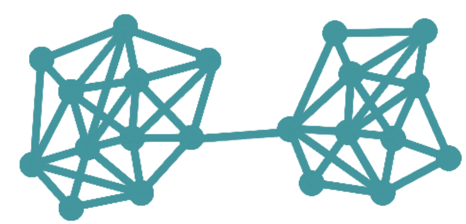


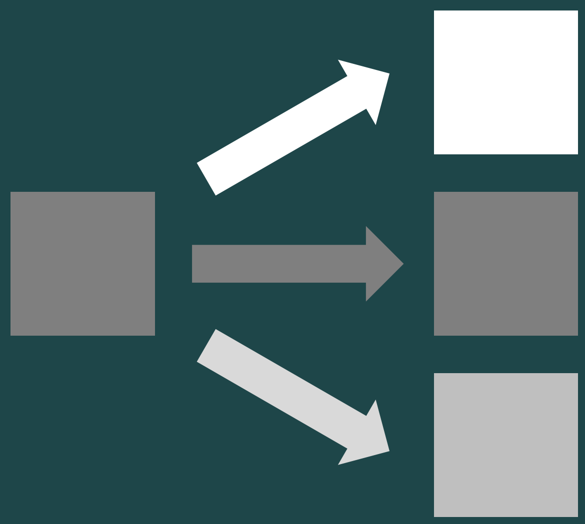
2 **entanglement**

testing and communication

3 **area law**

gaps, connections, correlations



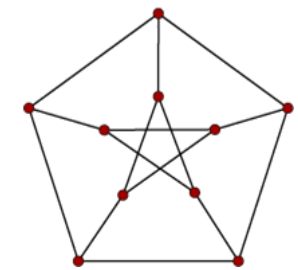


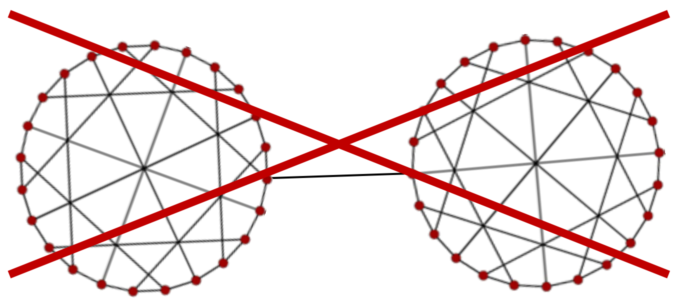
Quantum Expanders

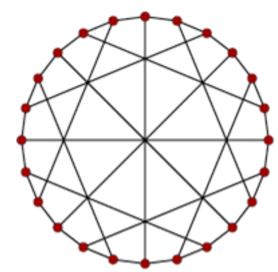
1 Classical expanders

- Aram Harrow's talk, QHC workshop at the
  [youtube Harrow quantum expanders]

- graphs that mix well
  divide in two? cut a lot (fraction) of edges!







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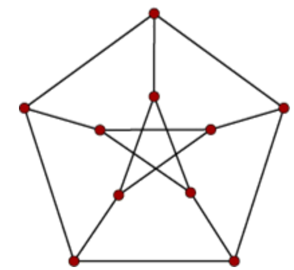
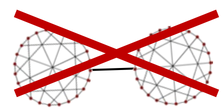
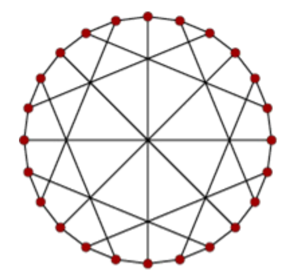


  examples: Cayley graphs





- normalized adjacency matrix
  second largest eigenvalue 1–λ

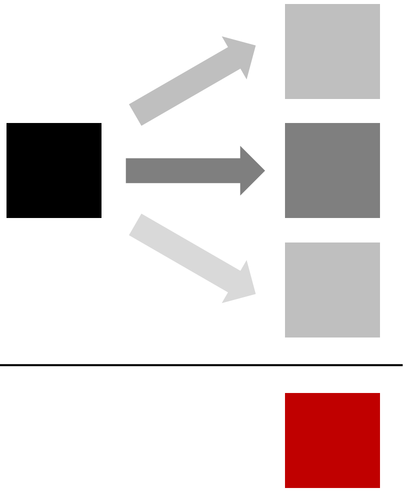
- a motivation for quantum expanders
  d-regular graphs, random permutations

$$A = \frac{1}{k}\sum_{i=1}^{k} \Pi_k$$

1 Mixing up something quantum

- applying random unitaries

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$



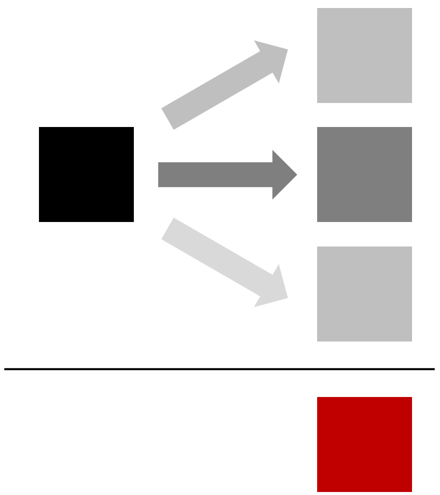
- a motivation for quantum expanders
  d-regular graphs, random permutations

$$A = \frac{1}{k}\sum_{i=1}^{k} \Pi_k$$

1 Quantum expanders

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$

- applying random unitaries from a set
  a discrete approximation to the Haar measure



- transform $N \times N$ matrices

- a small second largest singular value λ
  not far from the depolarizing channel

$$\|\mathcal{E} - D\|$$

- q. expander constructions, also for fixed k (8, 5, 3?)
  [Ben-Arroya+ 07, Hastings '07 , Gross & Eisert '08, Hastings & Harrow '09]

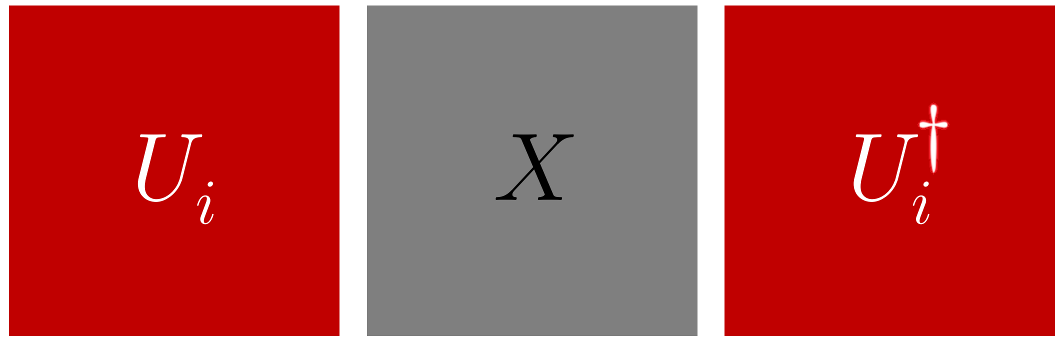
1 Quantum expanders

- transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^\dagger$$

- a matrix that doesn't change?

$$X = \mathbb{I}$$



$$U_i X = X U_i$$

1 Quantum expanders

- transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k} \sum_{i=1}^{k} U_i X U_i^\dagger$$

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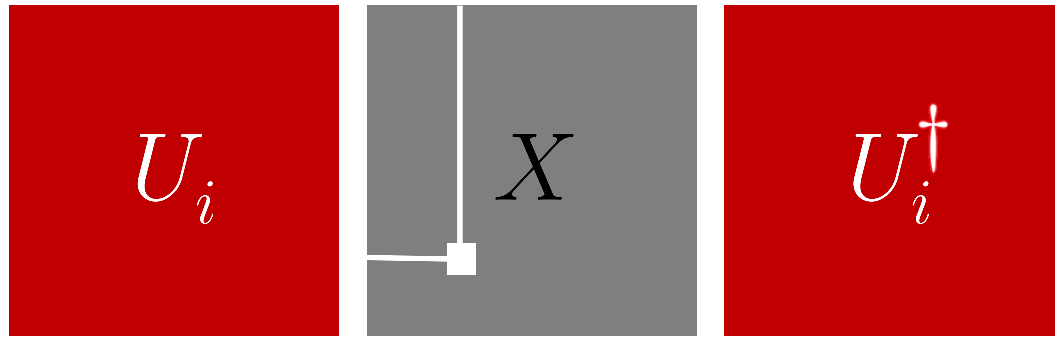

- interpreting matrices as 2-register states

$$\sum_{a,b} X_{ab} |a\rangle\langle b|$$

density matrix

1 Quantum expanders

- transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$

- a matrix that doesn't change?

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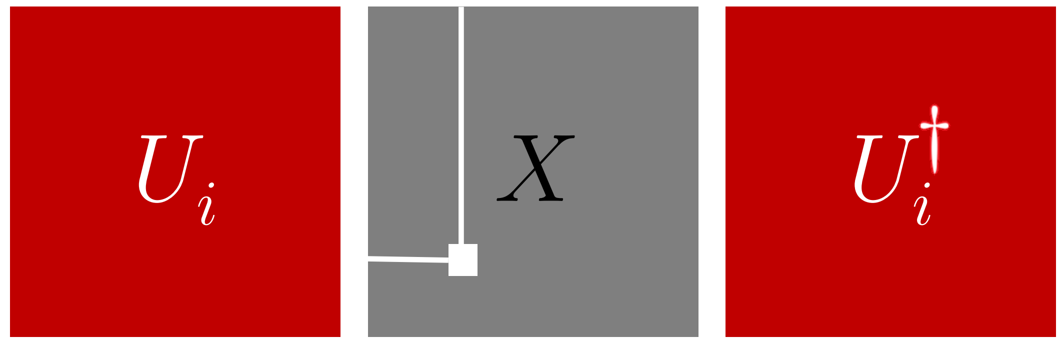


- interpreting matrices as 2-register states

$$\frac{1}{k}\sum_{i=1}^{k} (U_i \otimes U_i^*) \sum_{a,b} X_{ab}|a\rangle|b\rangle$$

pure state

- distributively applying an expander

$$\sum_{a,b} X_{ab}|a\rangle|b\rangle$$

stationary?

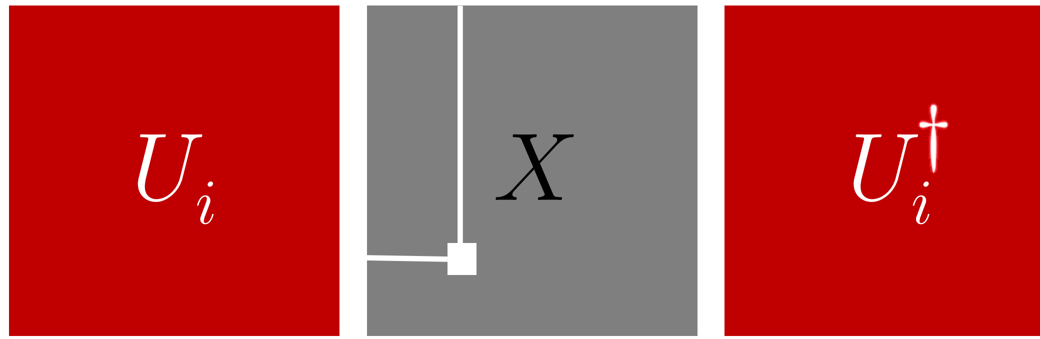
1 Quantum expanders

- transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$

- a matrix that doesn't change?

$$X = \mathbb{I}$$



- interpreting matrices as 2-register states

$$\frac{1}{k}\sum_{i=1}^{k}(U_i \otimes U_i^*)\sum_{a,b} X_{ab}|a\rangle|b\rangle$$

- distributively applying an expander

stationary? max. entangled!

$$|\Phi_N\rangle = \frac{1}{\sqrt{N}}\sum_{x=1}^{N}|x\rangle|x\rangle$$

local tests of global entanglement

2 EPR testing

- how costly is it to certify that we share a maximally entangled state?

$$\frac{1}{\sqrt{N}}\sum_{x=1}^{N}|x\rangle|x\rangle$$

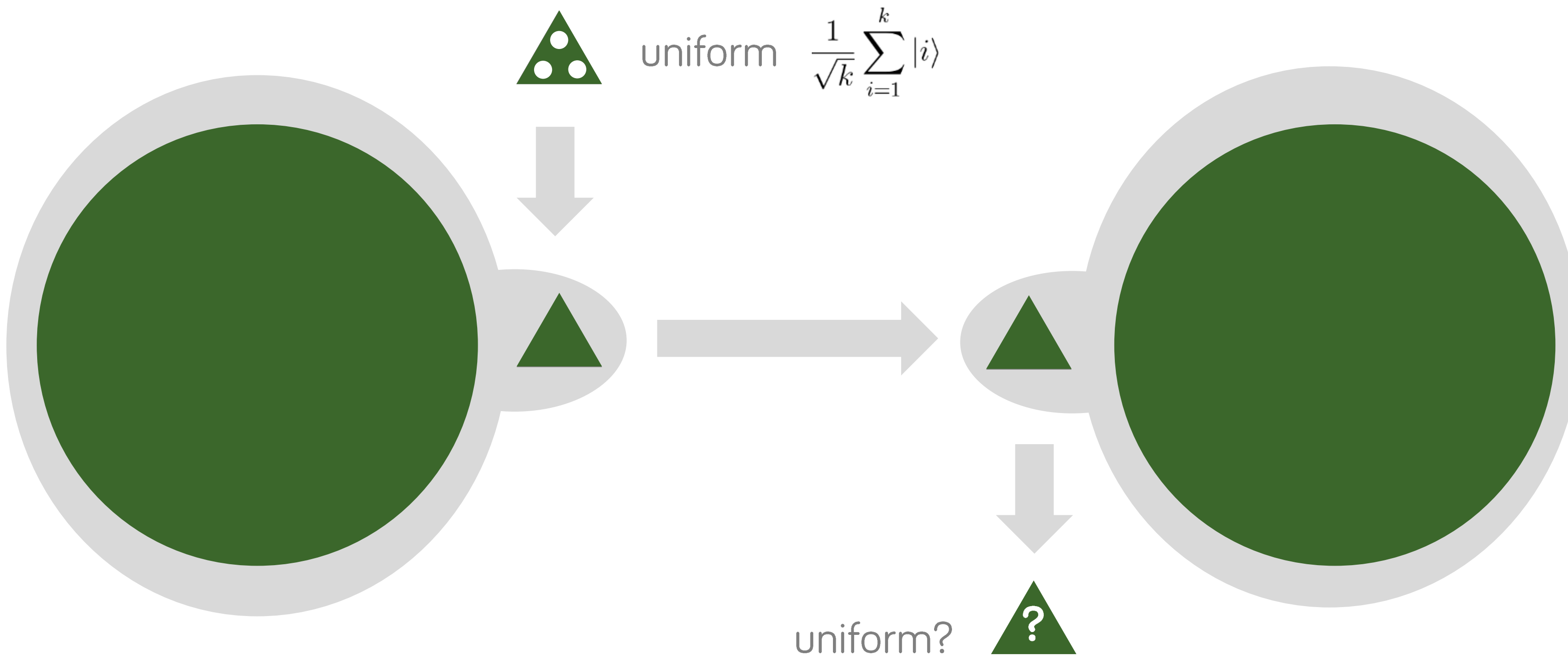
2 EPR testing

- how costly is it to certify that we share a maximally entangled state?

$$\frac{1}{\sqrt{N}}\sum_{x=1}^{N}|x\rangle|x\rangle$$

- apply a quantum expander distributively

$$U_i \otimes U_i^*$$



2 EPR testing

- action on states

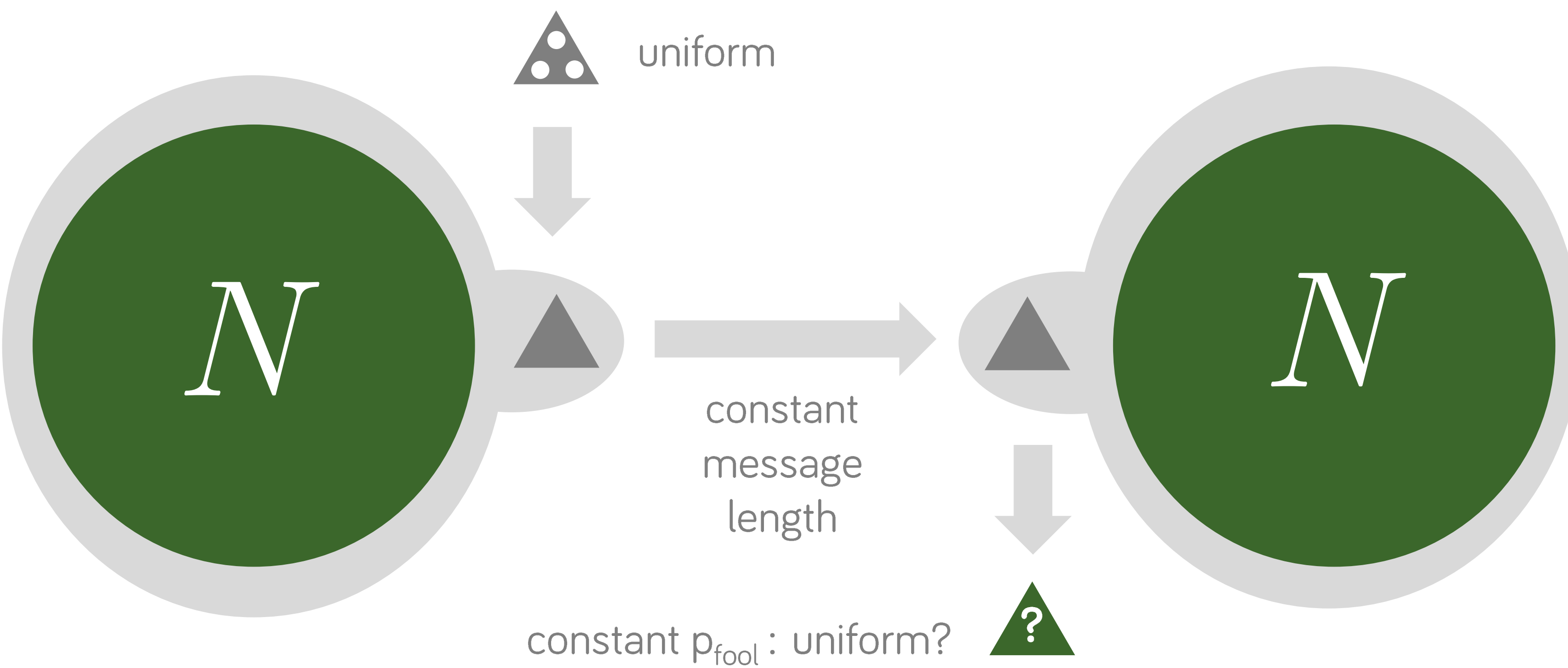
$$\frac{1}{\sqrt{k}}\sum_{i=1}^{k}|i\rangle\,(U_i\otimes U_i^*)\,|X\rangle$$

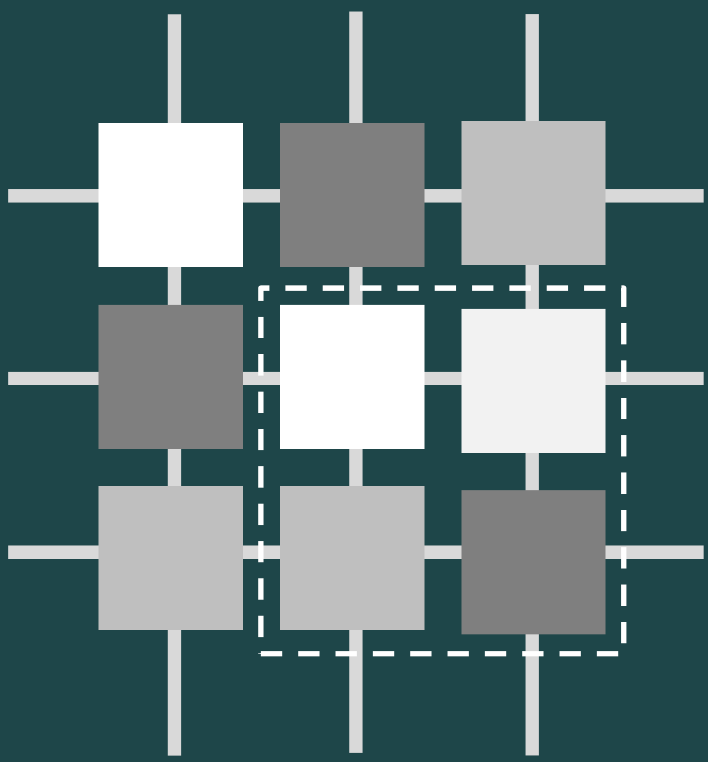
$$\sum_{a,b} X_{ab}|a\rangle|b\rangle$$

- does the qutrit remain uniform?
  does the matrix X commute with U_i?

U_i X $U_i^\dagger$

- quantum expander ... soundness





a counterexample to the

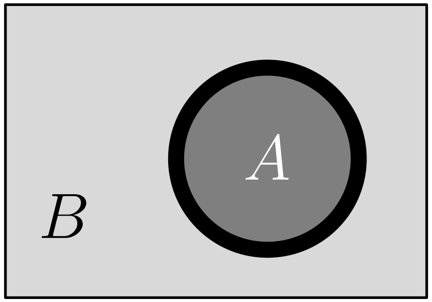
generalized area law

3 Area law: ground states of quantum spin systems

- entanglement entropy

$$S = -\mathrm{Tr}(\rho_A \ln \rho_A)$$ ~ ~~volume~~ area

Schmidt coefficients

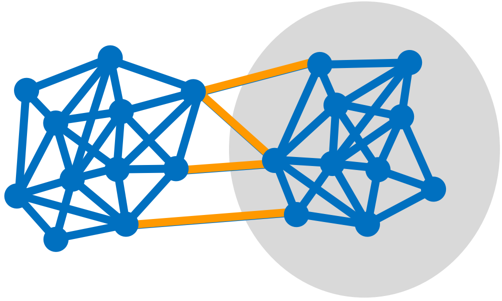


$$\rho_A = \mathrm{Tr}_B \rho$$

- 1D … algorithms [White 92, Vidal 03, Landau+ 13]
  theorems [Hastings 07, Arad+ 13]

- 2D … we're close
  small gap? large local dimension?

- generalized area conjecture
  entropy ~ cut size



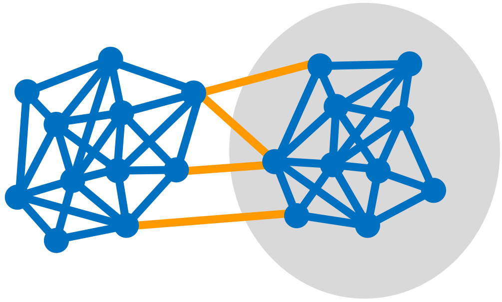
Not true.

a gap
a few links
O(1) terms

→

not much entanglement
(a “simple” ground state)

- generalized area conjecture
  entropy ~ cut size



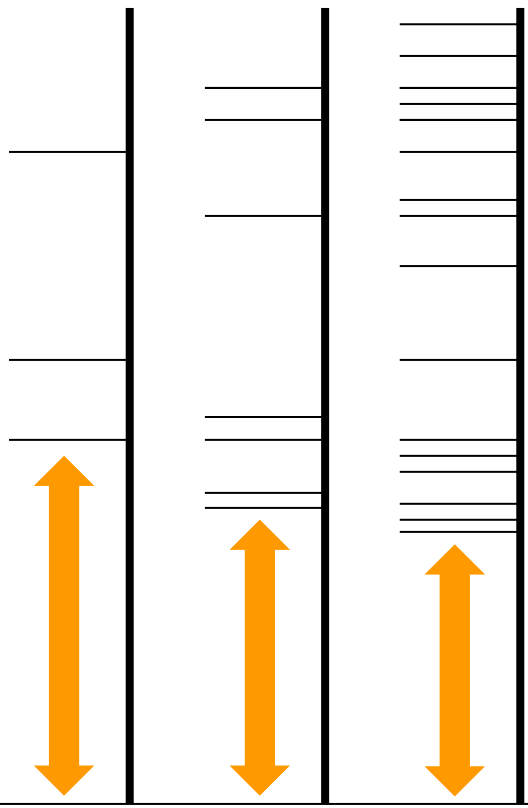
3 Gapped Hamiltonians

- Are there any states close to the ground state when we take the thermodynamic limit?

- local, $O(1)$ norm terms

~~an inverse-poly gap? $\Delta = \frac{c}{N} \to 0$~~

$N \to \infty$



3 Gapped Hamiltonians

- Nothing closer than Δ to the ground state.
- in 1D

a region

correlations with the outside fall off exponentially with distance

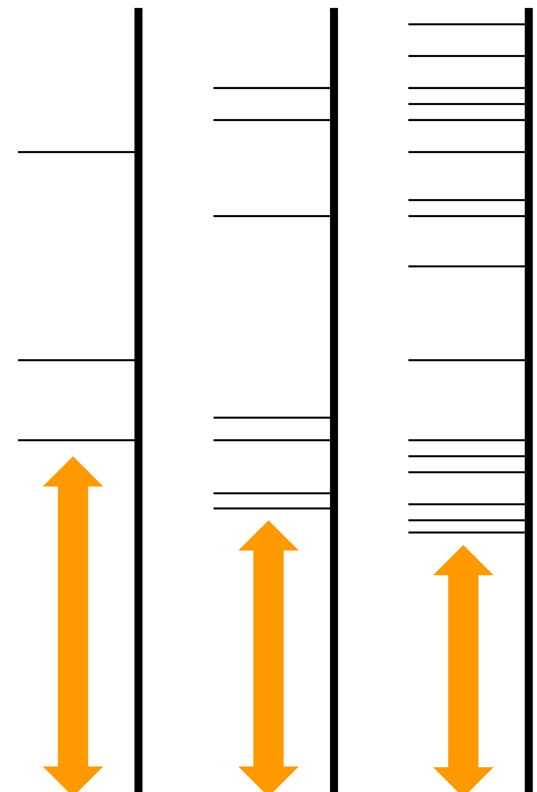
entropy from quantum correlations

$$S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$$

... constant

... log corrections for 1/poly gaps

$N \to \infty$



3 Gapped Hamiltonians

- Nothing closer than Δ to the ground state.

- in 1D

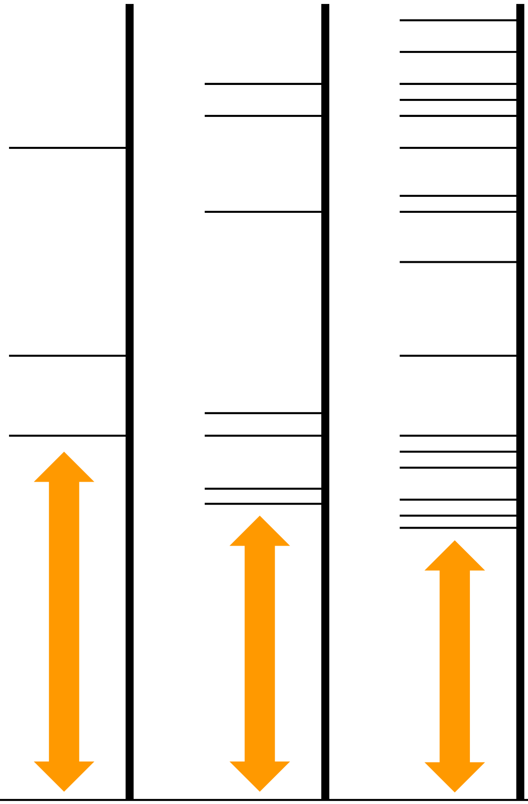
  the AKLT (spin-1) chain

  $$\sum_{j=1}^{N-1} \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3}\left(\vec{S}_j \cdot \vec{S}_{j+1}\right)^2$$

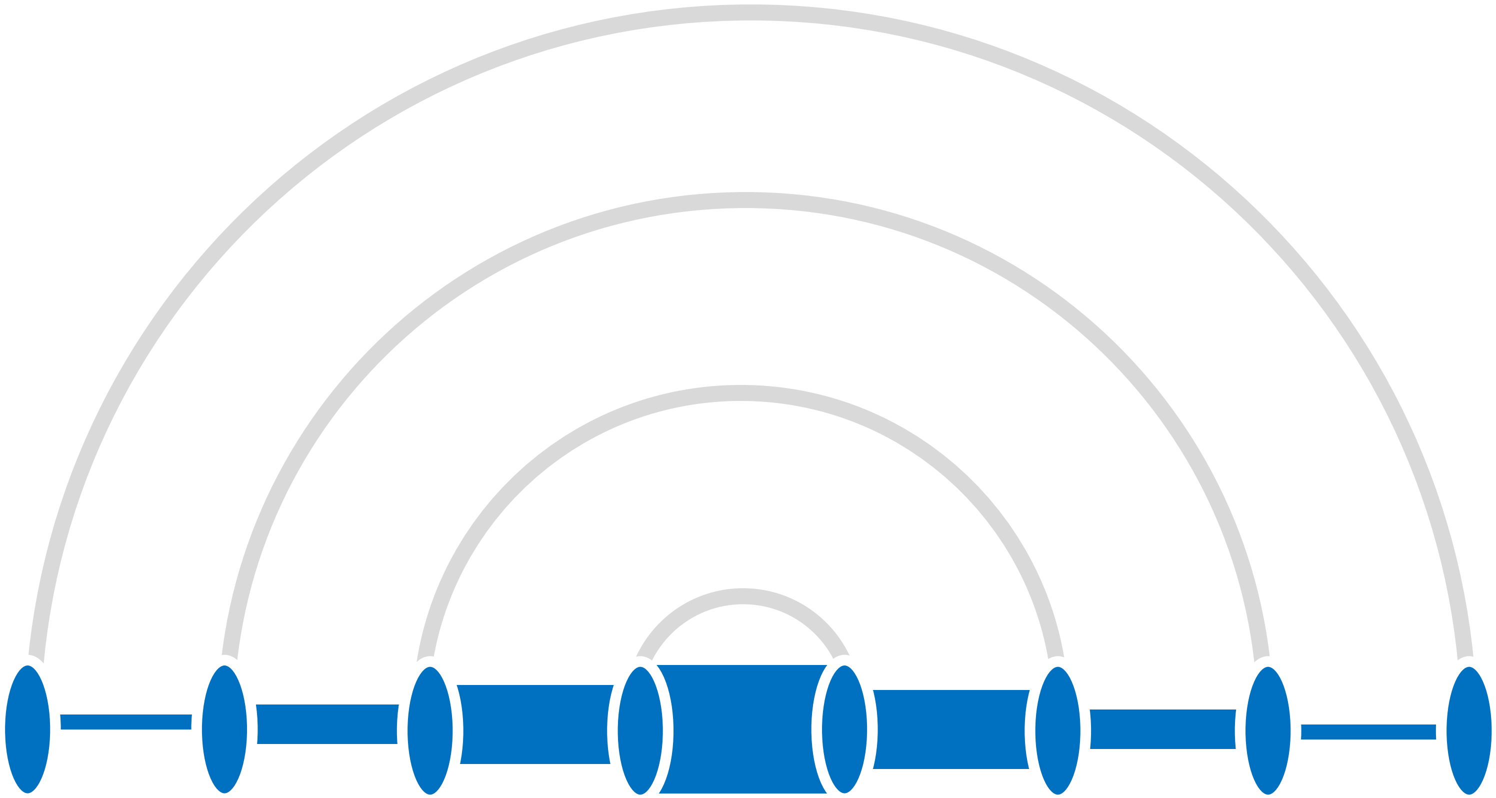
  a biased walk in 1D

  $$\sum_{j=1}^{N-1} \left(|j\rangle - B|j+1\rangle\right)\left(\langle j| - B\langle j+1|\right)$$

$$N \to \infty$$



- a large gap ... a simple (not too entangled) ground state?

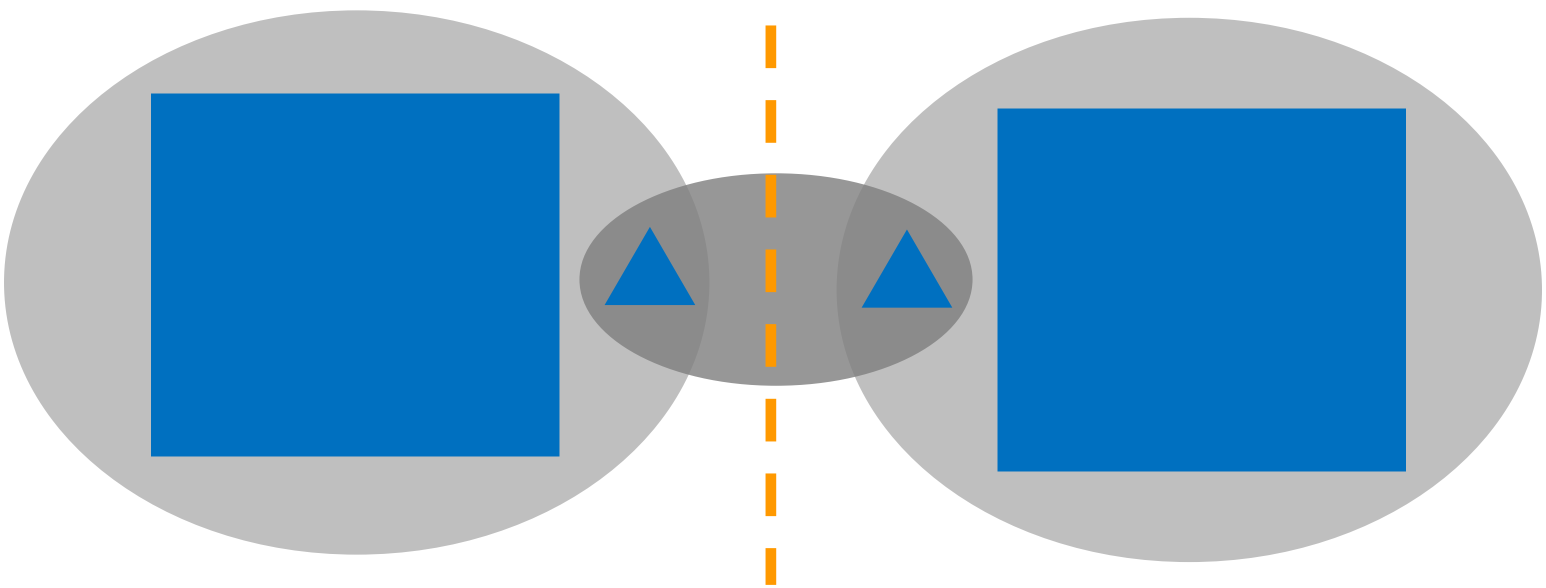


- without a gap, the entropy can be large

[Verstraete, Latorre+]

3 Our counterexample to the generalized area conjecture

- an $N \times 3 \times 3 \times N$ dimensional system
- a frustration-free, gapped, Hamiltonian
- a single O(1) interaction of two 3×3 subsystems
- a unique, very entangled ground state with O(N) entanglement entropy across the cut



3 A nonlocal Hamiltonian, $N \times 3 \times 3 \times N$

- a projector P_L with ground states $\frac{1}{\sqrt{3}}\left(|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle\right)$

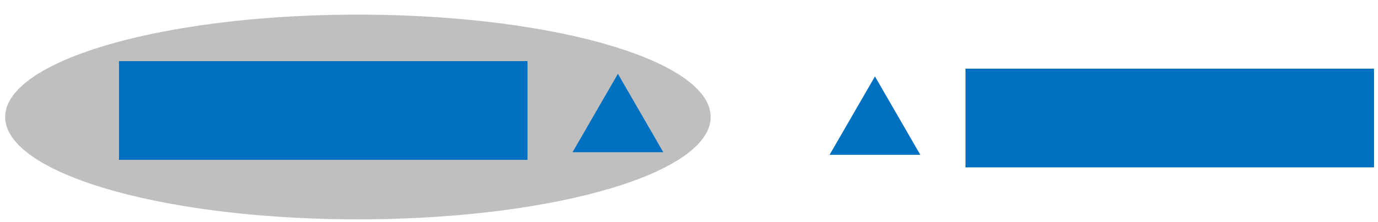
- as a vector

x
Ax
Bx

$\otimes|j\rangle \otimes |y\rangle$

as a matrix

X_1	X_2	X_3
AX_1	AX_2	AX_3
BX_1	BX_2	BX_3

P_L 

3 A nonlocal Hamiltonian, $N\times 3\times 3\times N$

- a projector P_R
  with ground states

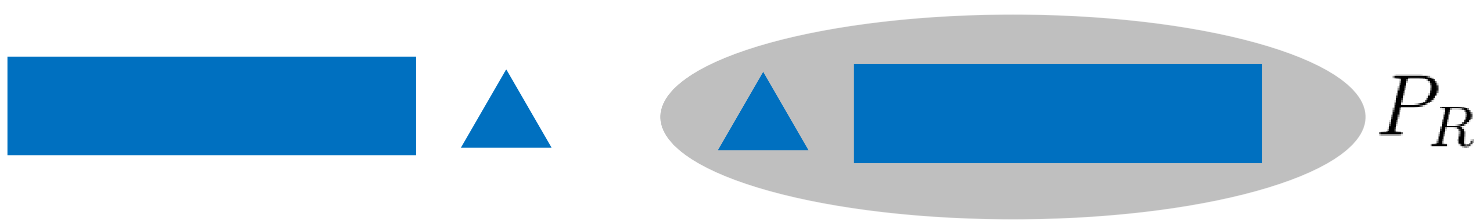
$$\frac{1}{\sqrt{3}}\left(|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle\right)$$

- as a vector $|i\rangle \otimes |x\rangle \otimes$

y	yA	yB

  as a matrix

Y_1	Y_1A	Y_1B
Y_2	Y_2A	Y_2B
Y_3	Y_3A	Y_3B

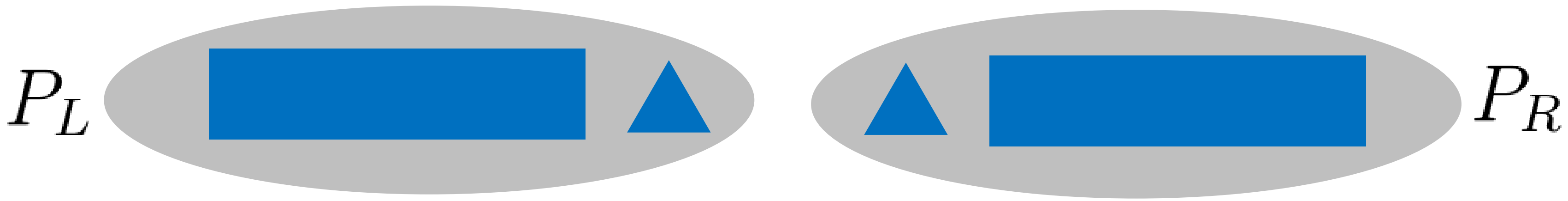


3 A nonlocal Hamiltonian, $N \times 3 \times 3 \times N$

- a projector P_L $\qquad (|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle)/\sqrt{3}$

  a projector P_R $\qquad (|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)/\sqrt{3}$

X_1	X_2	X_3	X	XA	XB	Y_1	Y_1A	Y_1B
AX_1	AX_2	AX_3	AX	AXA	AXB	Y_2	Y_2A	Y_2B
BX_1	BX_2	BX_3	BX	BXA	BXB	Y_3	Y_3A	Y_3B



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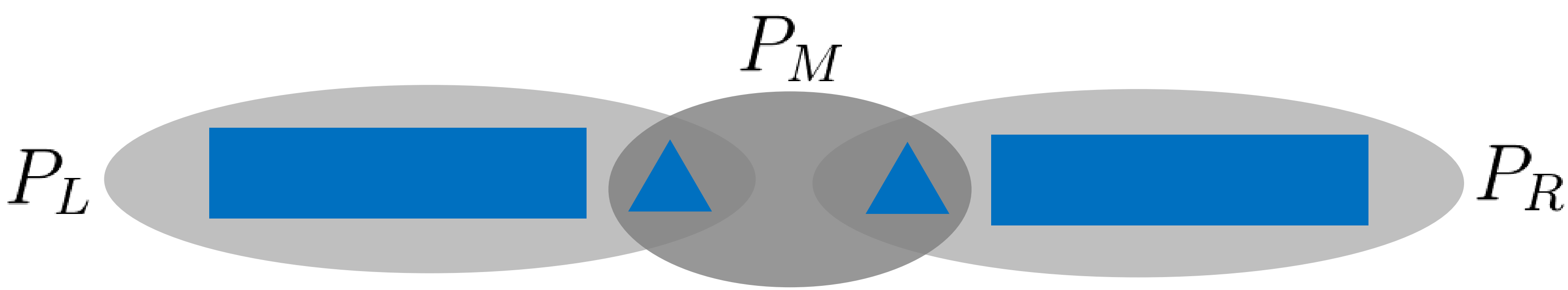
  a projector P_R $\qquad (|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)/\sqrt{3}$

  a projector P_M

X	XA	XB
AX	AXA	AXB
BX	BXA	BXB



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  a projector P_R $\quad (|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)\,/\sqrt{3}$

  a projector P_M

- who commutes with A and B?

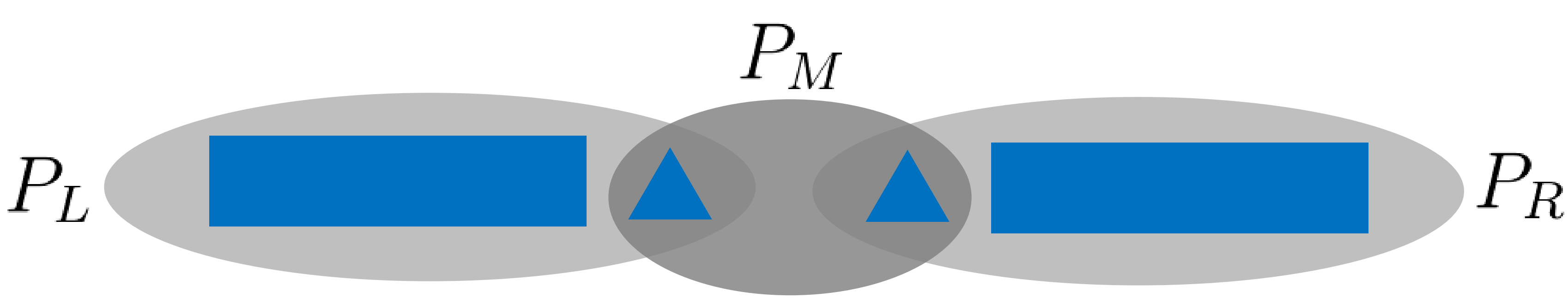
  only the identity, as $[I, A, B]$ are a q. expander

enforce symmetry

for 12 & 21

for 13 & 31

X	XA	XB
AX	AXA	AXB
BX	BXA	BXB



3 A nonlocal Hamiltonian, $N \times 3 \times 3 \times N$

- a projector P_L

  a projector P_R

  a projector P_M

$$(|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle)\,/\sqrt{3}$$

$$(|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)\,/\sqrt{3}$$

enforce symmetry for 12 & 21

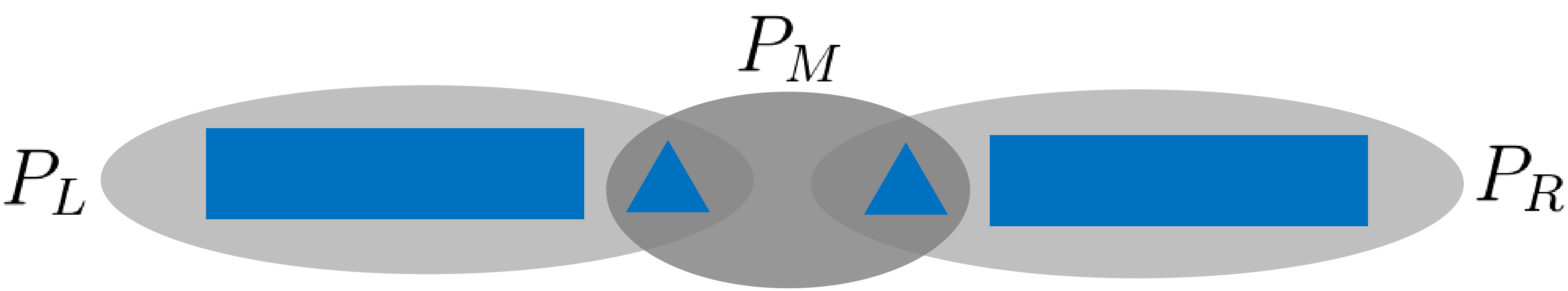
for 13 & 31

- who commutes with A and B?

  only the identity, as $[I, A, B]$ are a q. expander

I	A	B
A	AA	AB
B	BA	BB

$$\frac{1}{3\sqrt{N}}$$



3 A nonlocal Hamiltonian, $N \times 3 \times 3 \times N$

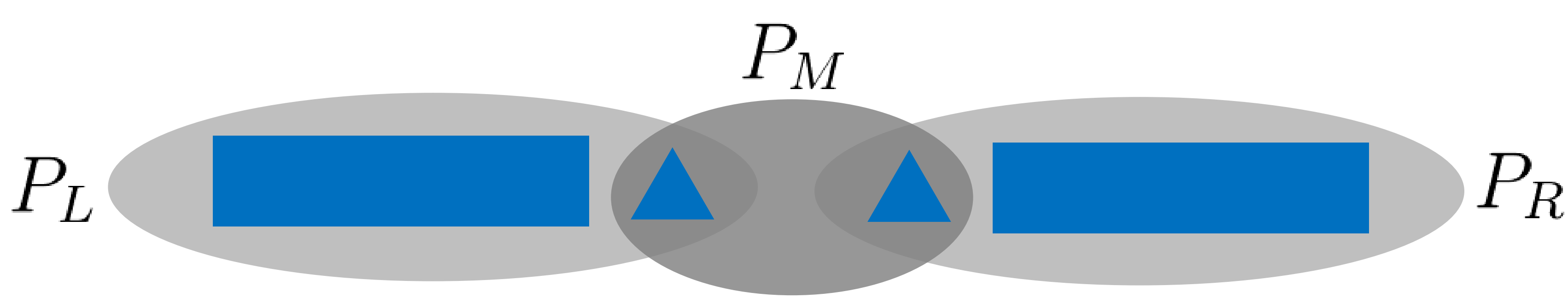
- ground state: unique
  very entangled

- Hamiltonian:
  frustration free
  gapped

I	A	B
A	AA	AB
B	BA	BB

$$\sum_{x=1}^{N} |x\rangle|x\rangle$$

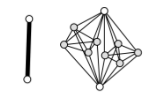
$$\frac{1}{3\sqrt{N}}$$

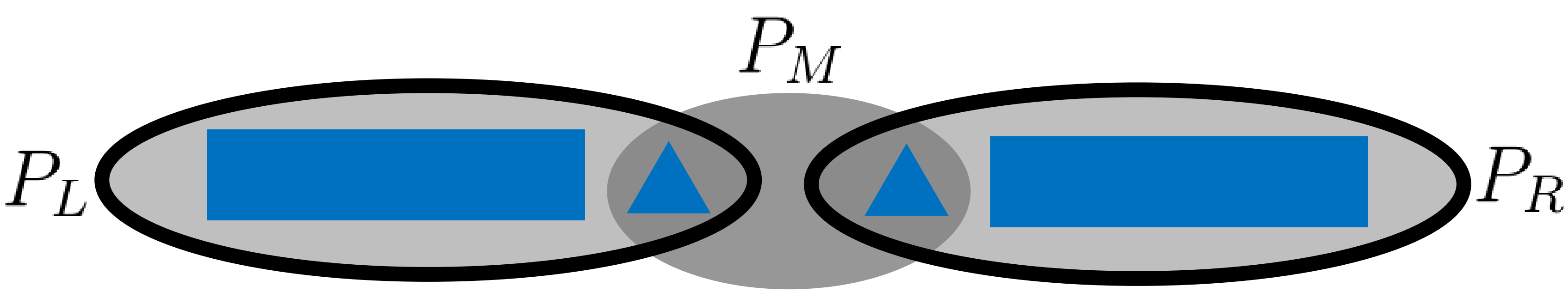


3 Making the counterexample local

- quantum expander $[I, A, B]$ … quantum circuits …
  ~~nonlocal projectors~~ … Kitaev's LH & history states 
  an approximate groundstate, a very small gap

- rescale P_L, P_R (not the middle!)
  huge, nonphysical couplings

- use new "strengthening gadgets" [N., Cao] 
  ~~large interaction strength~~ … extra particles, high degree

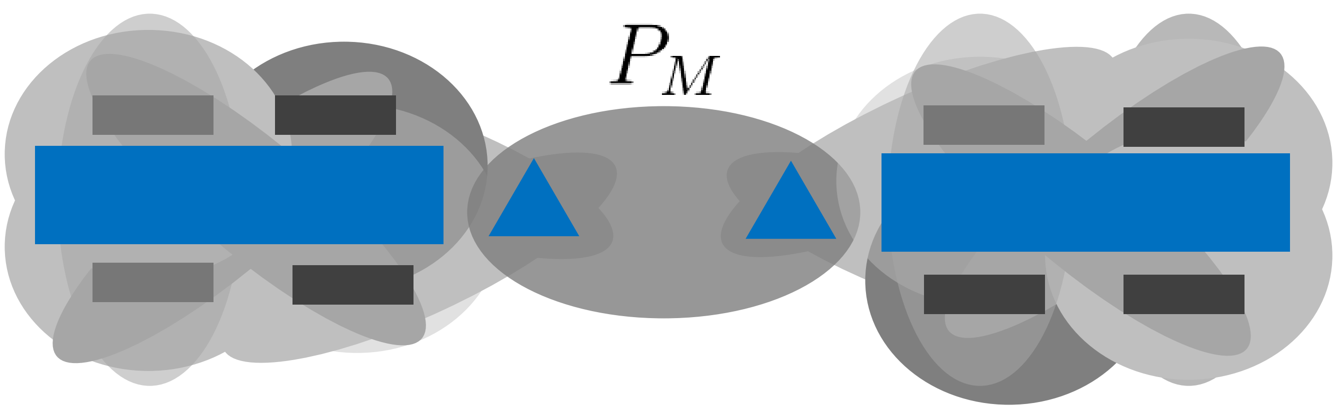


3 A local Hamiltonian, $(N{+}n)\times 3\times 3\times (N{+}n)$

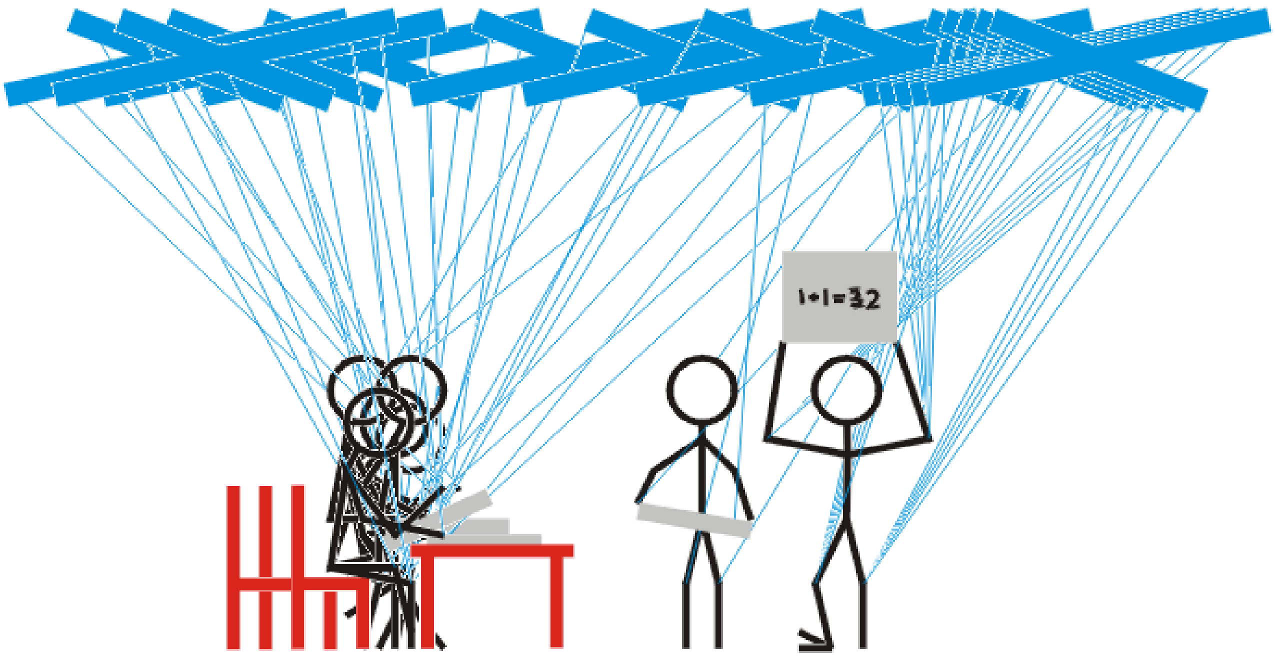
- frustrated, but still gapped
  O(1) norm terms

- a unique and
  still very entangled
  ground state

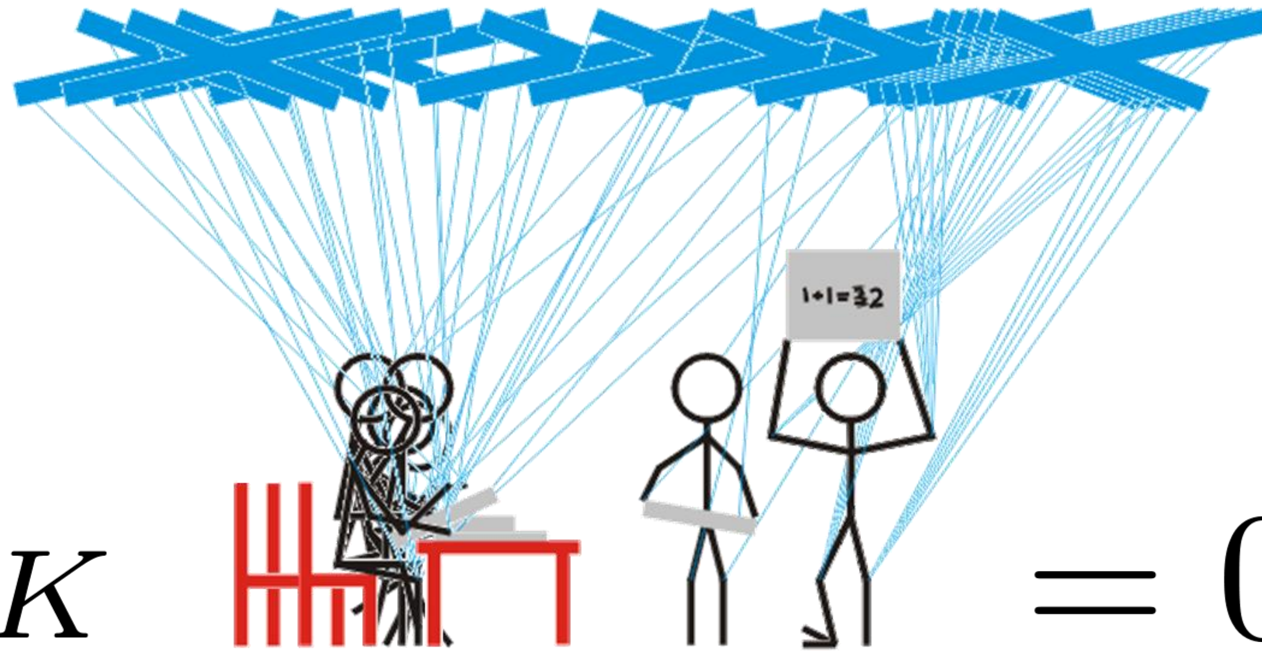
$$\approx |w\rangle \otimes \begin{matrix} I & A & B \\ A & AA & AB \\ B & BA & BB \end{matrix}$$



3 Implementing circuits locally: Feynman's computer



3 The history state: a ground state

H_K  $= 0$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} \underbrace{|\varphi_t\rangle}_{U_t \cdots U_1 |\varphi_0\rangle} \otimes |t\rangle$$

3 The history state: a ground state

k-local

c-o-n-d-i-t-i-o-n-s

clock encoding

state progression

initialization

idling

$$|\cdots\rangle|u\rangle \otimes |0\rangle$$

$$|\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

most of the state has the result

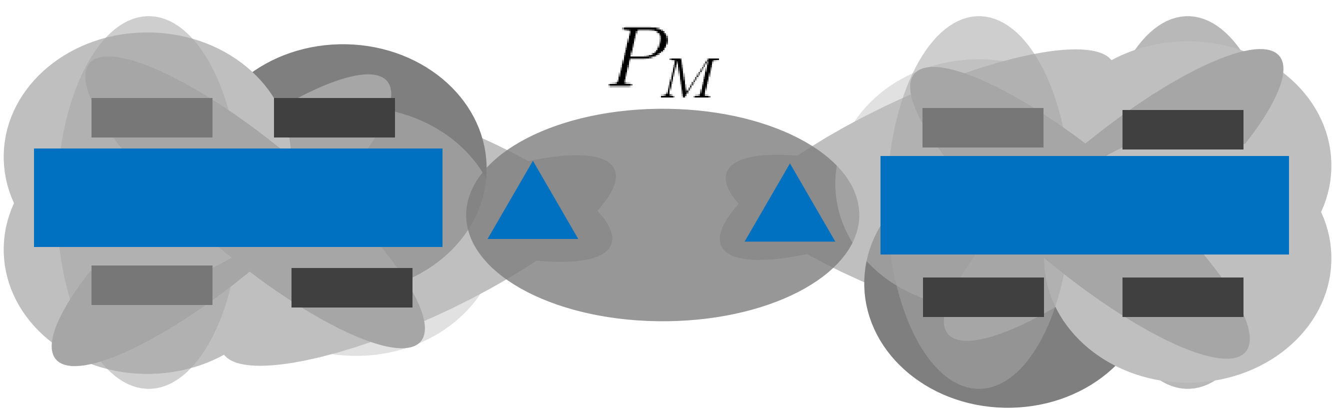
3 A local Hamiltonian, $(N{+}n)\times 3\times 3\times(N{+}n)$

- frustrated, but still gapped
  O(1) norm terms

- a unique and
  still very entangled
  ground state

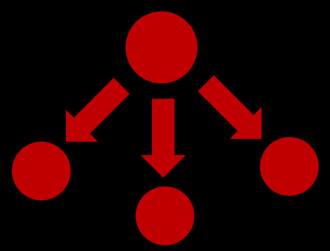
$$\approx |w\rangle \otimes$$

I	A	B
A	AA	AB
B	BA	BB



1 **q. expanders**

maximally entangled states

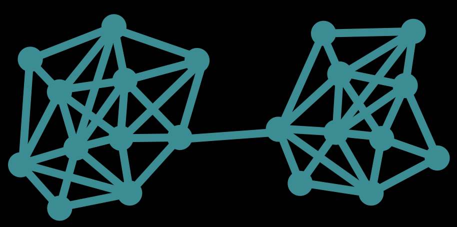


2 **entanglement**

testing and communication

3 **area law**

gaps, connections, correlations



Local tests of global entanglement

and a counterexample to the generalized area law.

M. Szegedy, U. Vazirani, Z. Landau, D. Nagaj, A. Harow, D. Aharonov

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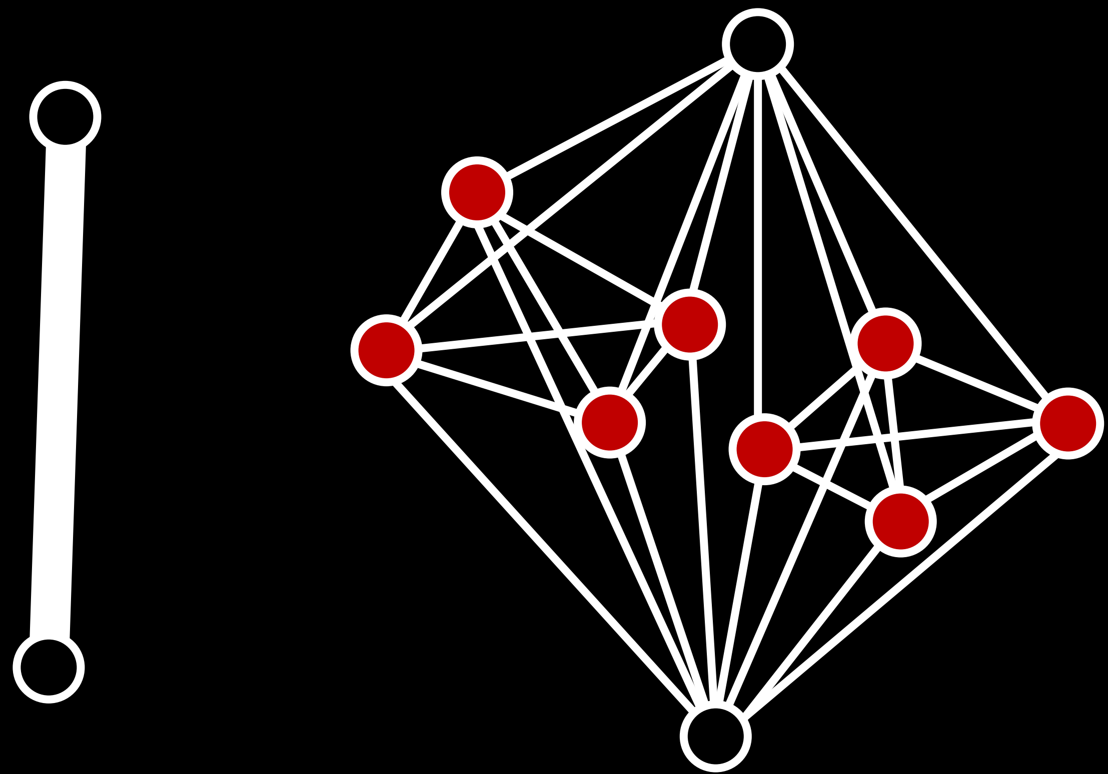
RC QI


29



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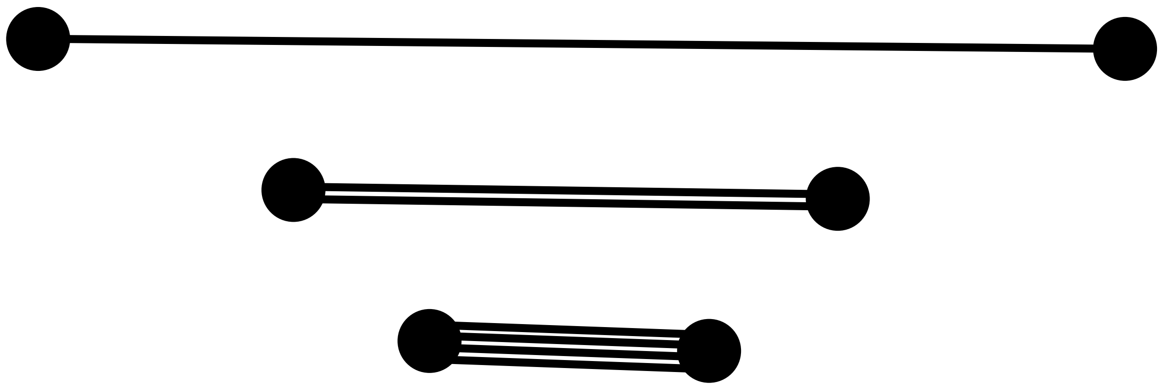
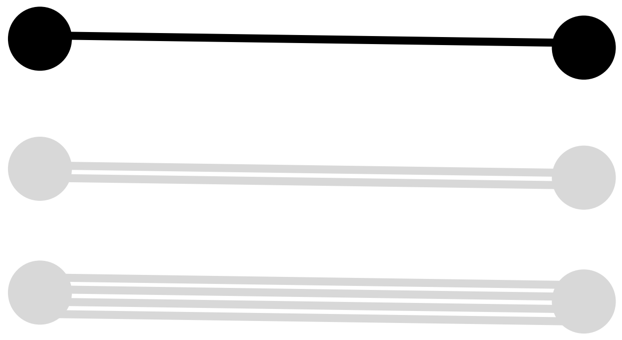
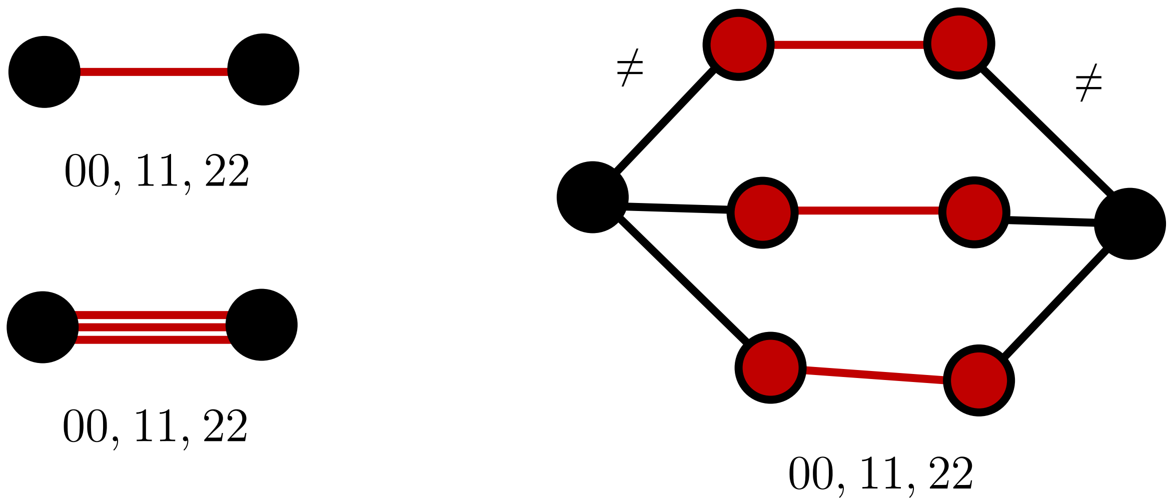
- interaction strength vs. distance



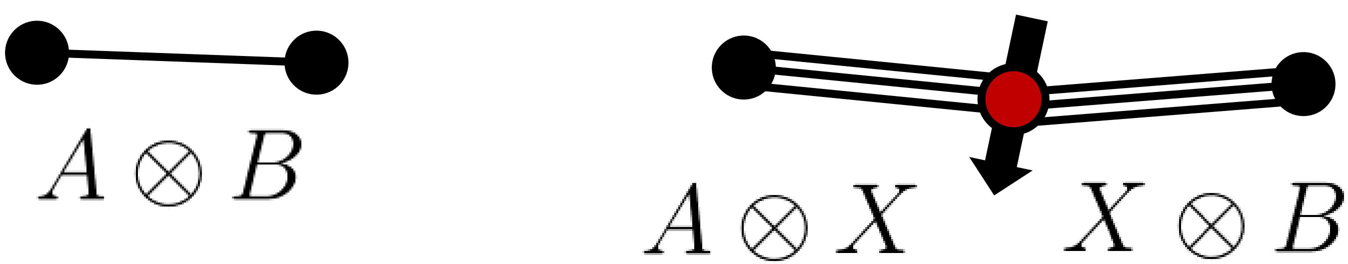
- limited interaction strength



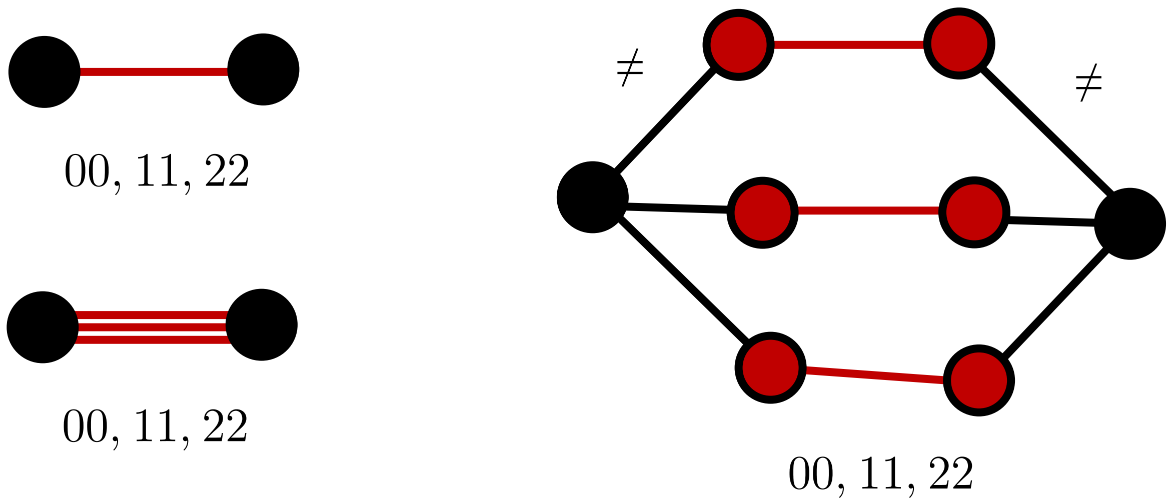
- classical gadgets



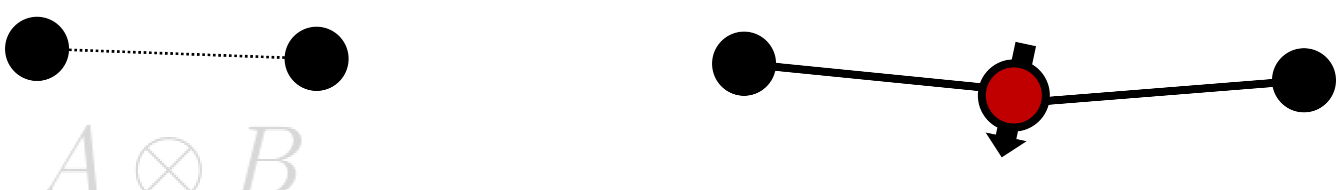
- quantum gadgets

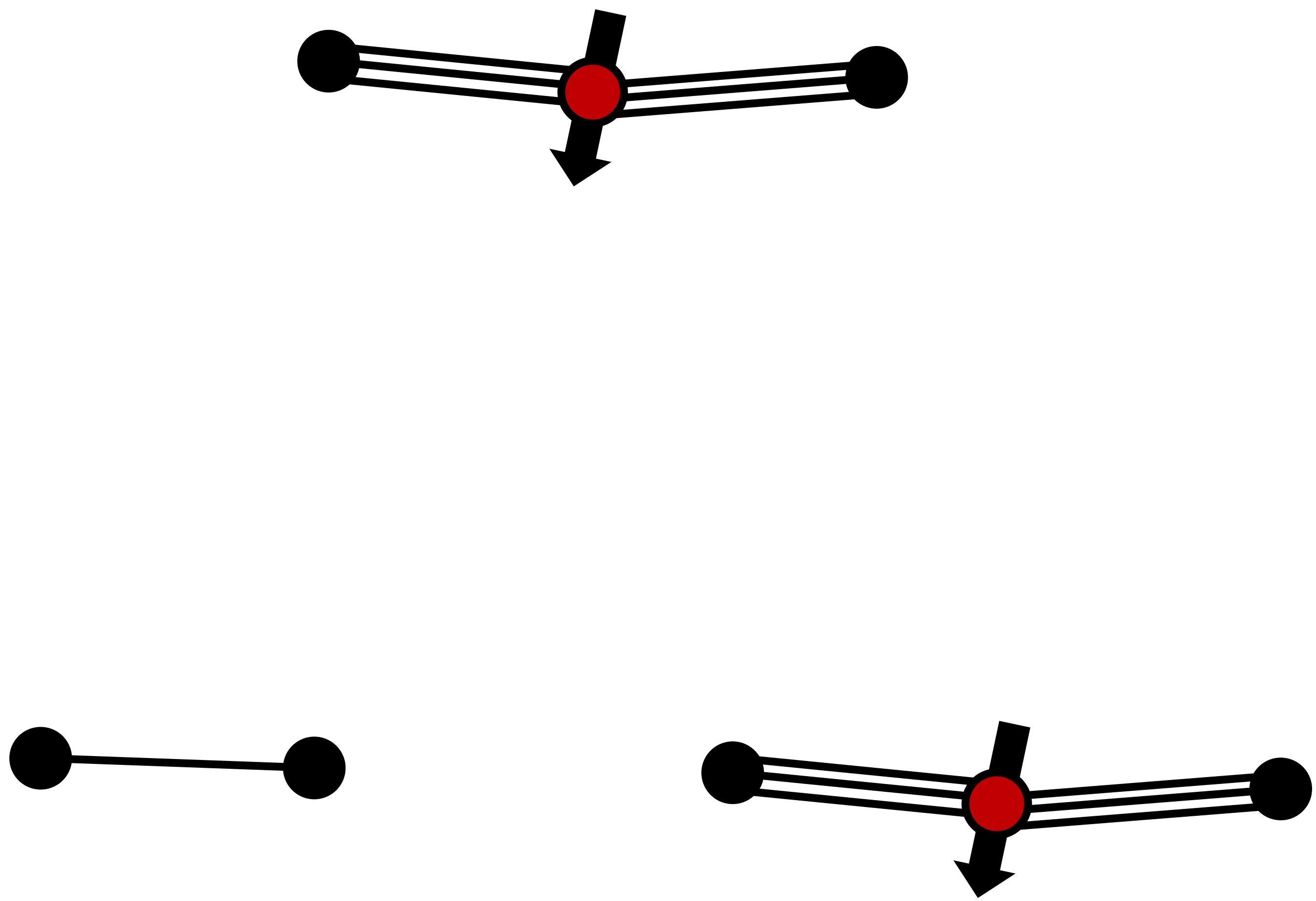


- classical gadgets

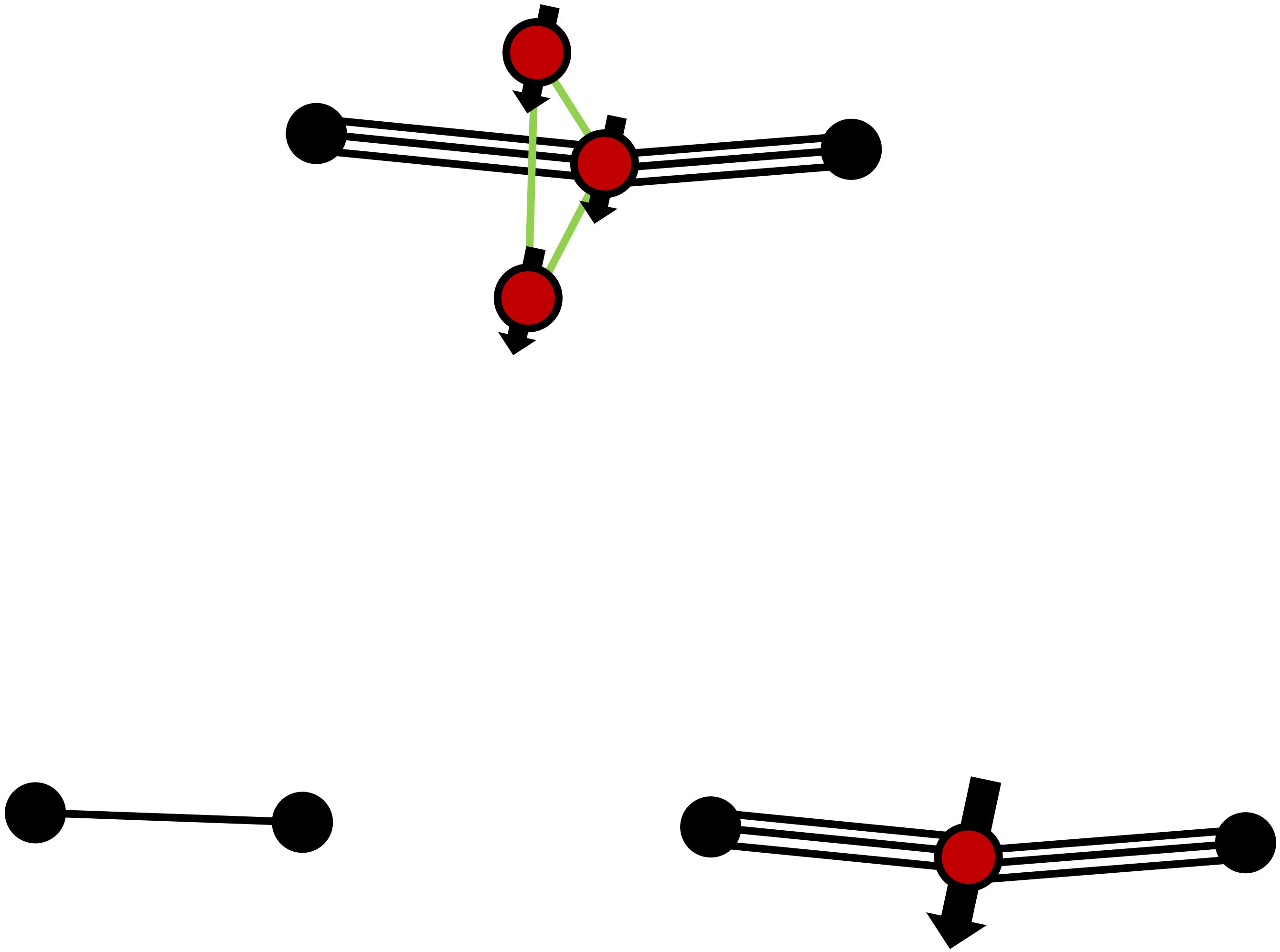


- quantum gadgets

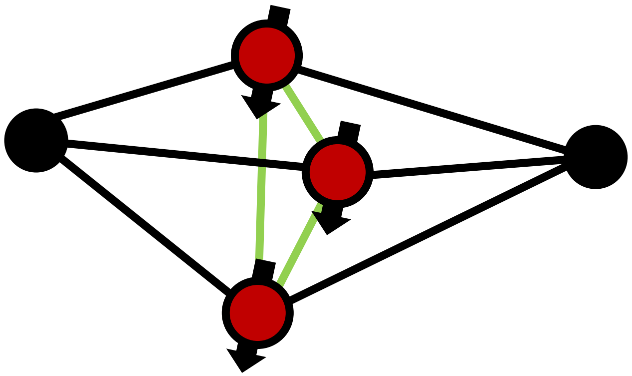


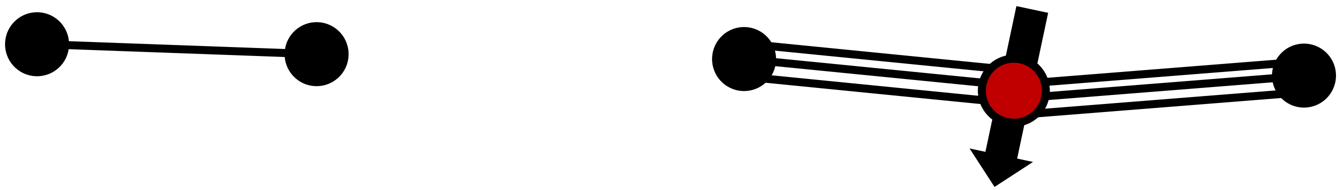


a “strong” field

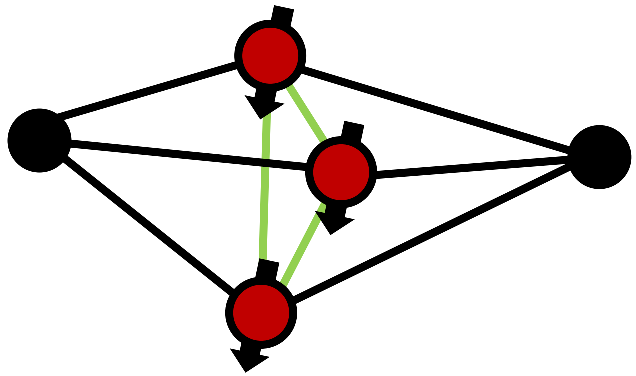


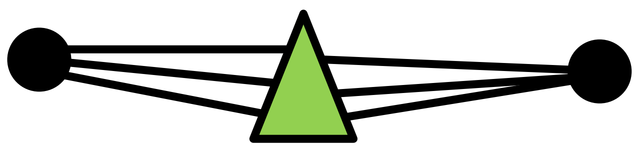
“strong” interactions





“strong” interactions

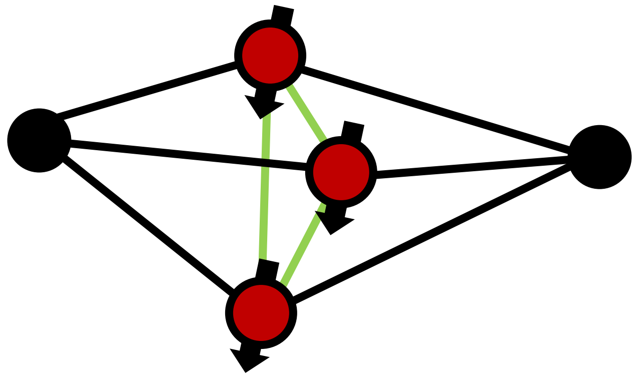


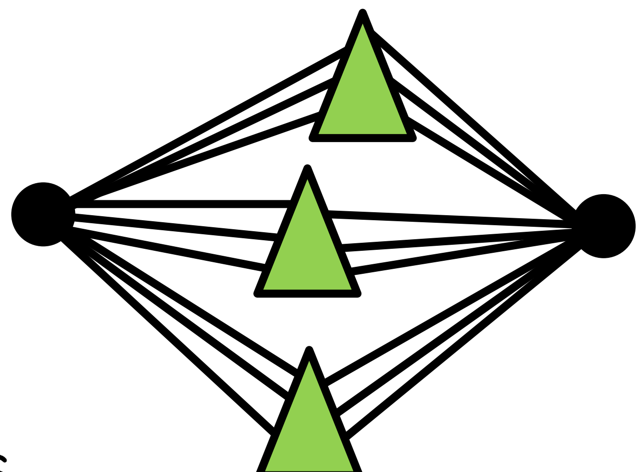


$A \otimes B$

one gadget

“strong” interactions





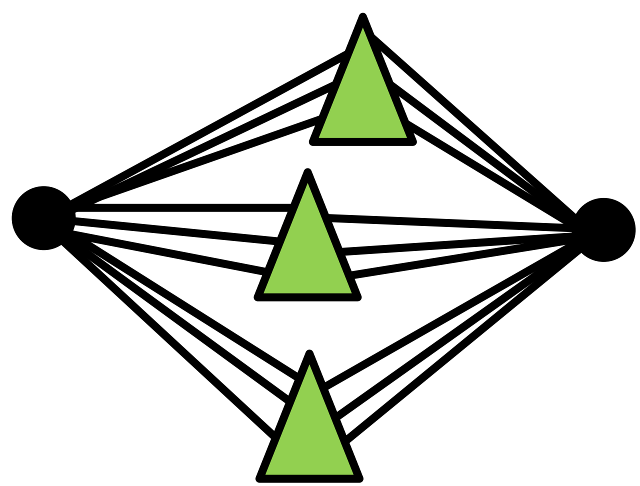
$$A \otimes B$$

several gadgets

“strong” interactions

weak components

new parallel composition

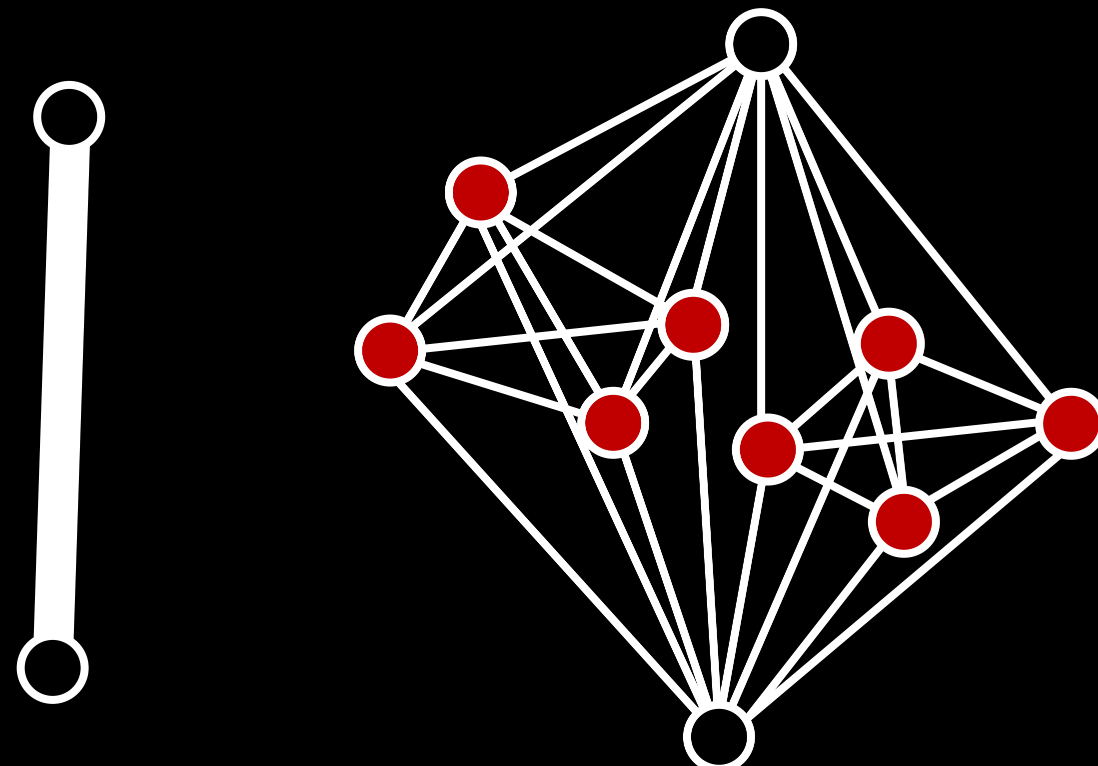


O(1) terms? QMA-complete.

1/poly gap? Constant gap.

High degree (poly).

Fractional gap? Worse.



Daniel Nagaj

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Yudong Cao

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