Local tests of global entanglement and a counterexample to the generalized area law.

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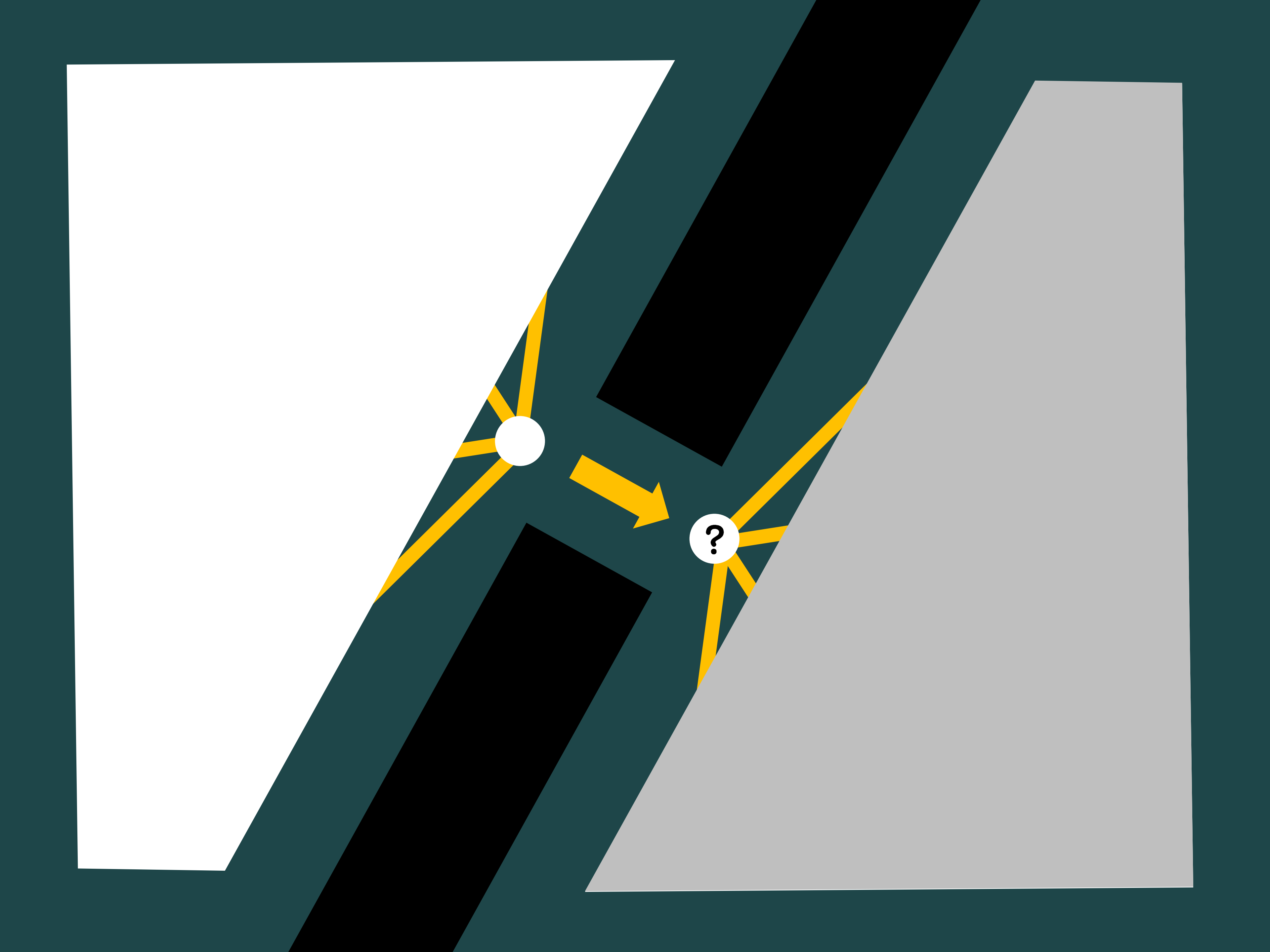
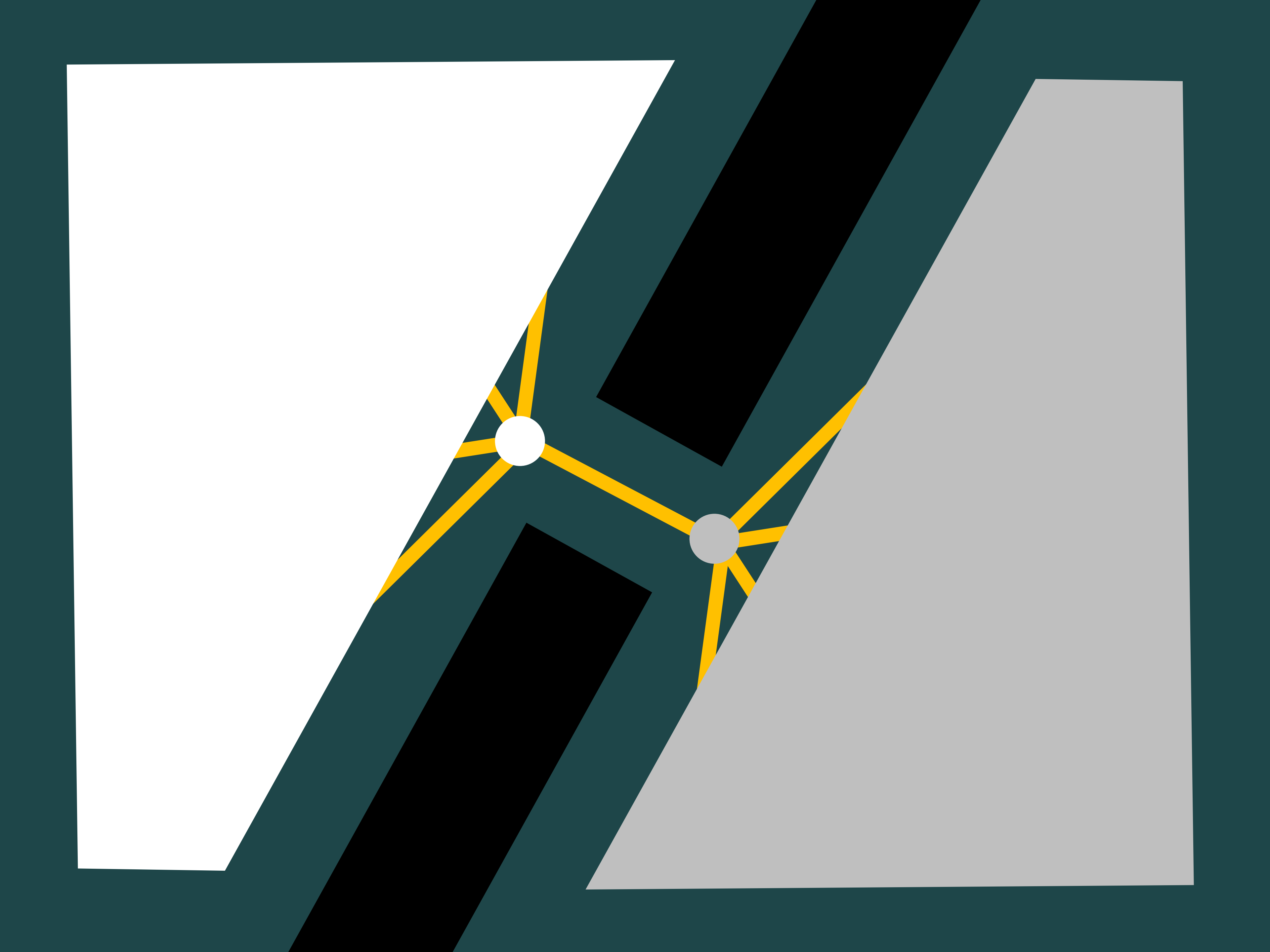
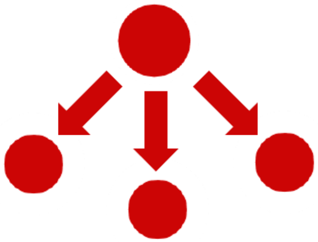
2014 | 5 | 6
IQIM @ Caltech





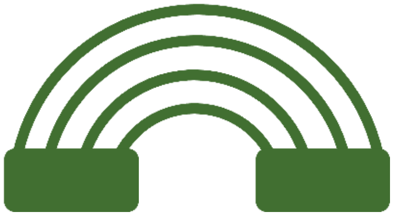
1 **q. expanders**

maximally entangled states



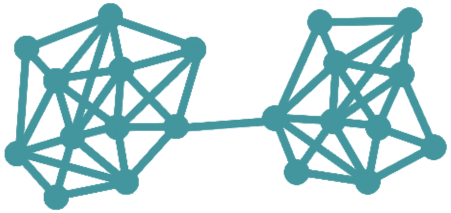
2 **entanglement**

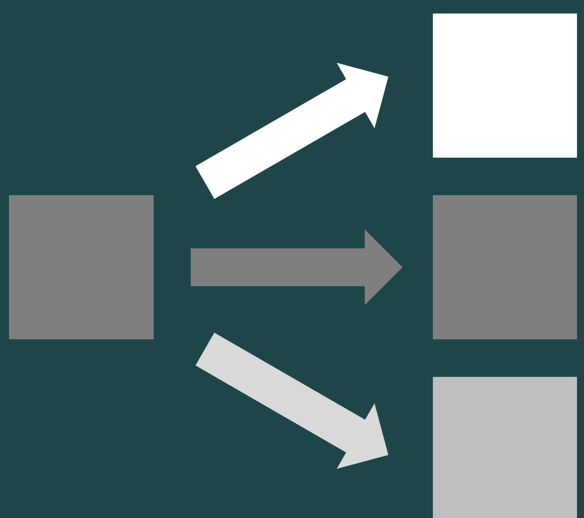
testing and communication



3 **area law**

gaps, connections, correlations



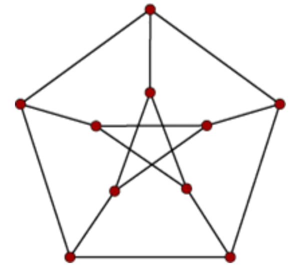


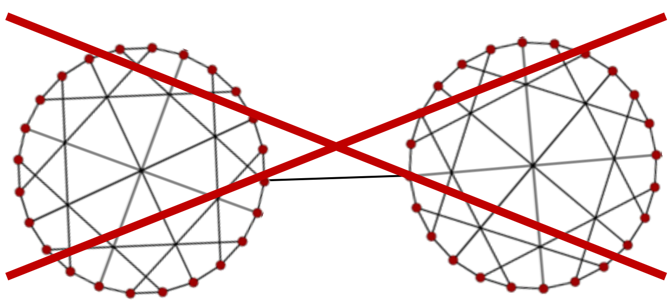
Quantum Expanders

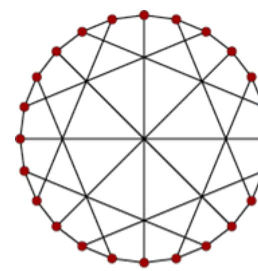
1 Classical expanders

- Aram Harrow's talk, QHC workshop at the
  [youtube Harrow quantum expanders]

- graphs that mix well
  divide in two? cut a lot (fraction) of edges!







1 Classical expanders

- Aram Harrow's talk, QHC workshop at the
  [youtube Harrow quantum expanders]

Simons Institute for the Theory of Computing

- graphs that mix well
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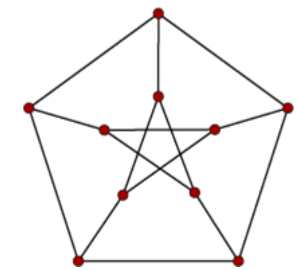
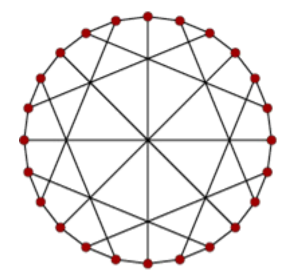


examples: Cayley graphs



- normalized adjacency matrix
  second largest eigenvalue 1–λ

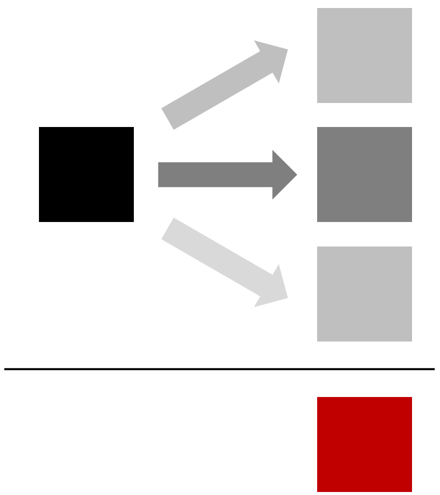
- a motivation for quantum expanders
  d-regular graphs, random permutations

$$A = \frac{1}{k}\sum_{i=1}^{k}\Pi_k$$

1 Mixing up something quantum

- applying random unitaries

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$



- a motivation for quantum expanders
  d-regular graphs, random permutations

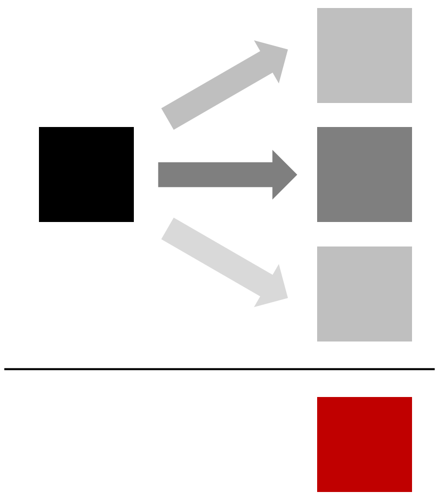
$$A = \frac{1}{k}\sum_{i=1}^{k} \Pi_k$$

1 An (N, k, λ) quantum expander

- transforming matrices

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^\dagger$$

- applying “random” unitaries
  a discrete approximation
  to the Haar measure



- a small second largest singular value λ
  not far from the depolarizing channel

$$\|\mathcal{E} - D\|$$

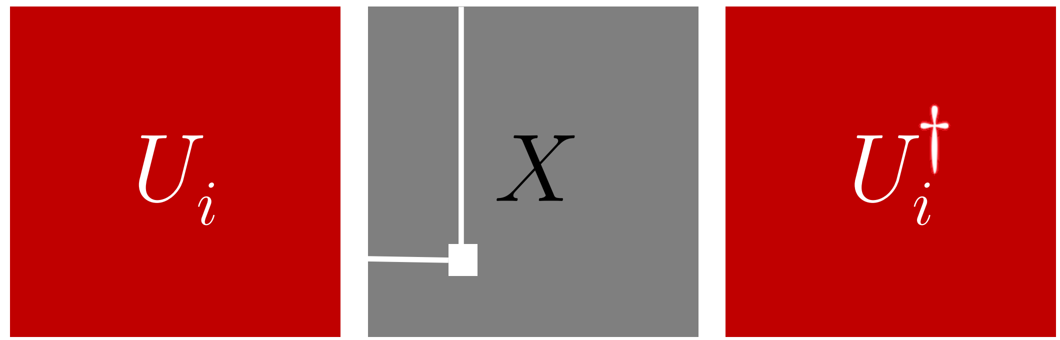
- quantum expander constructions, also fixed *k* (3)
  [Ben-Arroya+ 07, Hastings ’07 , Gross & Eisert ‘08, Hastings & Harrow ‘09]

1 Quantum expanders

- transform $N \times N$ matrices

$$\mathcal{E}(X) = \frac{1}{k}\sum_{i=1}^{k} U_i X U_i^{\dagger}$$

- interpreting matrices as states



- distributively applying an expander

$$\frac{1}{k}\sum_{i=1}^{k} (U_i \otimes U_i^{*}) \sum_{a,b} X_{ab}|a\rangle|b\rangle$$

- a matrix/state that doesn't change?

$X \sim I$ … maximally entangled!

$$|\Phi_N\rangle = \frac{1}{\sqrt{N}}\sum_{x=1}^{N} |x\rangle|x\rangle$$

local tests of global entanglement

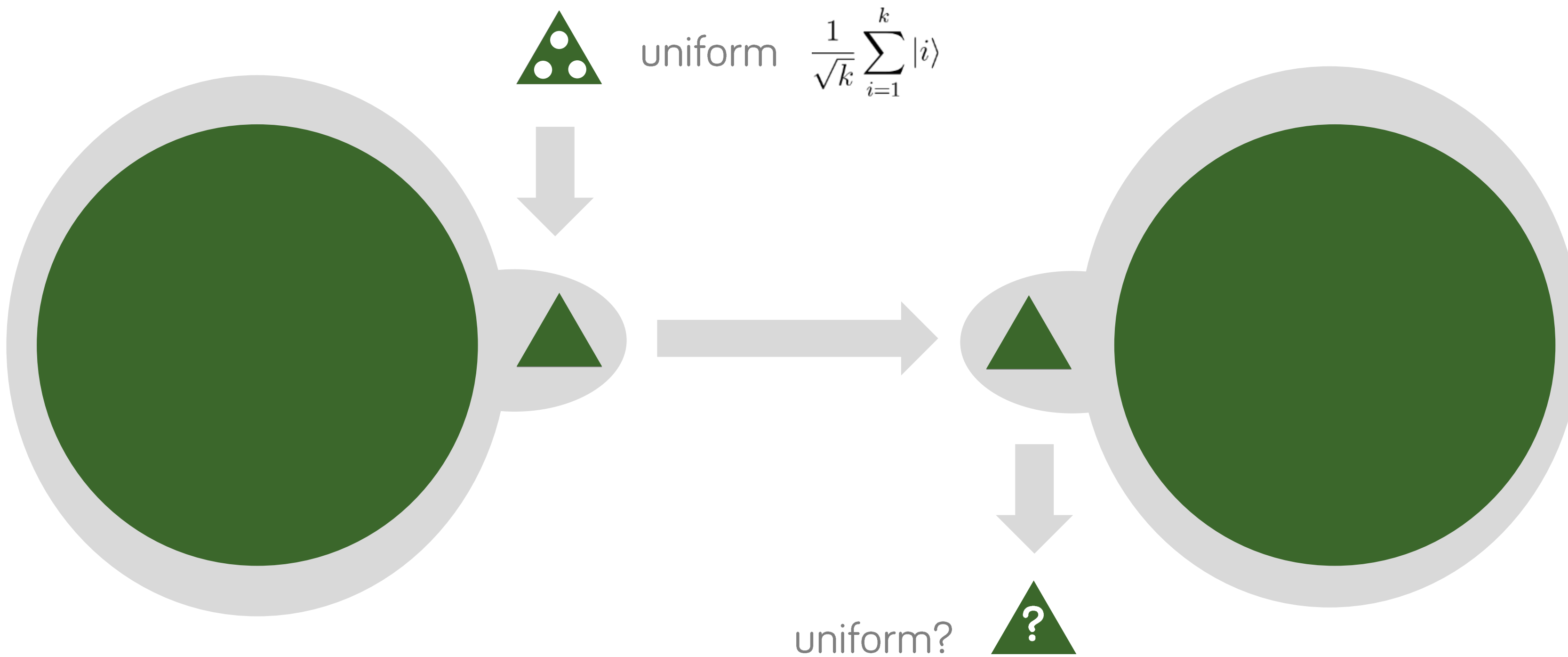
2 EPR testing

- how costly is it to certify that we share a maximally entangled state?

$$\frac{1}{\sqrt{N}}\sum_{x=1}^{N}|x\rangle|x\rangle$$

2 EPR testing

- how costly is it to certify that we share a maximally entangled state? $\frac{1}{\sqrt{N}}\sum_{x=1}^{N}|x\rangle|x\rangle$
- apply a quantum expander distributively $U_i \otimes U_i^*$



2 EPR testing

- action on states

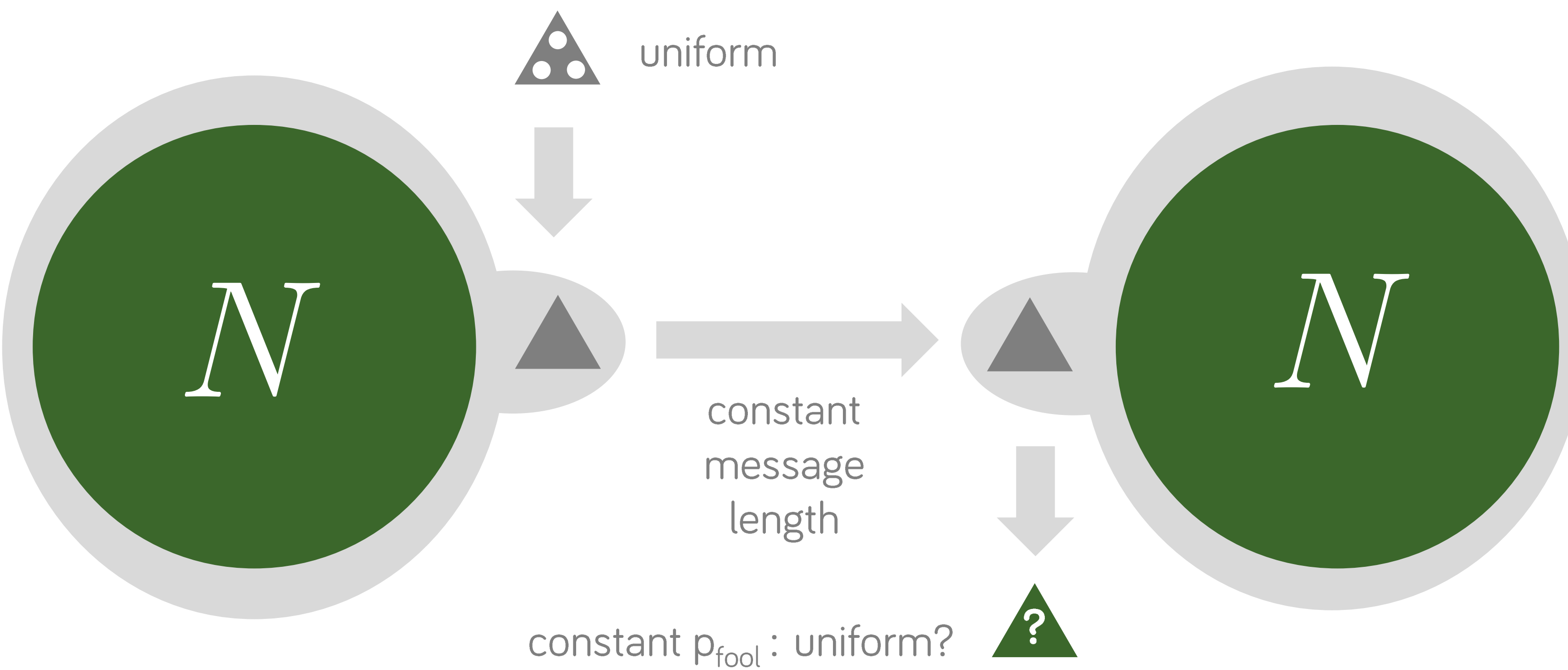
$$\frac{1}{\sqrt{k}}\sum_{i=1}^{k}|i\rangle\,(U_i\otimes U_i^*)\,|X\rangle$$

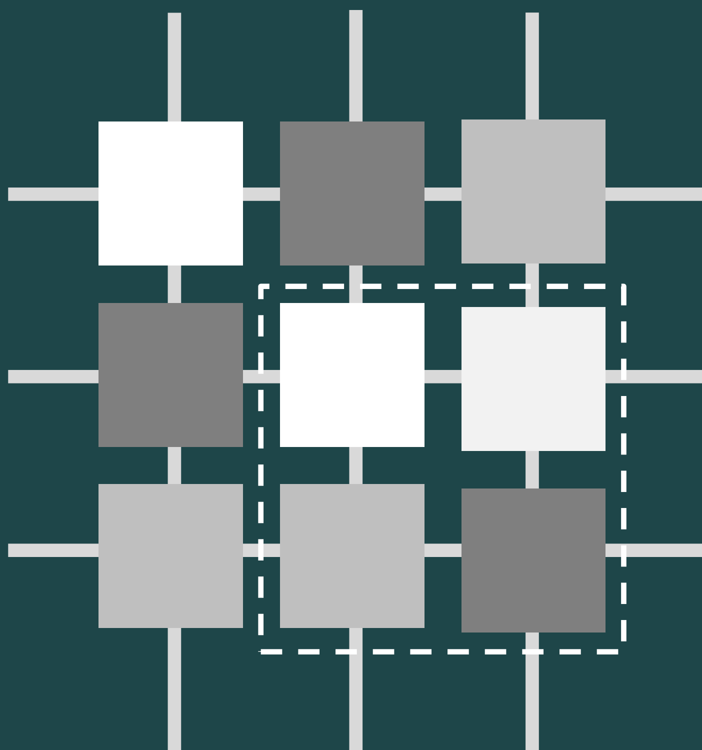
$$\sum_{a,b} X_{ab}|a\rangle|b\rangle$$

U_i X $U_i^\dagger$

- does the qutrit remain uniform?
  does the matrix X commute with U_i?

- quantum expansion … soundness





a counterexample to the

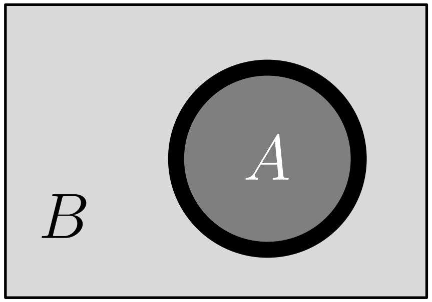
generalized area law

3 Area law: ground states of quantum spin systems

- entanglement entropy

$$S = -\mathrm{Tr}(\rho_A \ln \rho_A) \sim \sout{\text{volume}}\ \text{area}$$

Schmidt coefficients



$$\rho_A = \mathrm{Tr}_B \rho$$

- 1D … algorithms [White 92, Vidal 03, Landau+ 13]

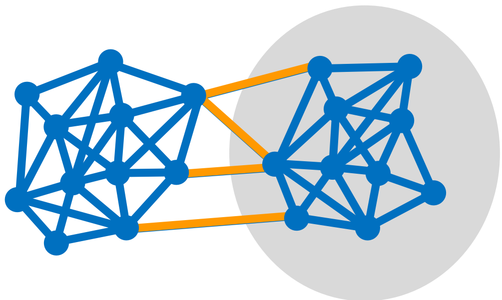
  theorems [Hastings 07, Arad+ 13]

- 2D … we're close

  small gap? large loc. dimension?

- generalized area conjecture

  entropy ~ cut size



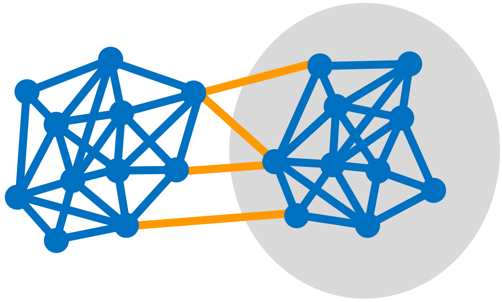
Not true.

few links
O(1) terms
a gap

→

not much
entanglement
(a "simple" ground state)

- generalized area conjecture
  entropy ~ cut size



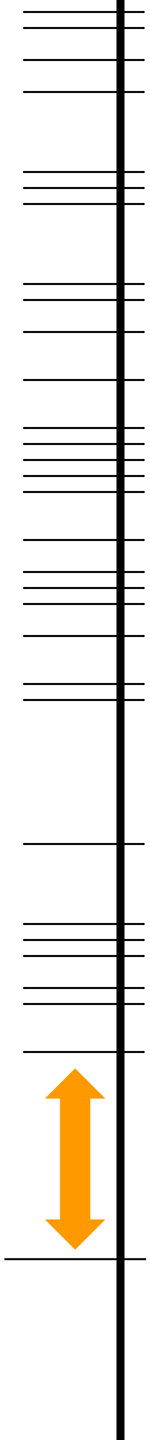
3 Gapped Hamiltonians

- local, $O(1)$ norm terms

Are there states
close to the ground state
when we take the
thermodynamic limit?

$$N \rightarrow \infty$$

~~an inverse-poly gap? $\Delta = \frac{c}{N} \rightarrow 0$~~



3 Gapped Hamiltonians

- Nothing closer than Δ to the ground state.

- in 1D

a region

correlations with the outside fall off exponentially with distance

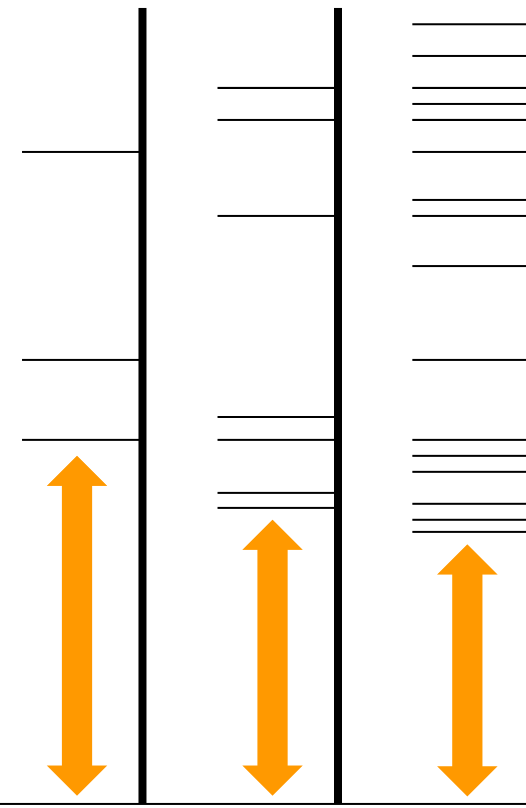
entropy from quantum correlations

$$S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$$

... constant

... log corrections

$$N \rightarrow \infty$$



3 Gapped Hamiltonians

- Nothing closer than Δ to the ground state.

- in 1D

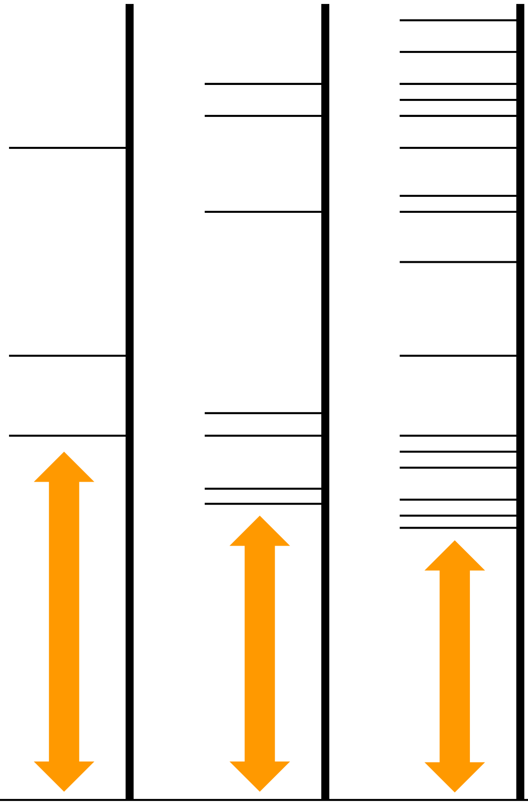
  the AKLT (spin-1) chain

$$\sum_{j=1}^{N-1} \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3}\left(\vec{S}_j \cdot \vec{S}_{j+1}\right)^2$$

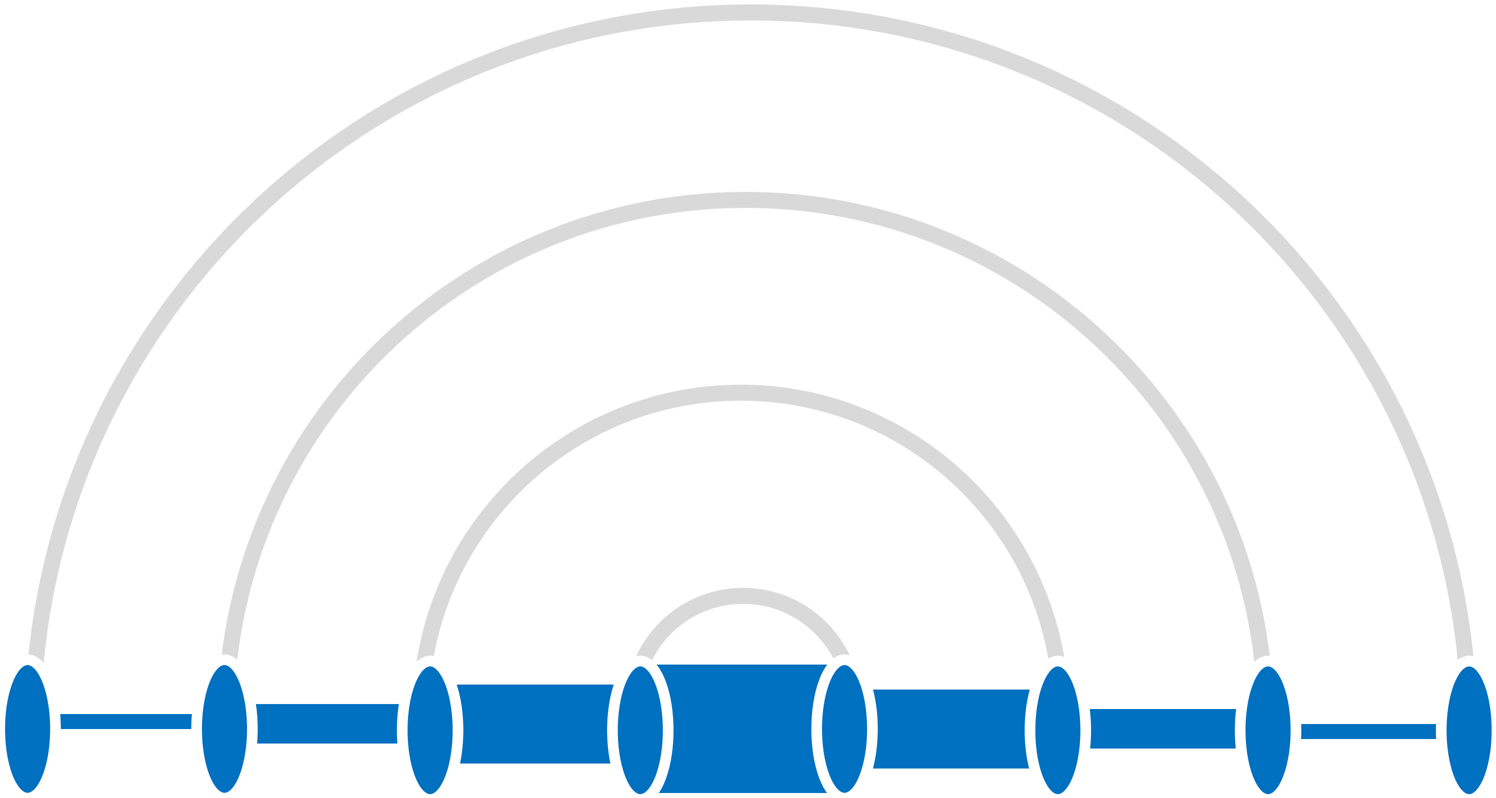
  a biased walk in 1D

$$\sum_{j=1}^{N-1} \left(|j\rangle - B|j+1\rangle\right)\left(\langle j| - B\langle j+1|\right)$$

$N \to \infty$



- a large gap ... a simple (not too entangled) ground state?

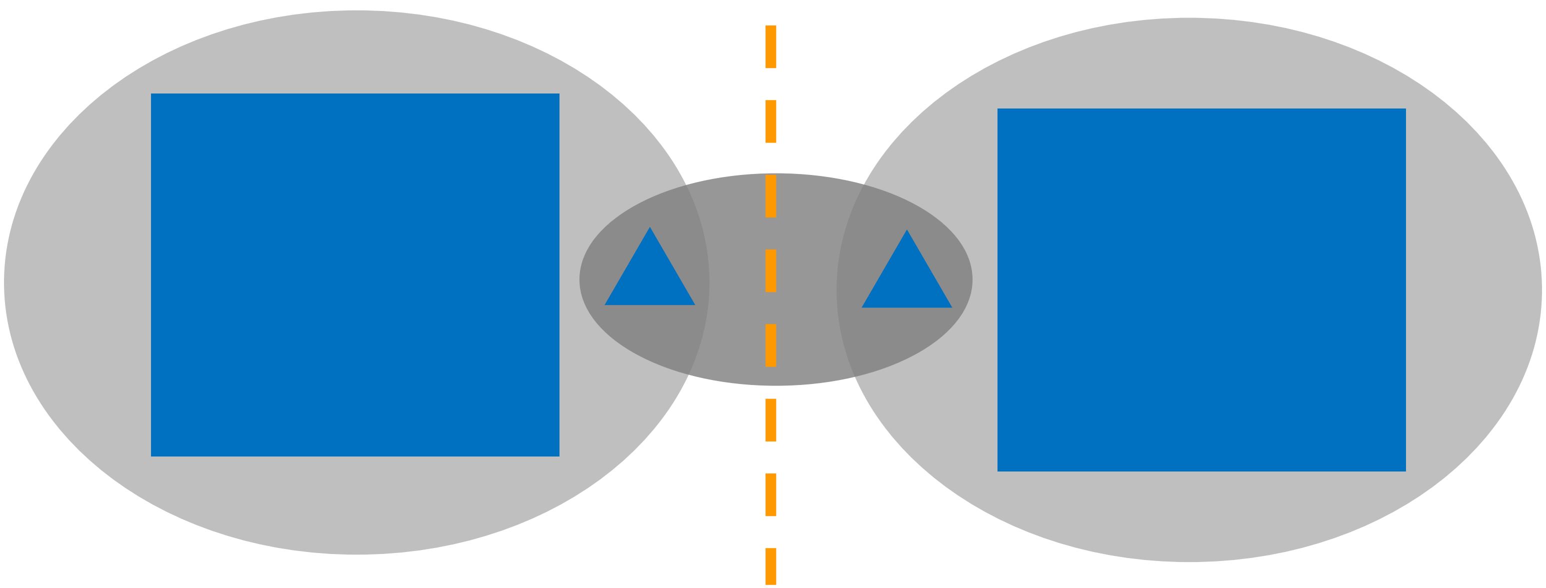


- without a gap, the entropy can be large

[Verstraete, Latorre+]

3 Our counterexample to the generalized area conjecture

- an $N \times 3 \times 3 \times N$ dimensional system
- a frustration-free, gapped, Hamiltonian
- a single O(1) interaction of two 3×3 subsystems
- a unique, very entangled ground state with O(N) entanglement entropy across the cut



3 A nonlocal Hamiltonian, $N \times 3 \times 3 \times N$

- a projector P_L with ground states $$\frac{1}{\sqrt{3}}\left(|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle\right)$$

- as a vector

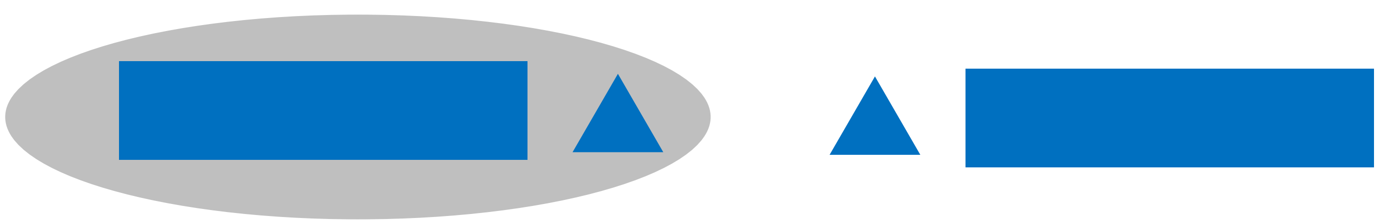
x
Ax
Bx

$\otimes |j\rangle \otimes |y\rangle$

as a matrix

X_1	X_2	X_3
AX_1	AX_2	AX_3
BX_1	BX_2	BX_3

P_L



3 A nonlocal Hamiltonian, $N\times 3\times 3\times N$

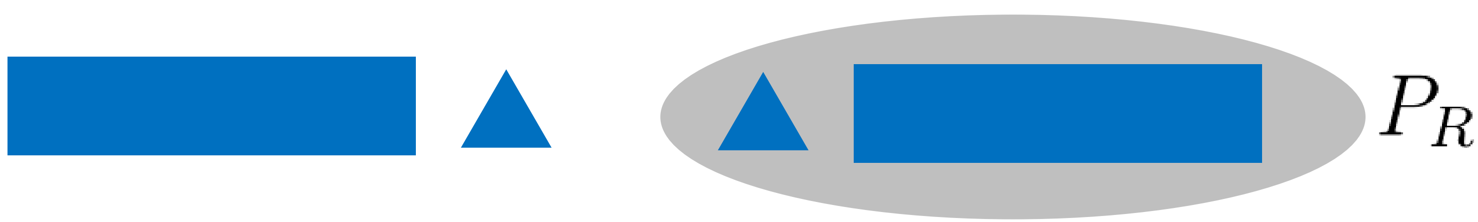
- a projector P_R
  with ground states $\frac{1}{\sqrt{3}}\left(|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle\right)$

- as a vector $|i\rangle \otimes |x\rangle \otimes$

y	yA	yB

  as a matrix

Y_1	$Y_1 A$	$Y_1 B$
Y_2	$Y_2 A$	$Y_2 B$
Y_3	$Y_3 A$	$Y_3 B$



3 A nonlocal Hamiltonian, $N \times 3 \times 3 \times N$

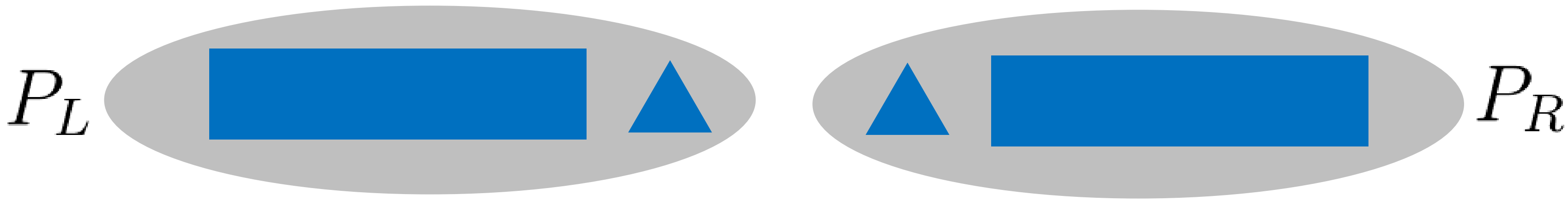
- a projector P_L $\qquad (|1\rangle|x\rangle + |2\rangle A|x\rangle + |3\rangle B|x\rangle)/\sqrt{3}$

  a projector P_R $\qquad (|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)/\sqrt{3}$

X_1	X_2	X_3
AX_1	AX_2	AX_3
BX_1	BX_2	BX_3

X	XA	XB
AX	AXA	AXB
BX	BXA	BXB

Y_1	Y_1A	Y_1B
Y_2	Y_2A	Y_2B
Y_3	Y_3A	Y_3B



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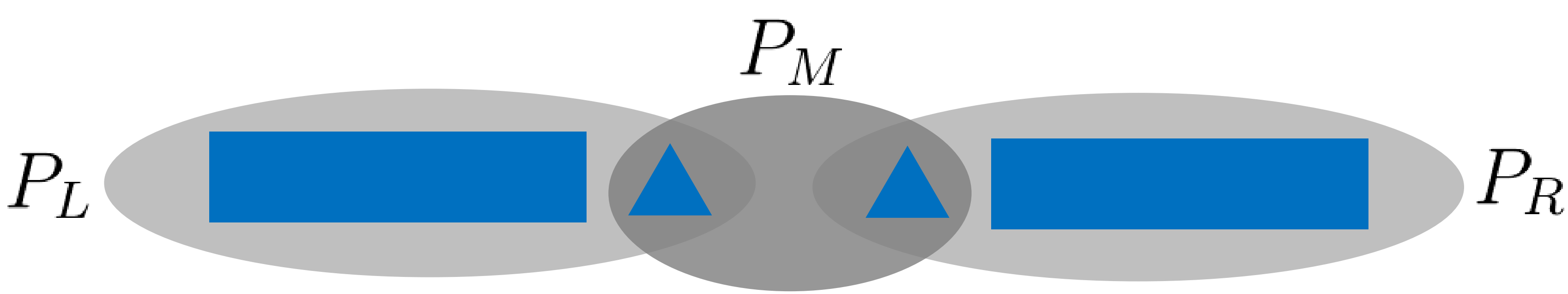
  a projector P_R $\qquad (|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)/\sqrt{3}$
  a projector P_M

X	XA	XB
AX	AXA	AXB
BX	BXA	BXB



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  a projector P_R $\quad (|1\rangle|y\rangle + |2\rangle A|y\rangle + |3\rangle B|y\rangle)/\sqrt{3}$

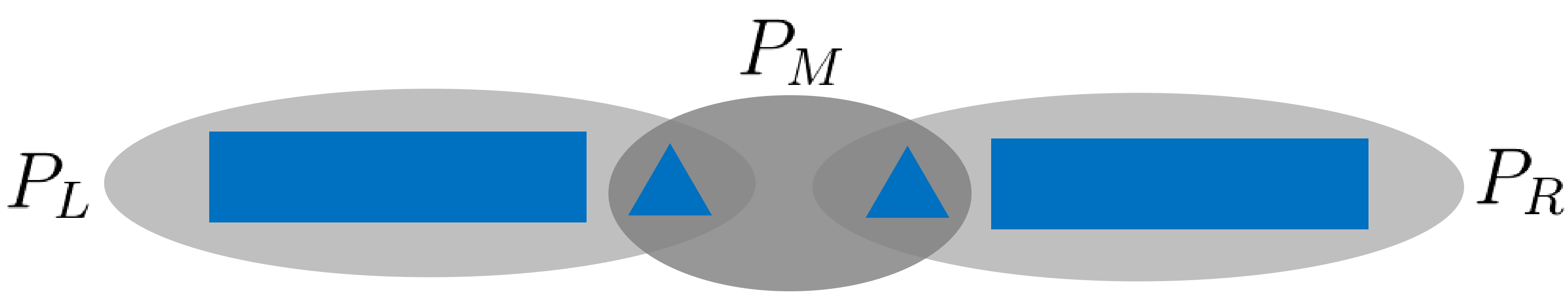
  a projector P_M enforcing symmetry for 12 & 21

  for 13 & 31

- who commutes with A and B?

  only the identity, as $[I, A, B]$ are a q. expander

X	XA	XB
AX	AXA	AXB
BX	BXA	BXB



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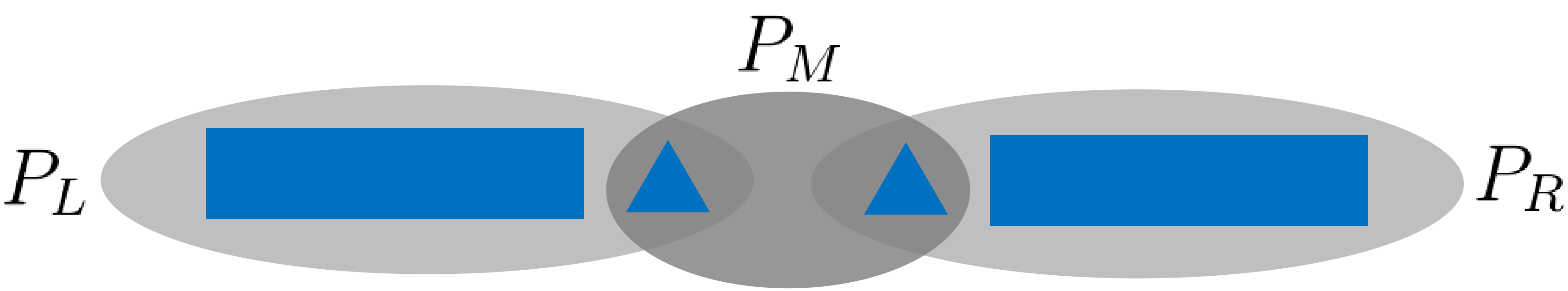
  a projector P_M enforcing symmetry for 12 & 21
  for 13 & 31

- who commutes with A and B?

  only the identity, as $[I, A, B]$ are a q. expander

I	A	B
A	AA	AB
B	BA	BB

$$\frac{1}{3\sqrt{N}}$$



3 A nonlocal Hamiltonian, $N \times 3 \times 3 \times N$

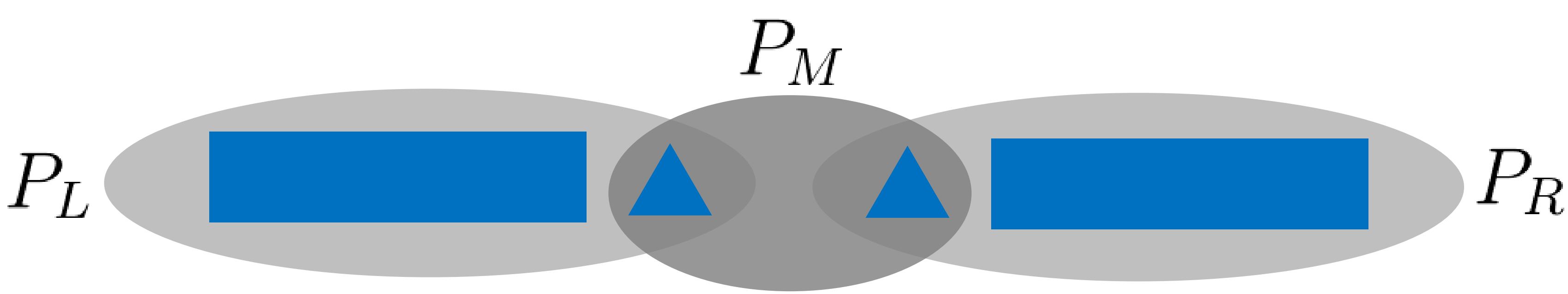
- ground state: unique
  very entangled

- Hamiltonian:
  frustration free
  gapped

I	A	B
A	AA	AB
B	BA	BB

$$\sum_{x=1}^{N} |x\rangle|x\rangle$$

$$\frac{1}{3\sqrt{N}}$$

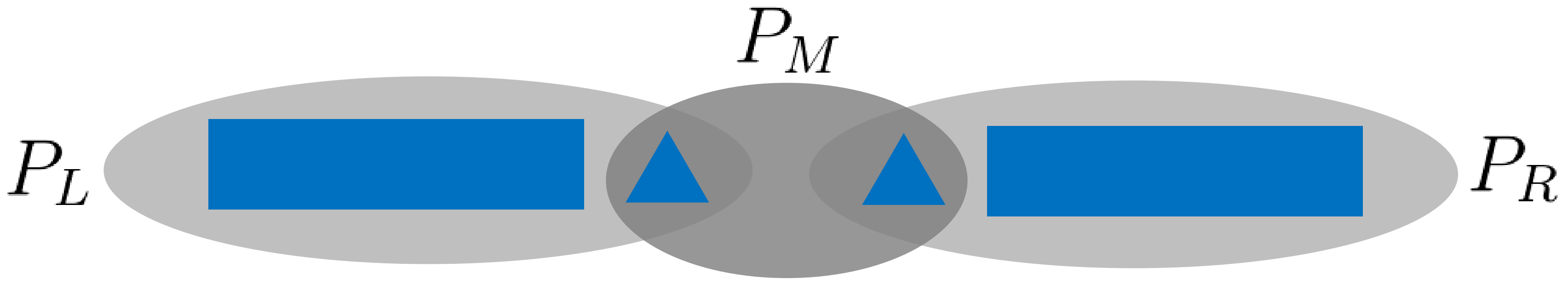


3 Making the counterexample local

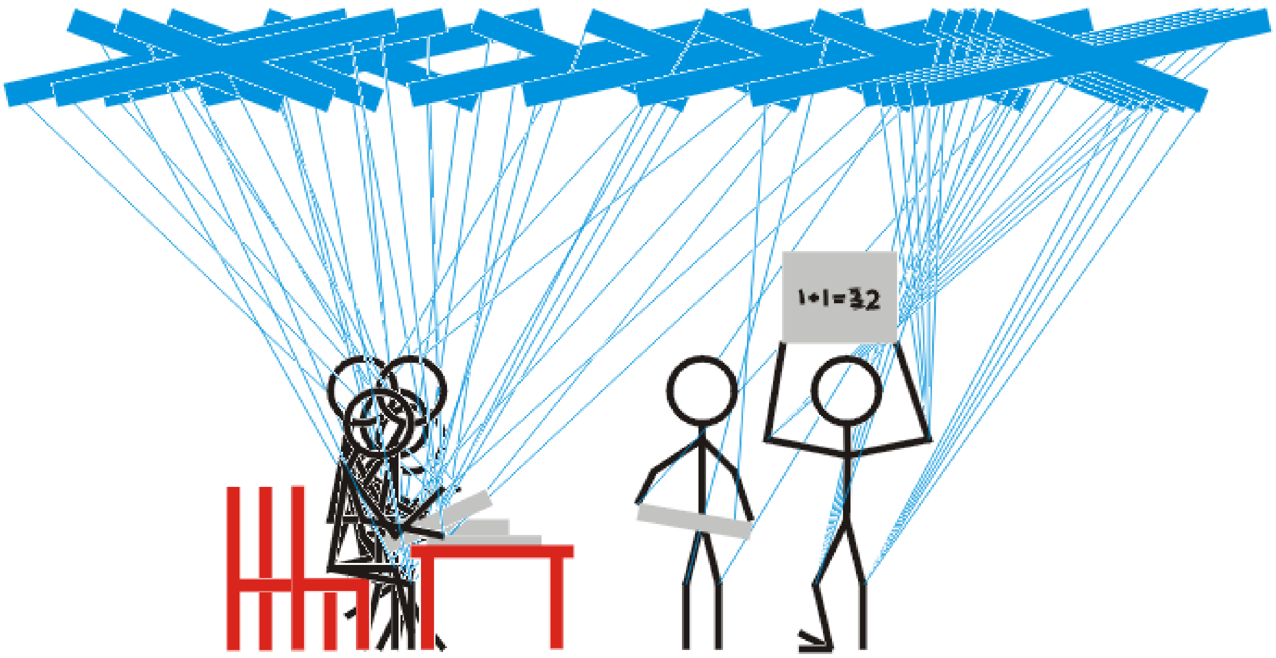
- quantum expander $[I, A, B]$ … quantum circuits …
  ~~nonlocal projectors~~ … Kitaev's LH & history states
  approximate g. s., the gap becomes very small

- rescale P_L, P_R (not the middle!)
  huge, nonphysical couplings … do they matter?

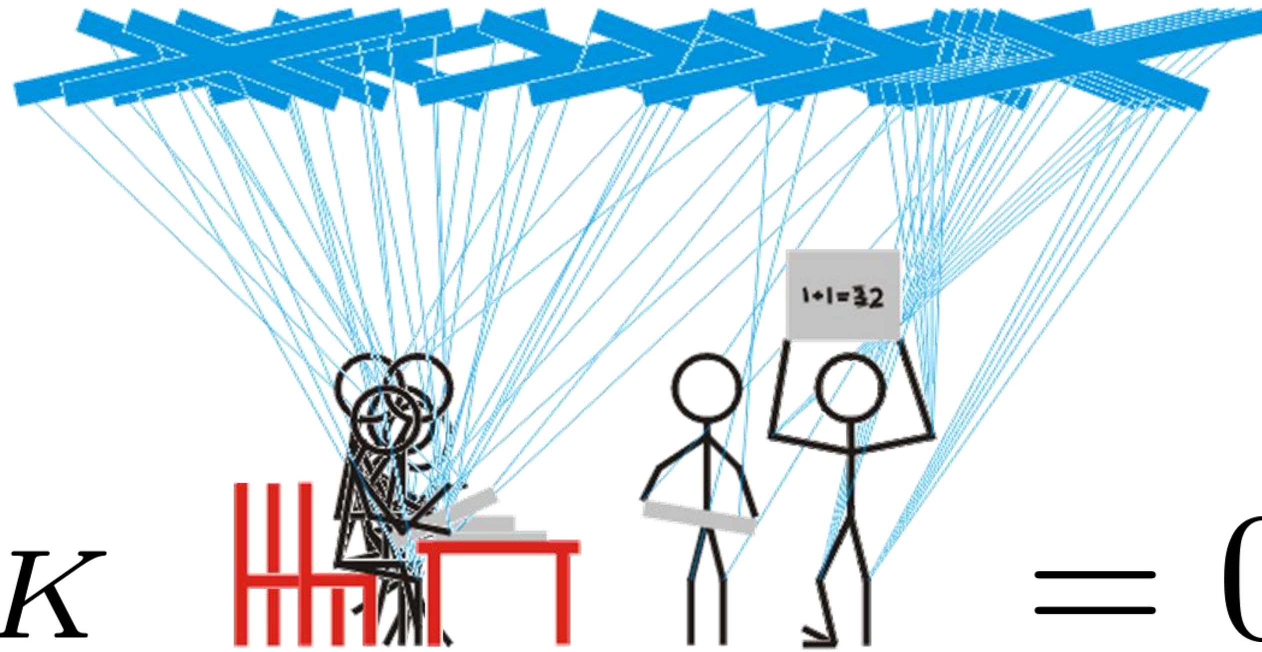
- use new "strengthening gadgets" [N., Cao]
  ~~large interaction strength~~ … extra particles, high degree



3 Implementing circuits locally: Feynman's computer



3 The history state: a ground state

H_K  $= 0$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_t\rangle = U_t \cdots U_1 |\varphi_0\rangle$$

3 The history state: a ground state

k-local

c-o-n-d-i-t-i-o-n-s

clock encoding

state progression

initialization

idling

$$|\cdots\rangle|u\rangle \otimes |0\rangle$$

$$|\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

most of the state has the result

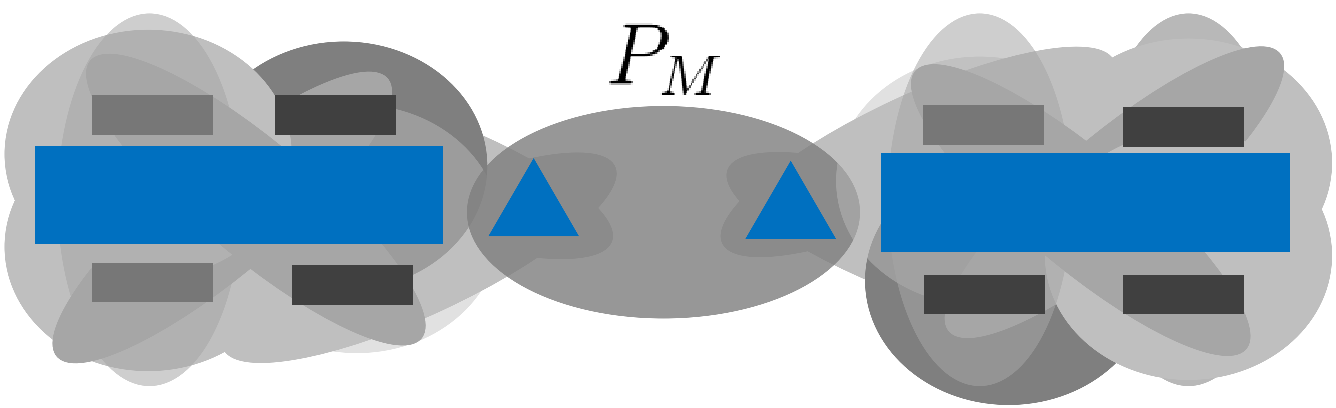
3 A local Hamiltonian, $(N+n) \times 3 \times 3 \times (N+n)$

- frustrated, but still gapped
  O(1) norm terms

- a unique and
  still very entangled
  ground state

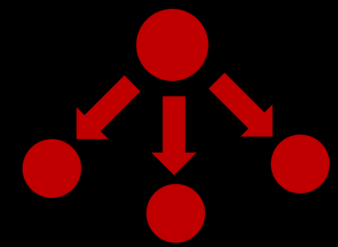
$$\approx |w\rangle \otimes$$

I	A	B
A	AA	AB
B	BA	BB



1 **q. expanders**

maximally entangled states

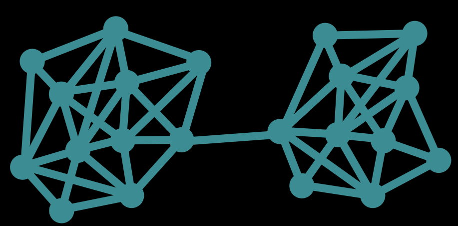


2 **entanglement**

testing and communication

3 **area law**

gaps, connections, correlations



a counterexample to the generalized area law

MS

UV

ZL

DN

DA

AH

2014 | 5 | 6

IQIM @ Caltech

Local tests of global entanglement and a counterexample to the generalized area law.

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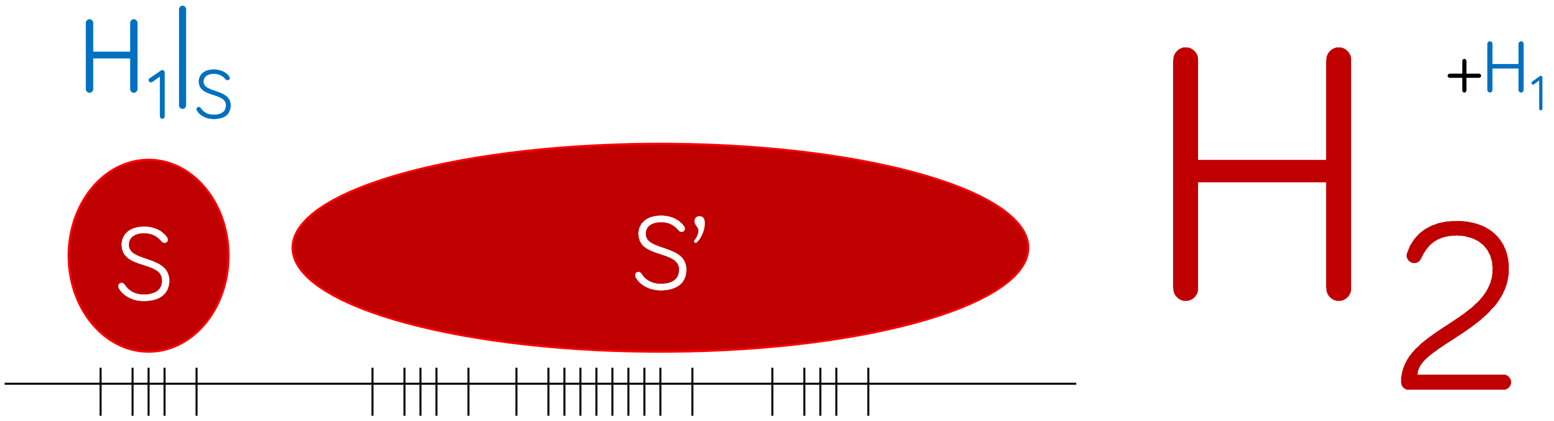
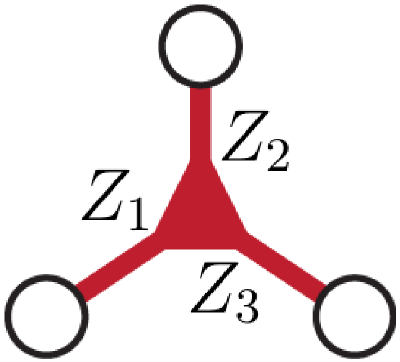
projections
& gadgets

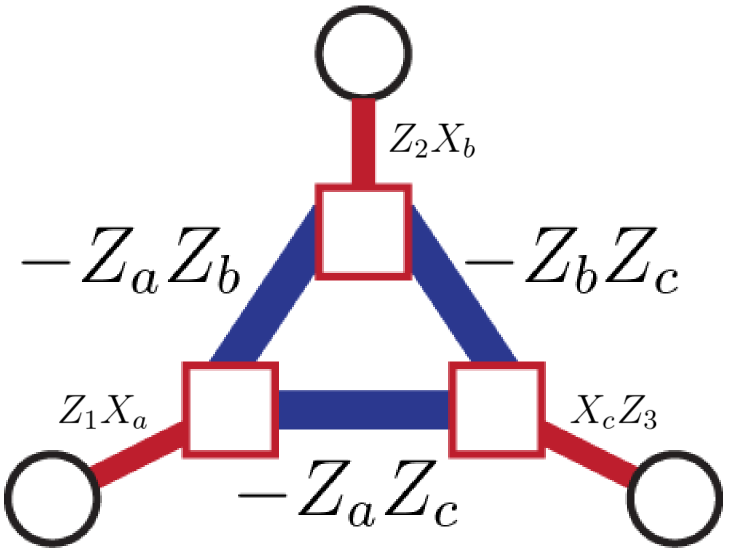
4 The projection lemma: a useful tool



- a HIGH energy penalty for “illegal” states?
- the low energy states live near the “legal” subspace

4 Going further: Quantum gadgets ("3 from 2")





- strongly coupled ancillas (a new energy scale)
- perturbation theory

$$G'(z) = \left(z\mathbb{I} - H'\right)^{-1}$$

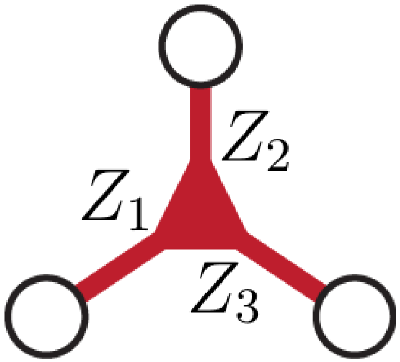
$$H' = H + V$$

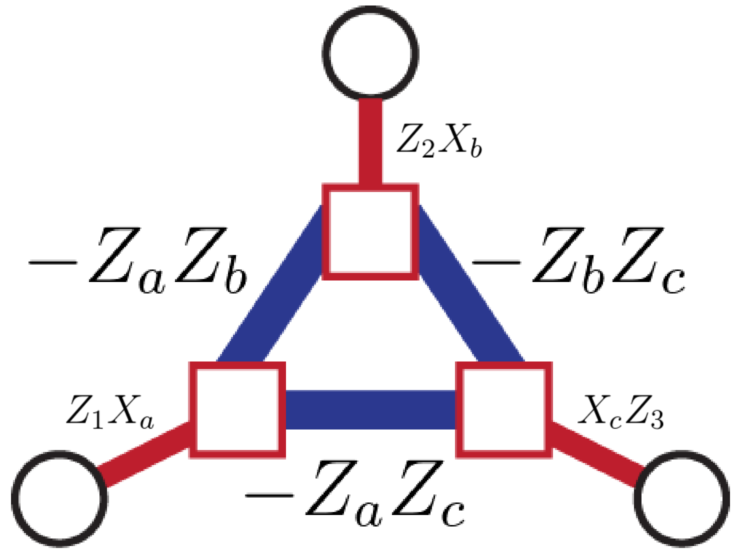
$$||H|| \gg ||V||$$

$$S = \text{span}\left\{|000\rangle, |111\rangle\right\}$$

[Kempe, Kitaev, Regev '03]

4 Going further: Quantum gadgets ("3 from 2")





- strongly coupled ancillas (a new energy scale)
- perturbation theory gives us an effective Hamiltonian

$$H' = H + V$$

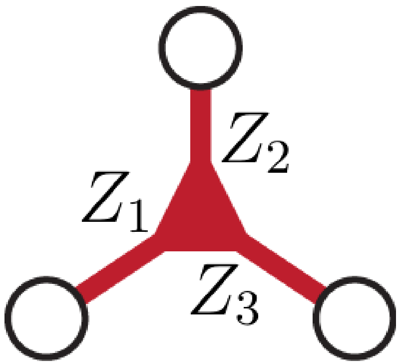
$$||H|| \gg ||V||$$

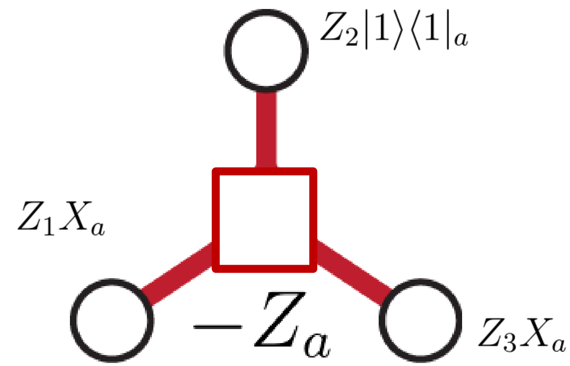
$$S = \text{span}\{|000\rangle, |111\rangle\}$$

$V\big\|_S$	$V^2\big\|_S$	$V^3\big\|_S$
projection lemma	unwanted (subtract)	the effective 3-local term

[Kempe, Kitaev, Regev '03]

4 STILL HUGE fields, LARGE interactions [Cao et al., 1311.2555]





- strongly bound a single ancilla
  still needs strong interactions

$$S = \{|0\rangle\}$$

- perturbation theory gives us an effective Hamiltonian

$$H' = H + V$$

$$||H|| \gg ||V||$$

$$V\Big|_S \qquad V^2\Big|_S \qquad V^3\Big|_S$$

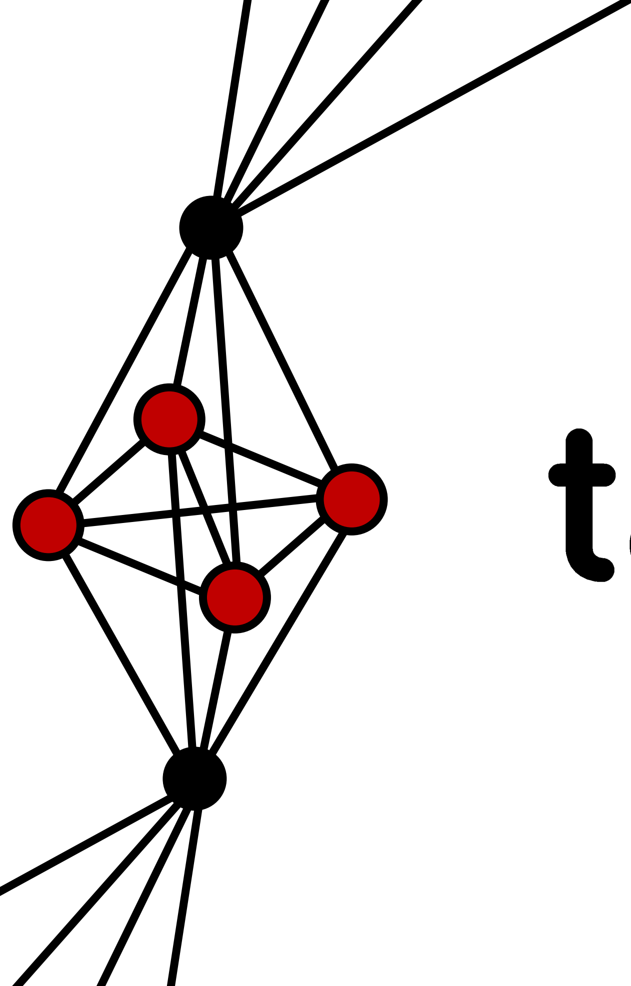
projection lemma

unwanted (subtract)

the effective 3-local term

special cases (Z-basis) exact gadgets!

[Biamonte 0801.3800]

the  talk

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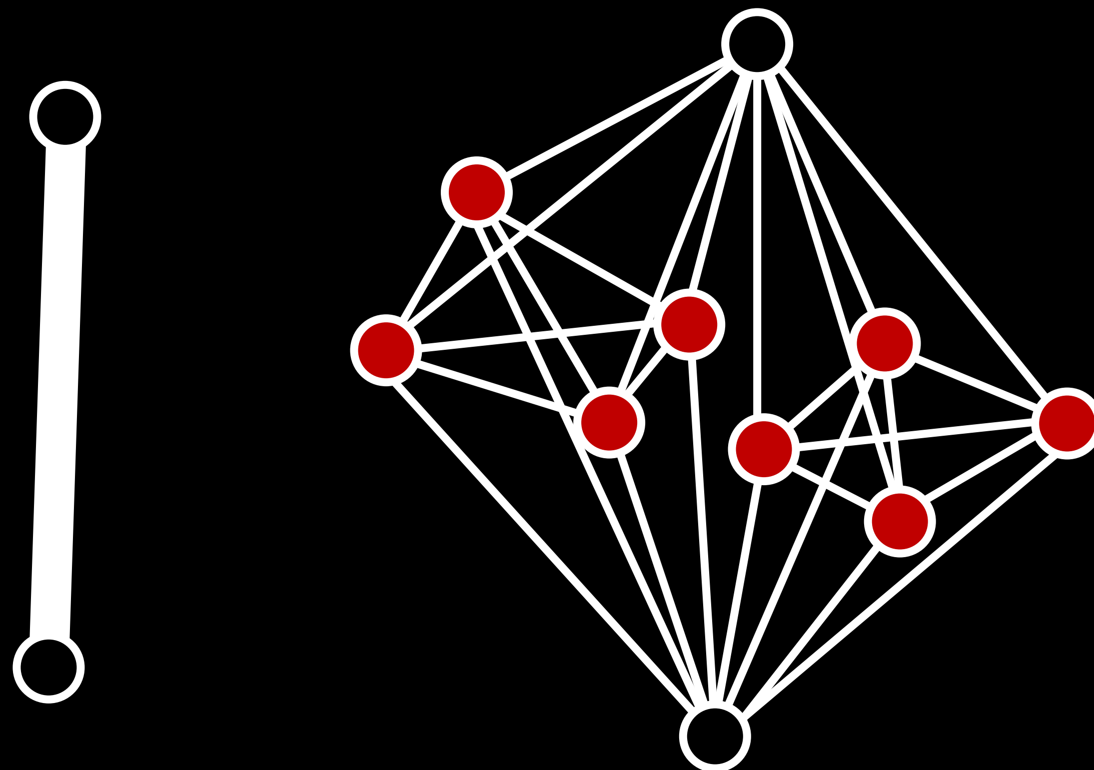

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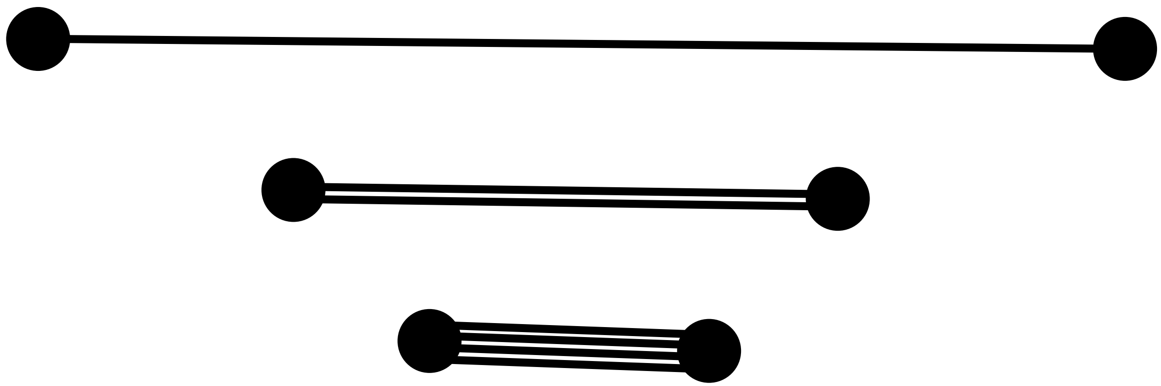
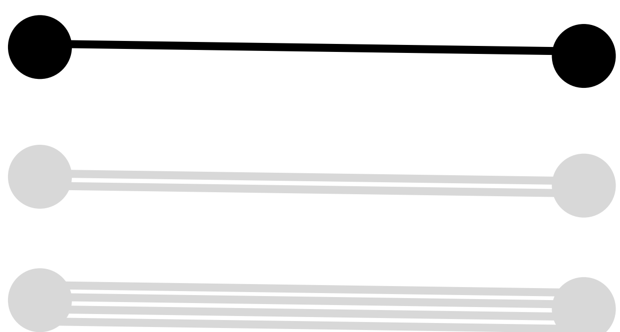
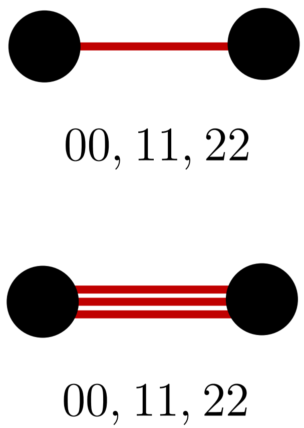
- interaction strength vs. distance

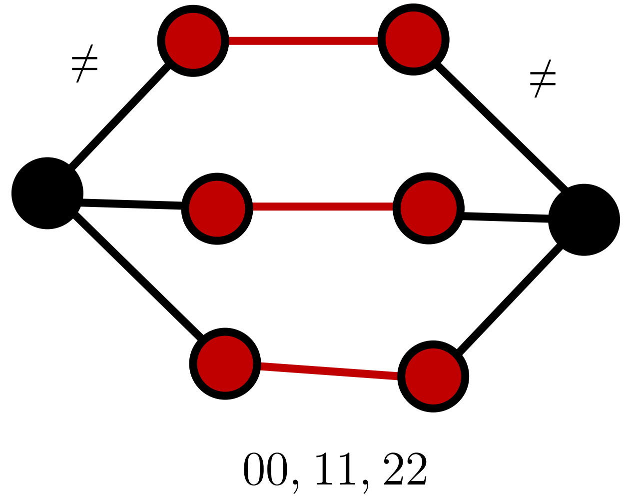


- limited interaction strength

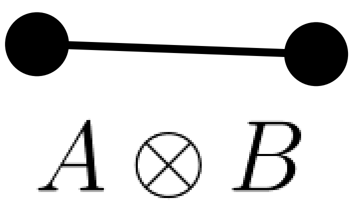


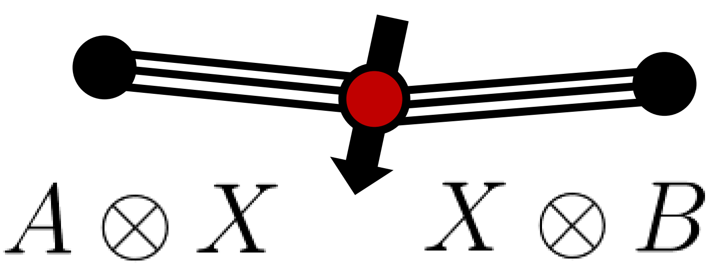
- classical gadgets



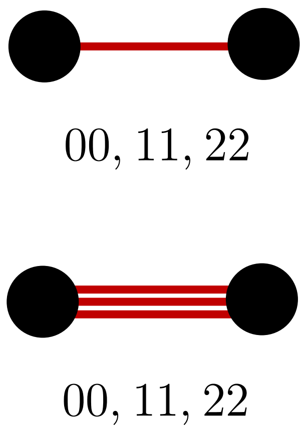


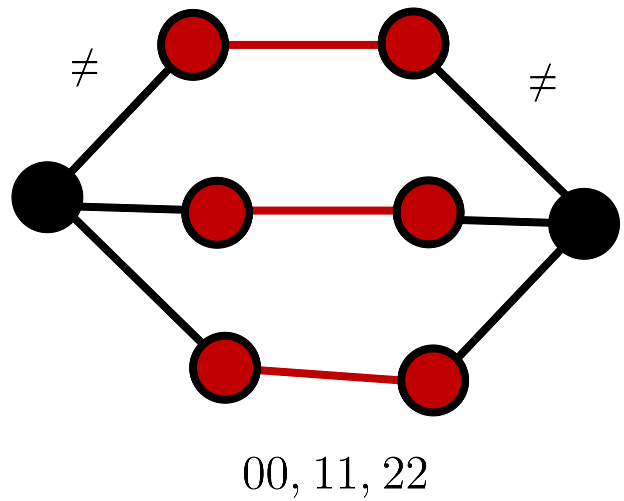
- quantum gadgets



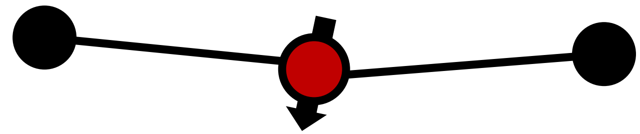


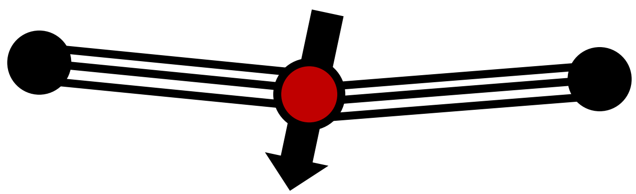
- classical gadgets

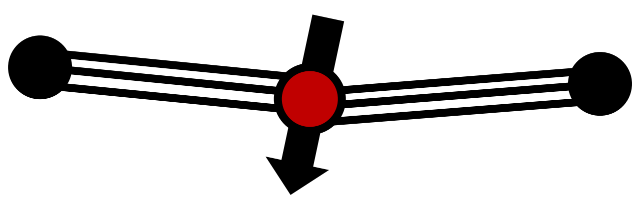




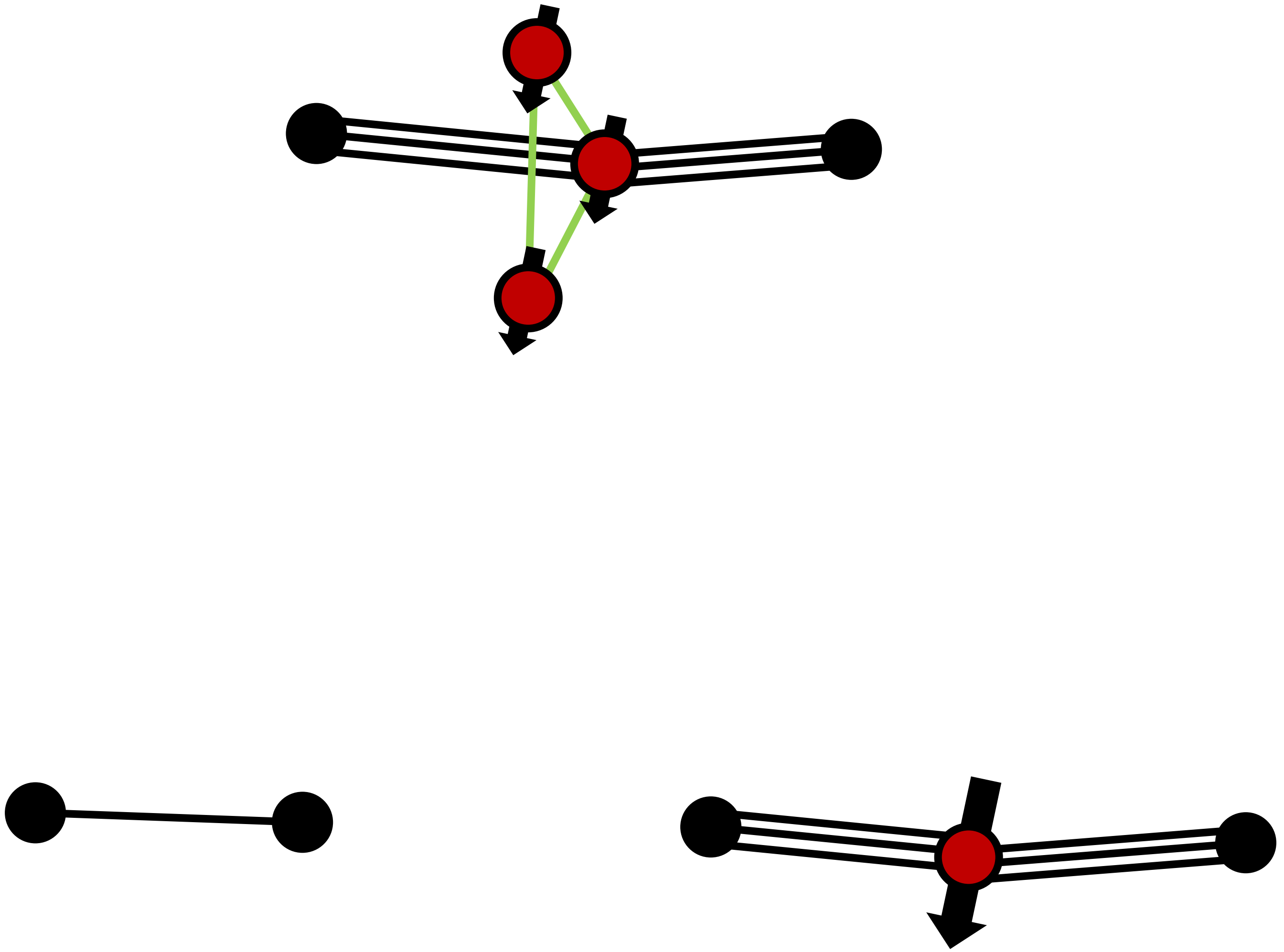
- quantum gadgets



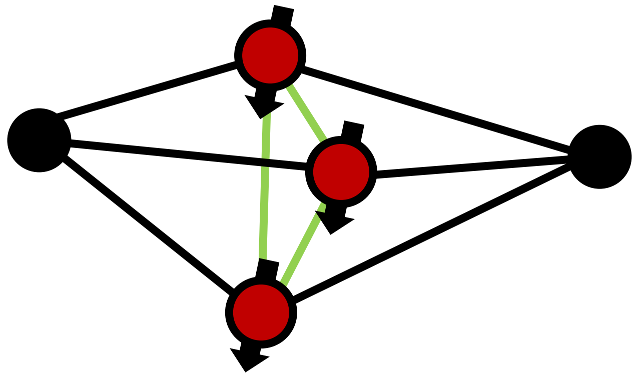


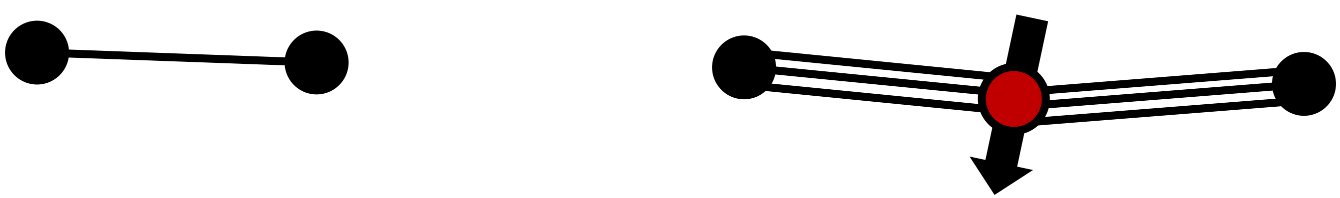


a “strong” field

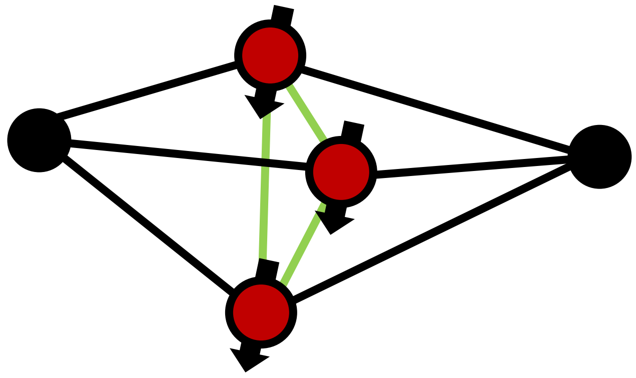


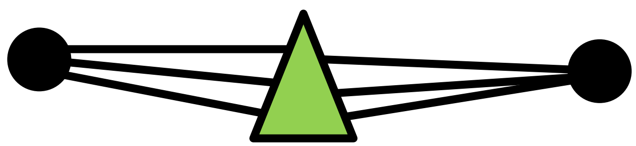
“strong” interactions





“strong” interactions

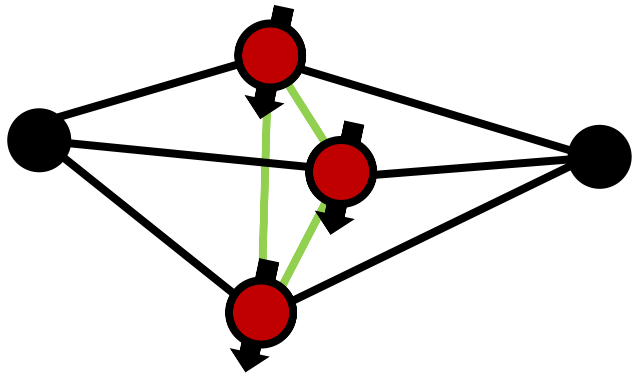


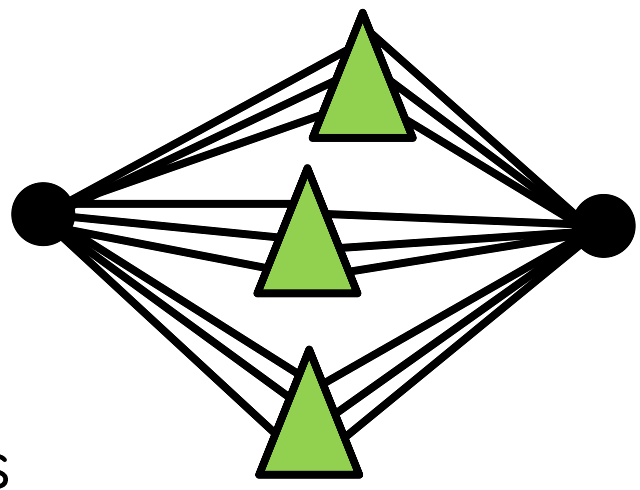


$A \otimes B$

one gadget

“strong” interactions





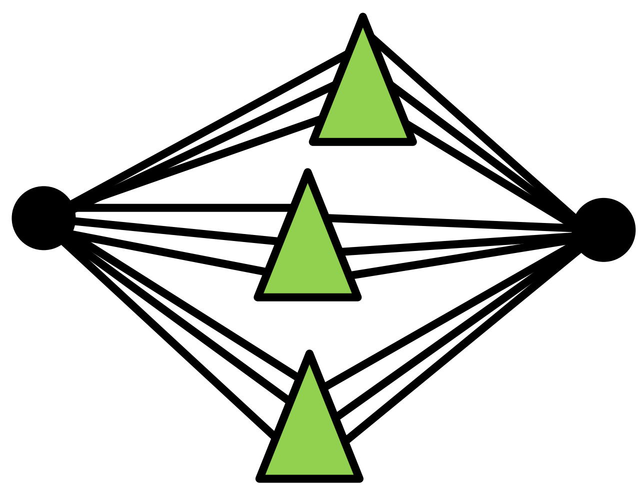
$$A \otimes B$$

several gadgets

“strong” interactions

weak components

new parallel composition

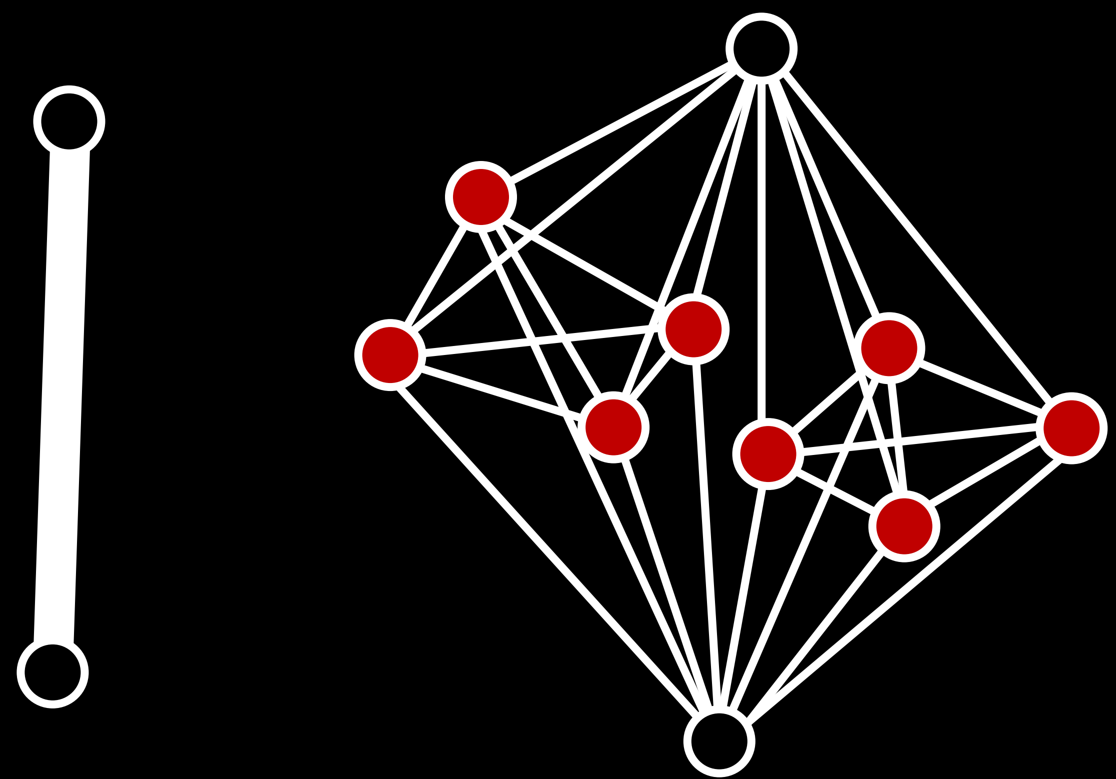


O(1) terms? QMA-complete.

1/poly gap? Constant gap.

High degree (poly).

Fractional gap? Worse.



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