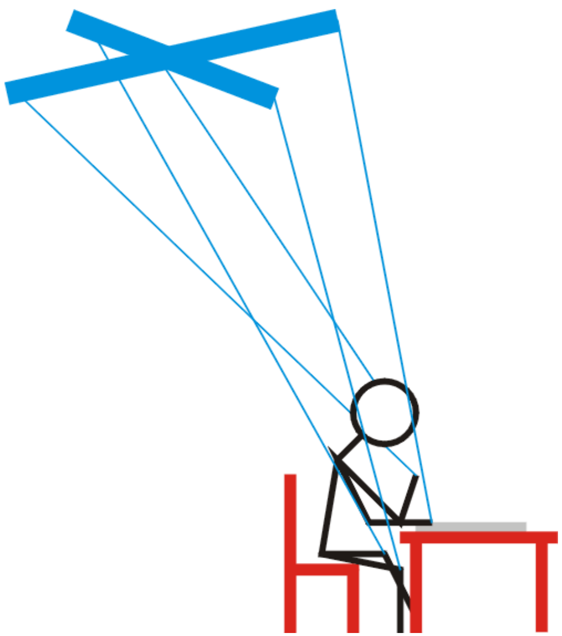
An introduction to

Local Hamiltonians & Quantum Complexity



D. Gosset, S. Hallgren, S. Narayanaswami,
S. Mozes, P. Wocjan, O. Regev, P. Love, P. Shor, S. Jordan,
S. Lloyd, A. Landahl, S. Irani, D. Gottesman, S. Bravyi,
E. Farhi, J. Goldstone, D. Aharonov, T. Morimae,
R. Movassagh, F. Brandao, J. Eisert, J. Whitfield, ...

Daniel Nagaj

universität wien

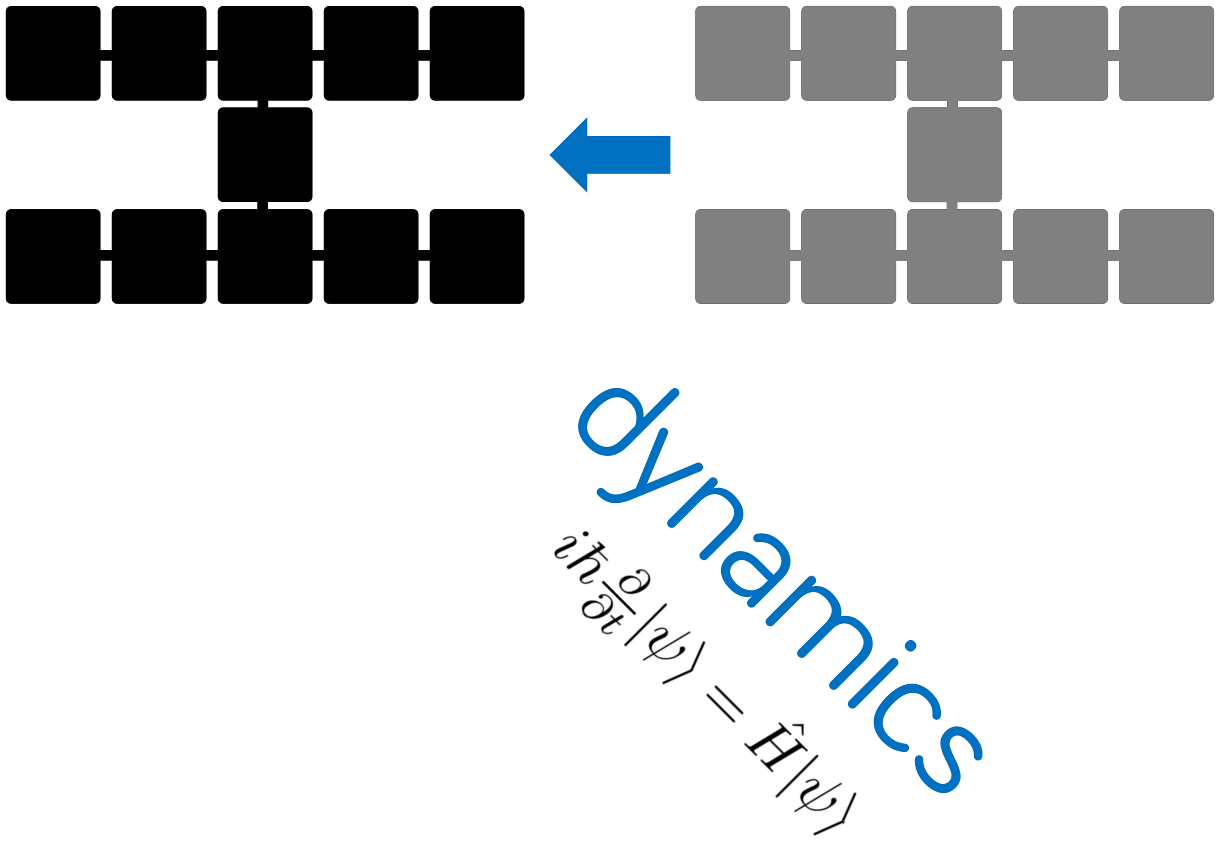

如意吉祥

Local

Hamiltonian

[wisegeek.com]

Hamiltonians


dynamics
$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$

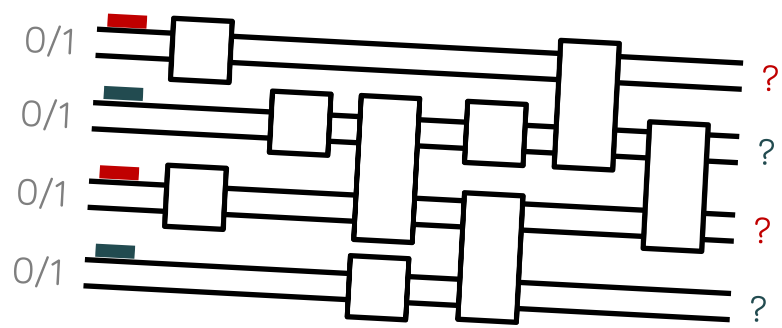

scattering
[fermilab]

scattering

Universal computation by multi-particle quantum walk

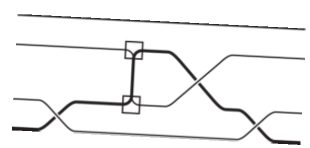
- dual-rail encoding
  N wavepackets

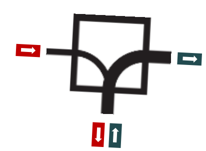
$$a_j^\dagger a_k + a_k^\dagger a_j$$

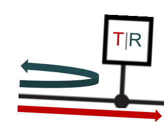


- CPHASE: interaction

$$a_j^\dagger a_k^\dagger a_j a_k$$

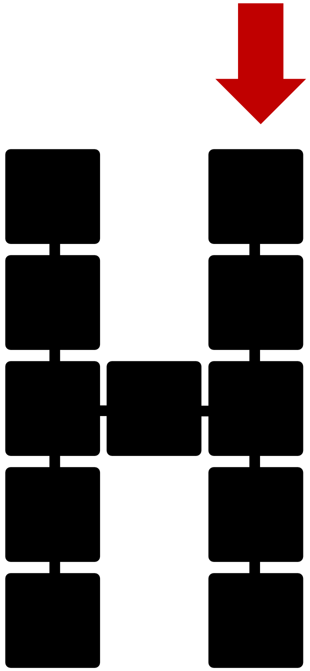


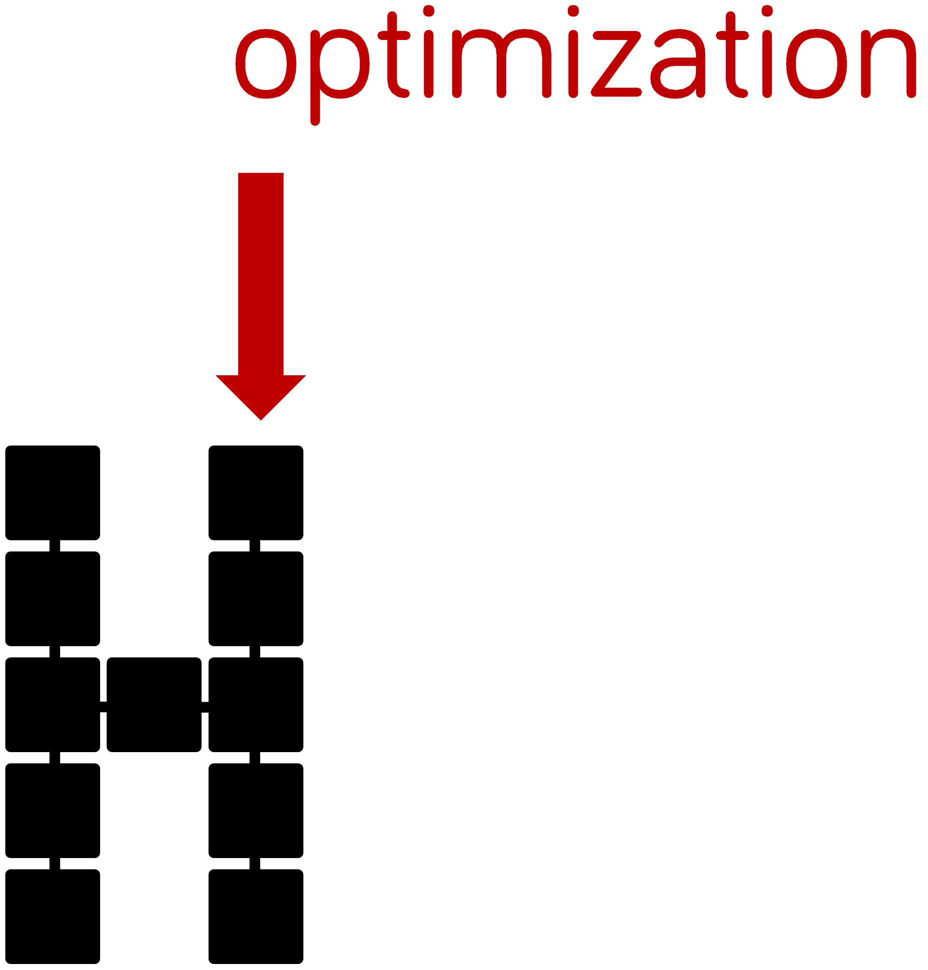


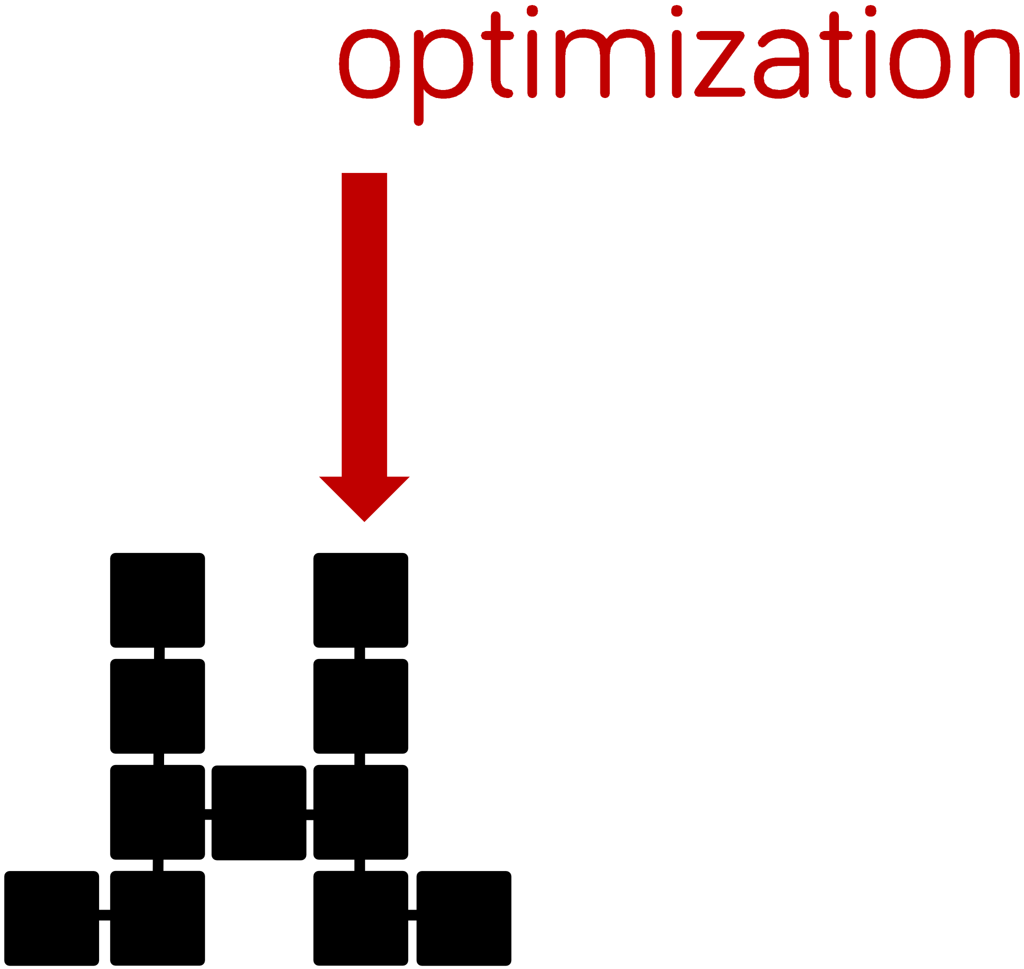


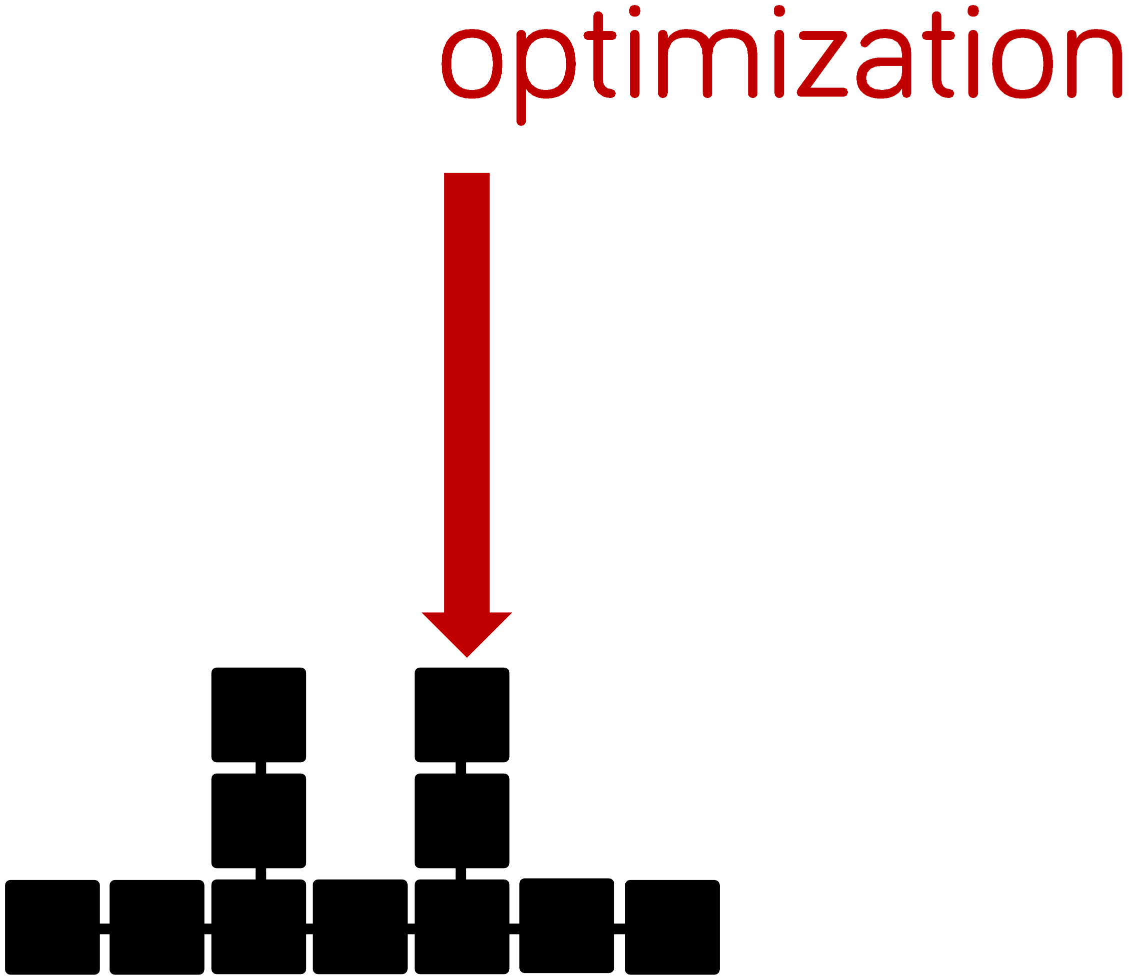
[Childs, Gosset, Webb, Science 339, 791 (2013)]

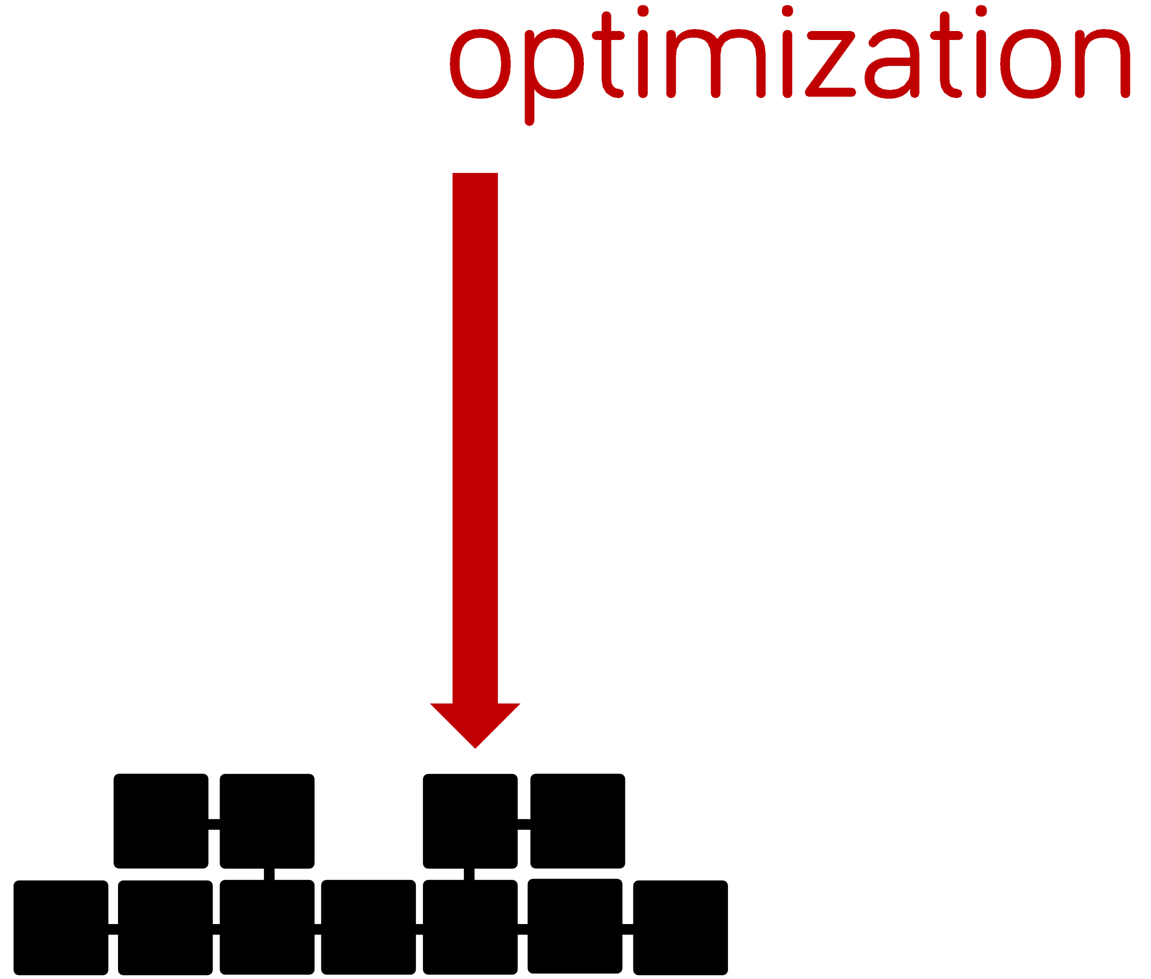
optimization




optimization

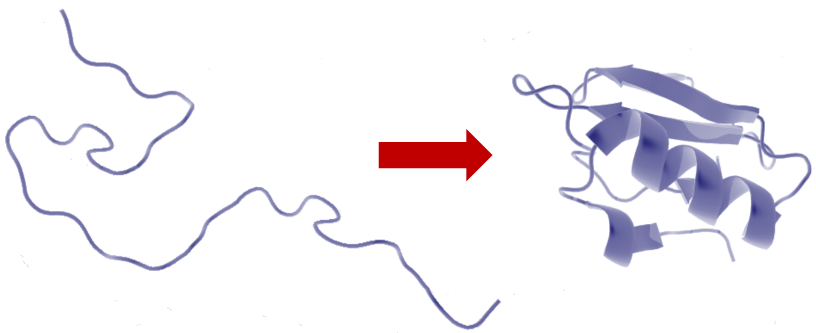

optimization


optimization

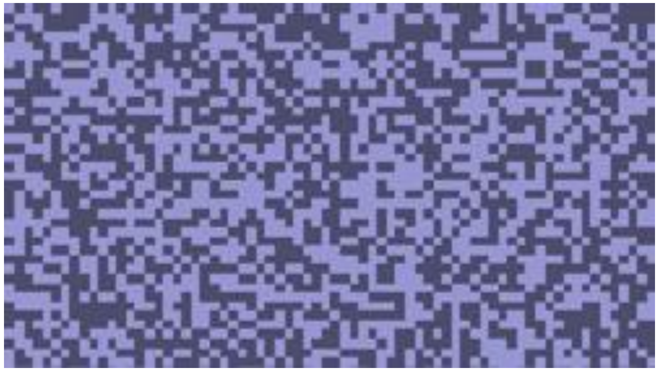

optimization

protein folding

spin glasses



[wikipedia]



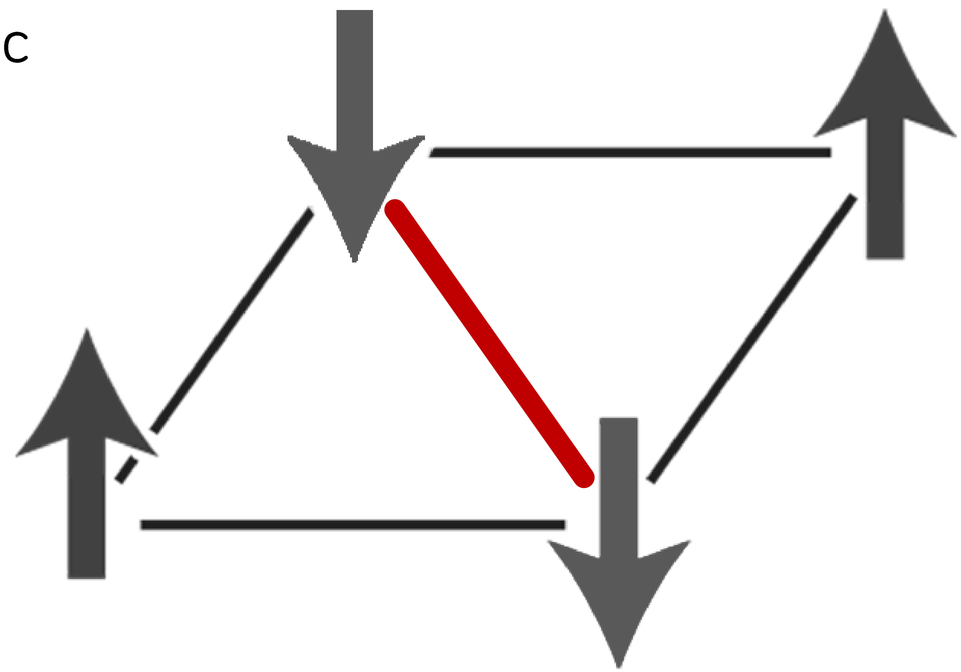
[uni-koeln.de]

local Hamiltonians

0 Frustrated systems

You can't make everybody happy.

antiferromagnetic
spin glass



a global
ground state

HARD?

find & describe it?
is it entangled?


frustrated
FRUST
RATED

0 Local Hamiltonians

Many local questions.

- What are they like?

  ground state (energy)

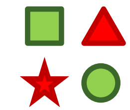
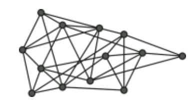
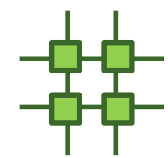
  QMA-complete problems

$$H(t) = \sum_j H_j(t)$$

- What are they good for?

  quantum computing, chemistry, control, transport

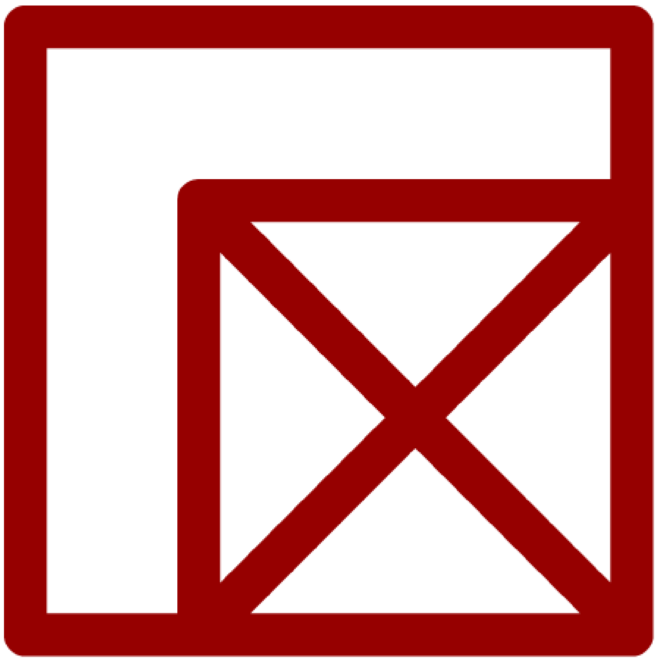
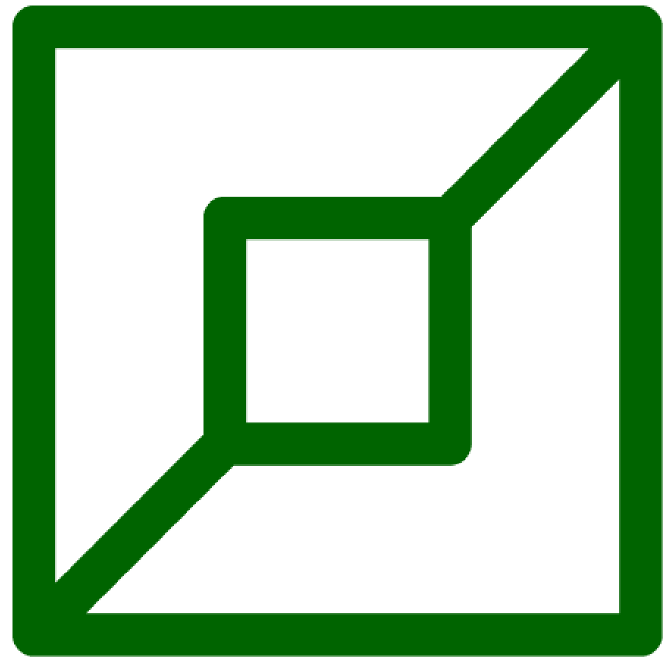
  BQP universality

- local particle dimension 
- interaction geometry 
- time independence 
- translational invariance 
- promise and eigenvalue gaps 
- energy × time cost 

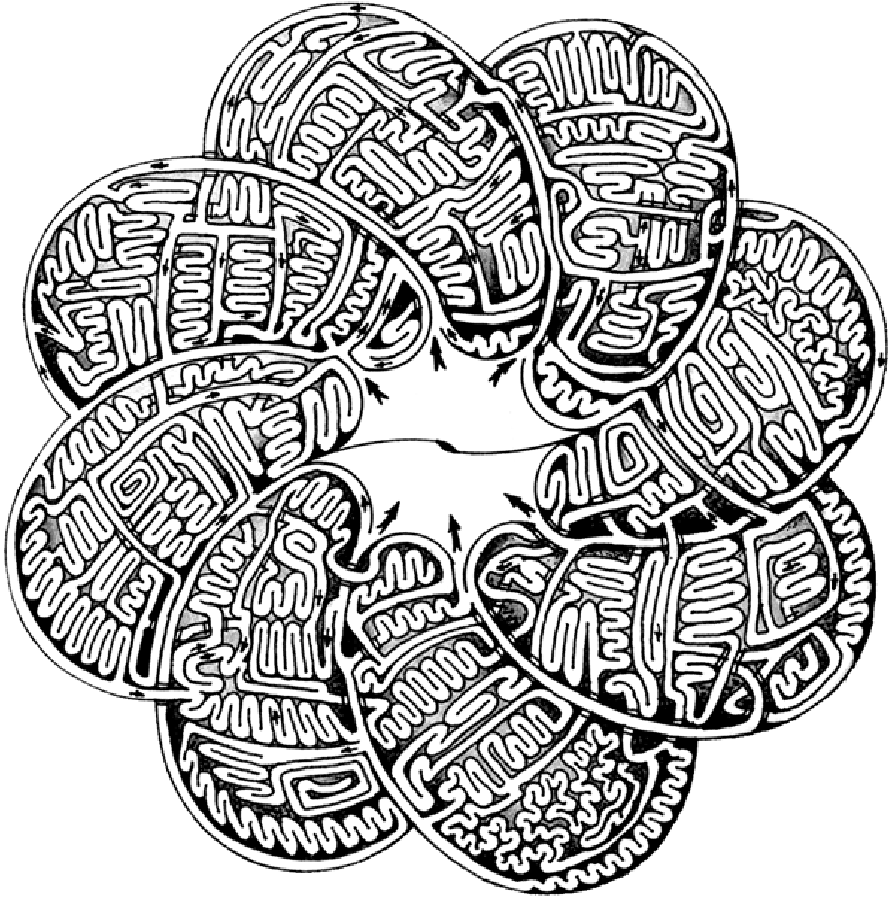
how hard
is this
question

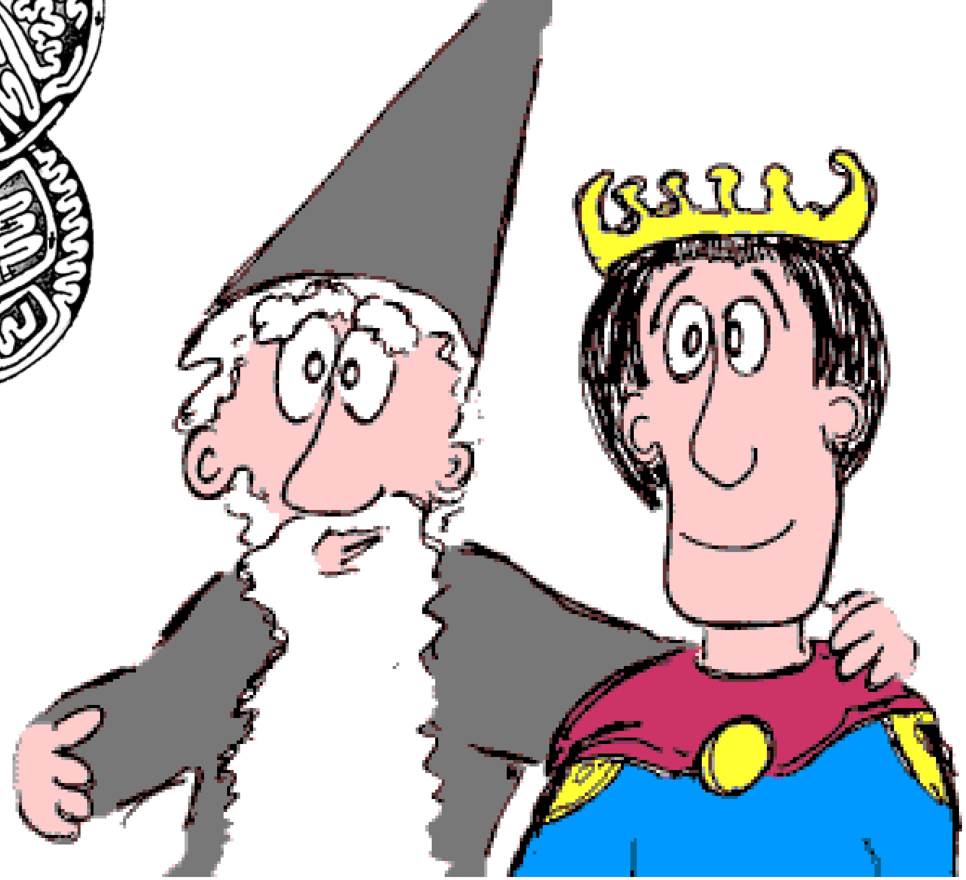
[J. Howard Miller]

P

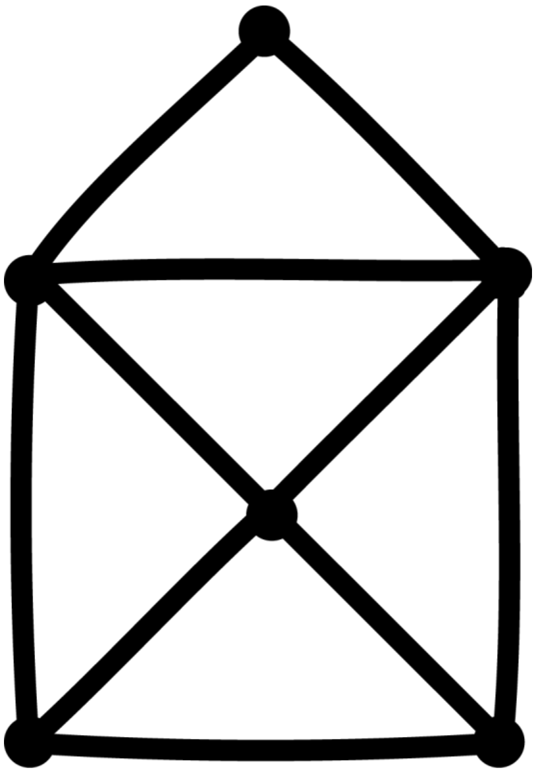
NP

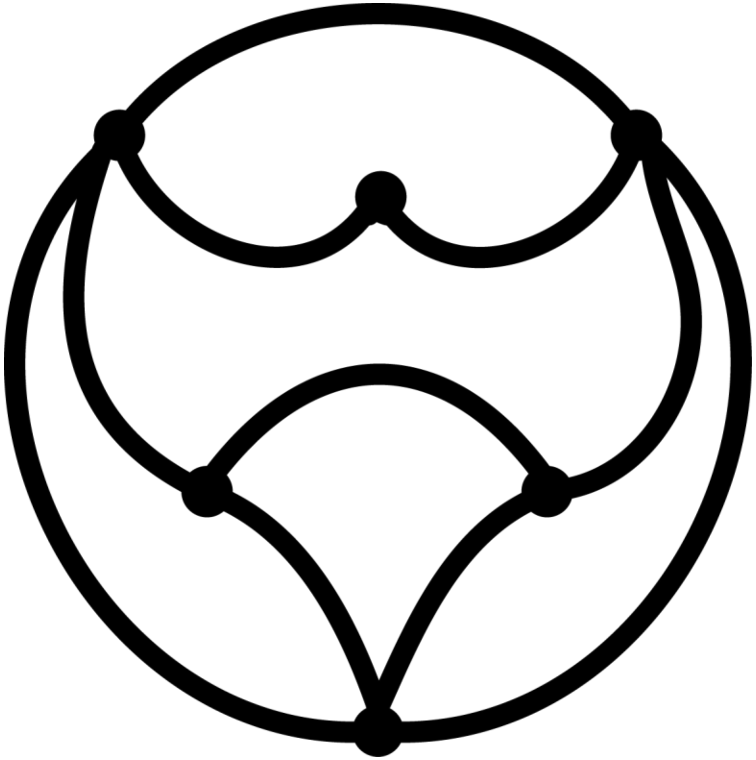




[maze: Andrew Bernhardt]
[A+M: primaryresources.co.uk]

1 A graph isomorphism puzzle





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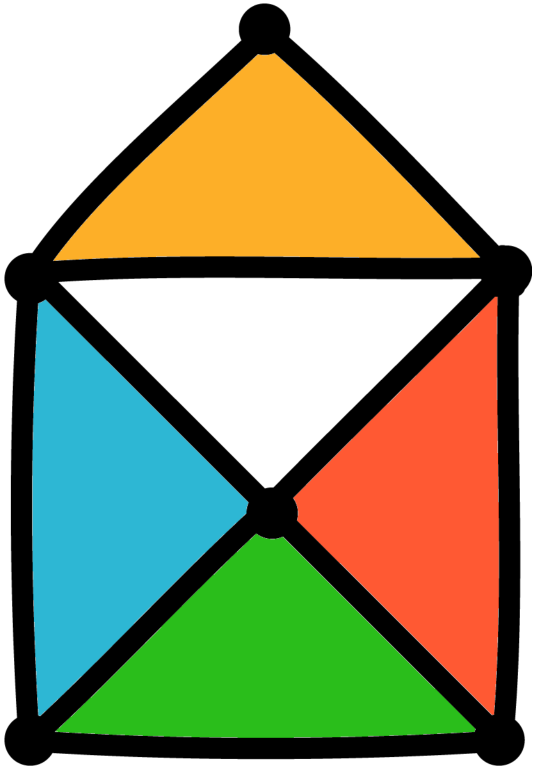


1 A cryptarithmetic puzzle

```
  DID
+DINOS
------
 CROAK
```

?

[www.iread.it/cryptarithms.php]

1 A cryptarithmetic puzzle

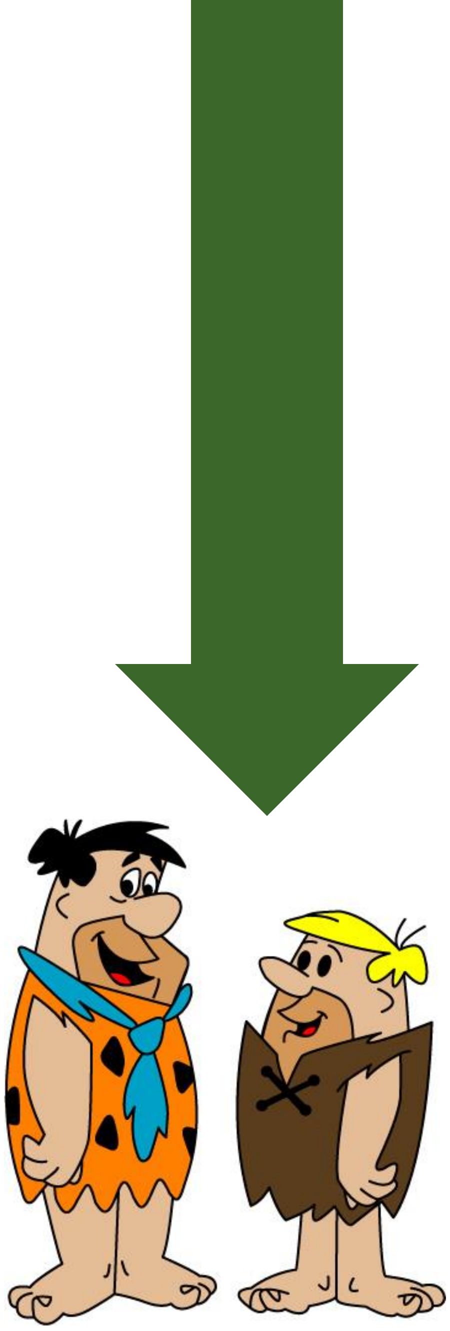
$$
\begin{array}{r}
595 \\
+\ 59842 \\
\hline
60437
\end{array}
$$

!

Did dinosaurs exist?

?

a proof



a witness

Yes/no questions, easy to verify solutions.

a verification circuit

from a uniform family

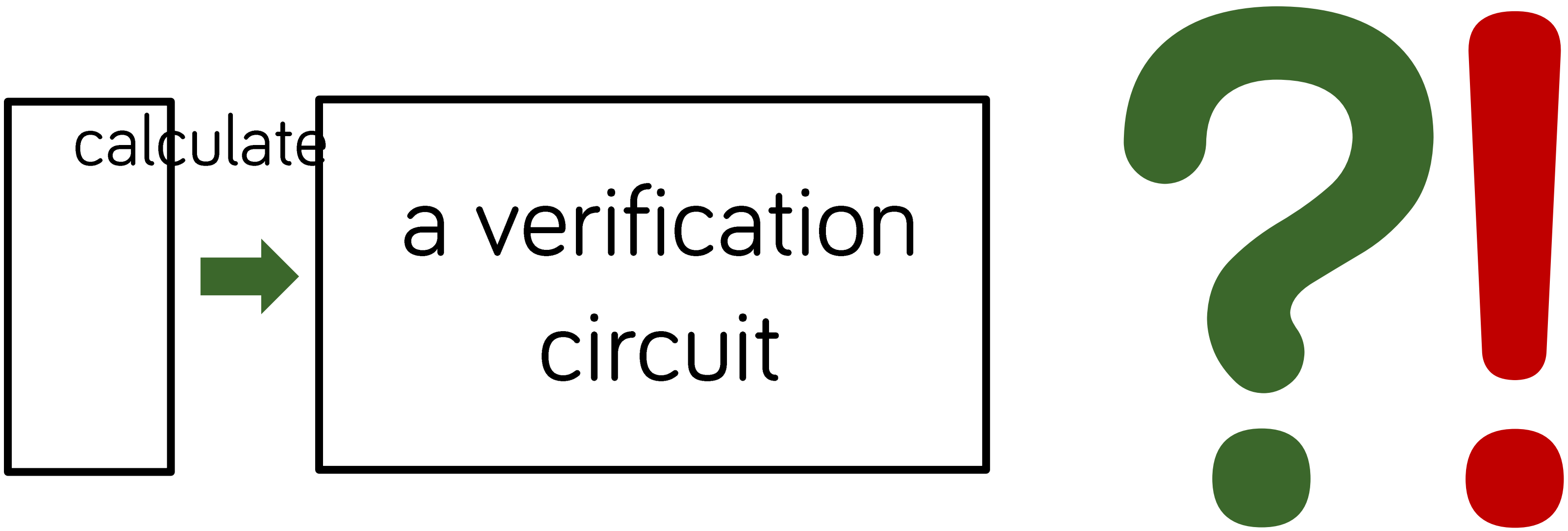
?!

YES? Accept a good proof.

NO? Reject any witness.

1 The class P

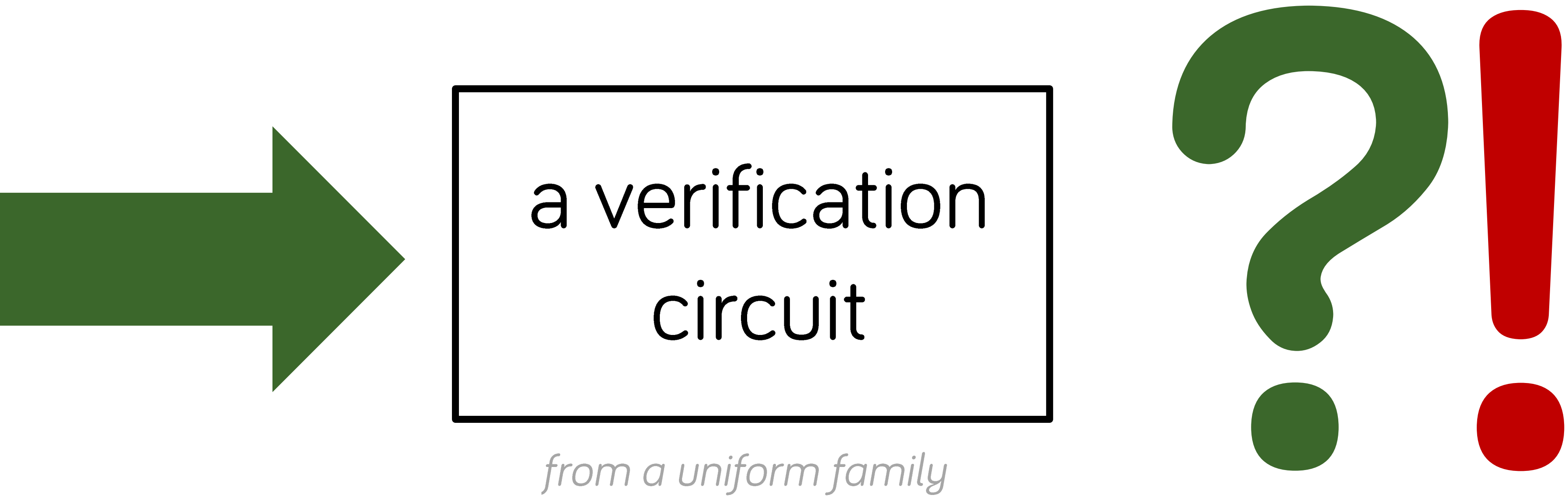
Yes/no questions that we can answer.



YES? Figure it out by yourself.

NO? Figure it out by yourself.

Yes/no questions, easy to verify solutions.

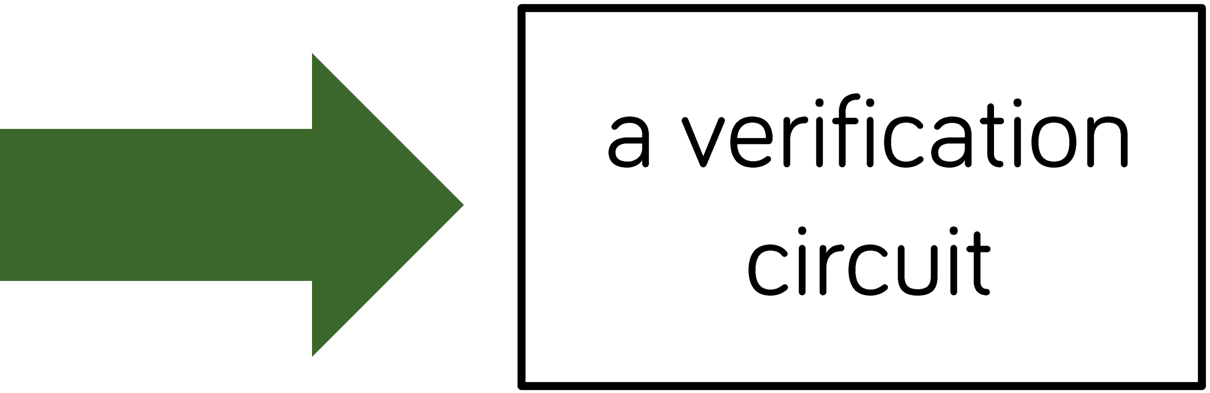


YES? Accept a good proof.

NO? Reject any witness.

1 An NP-hard problem

- An NP-hard problem solver solves anything in NP.



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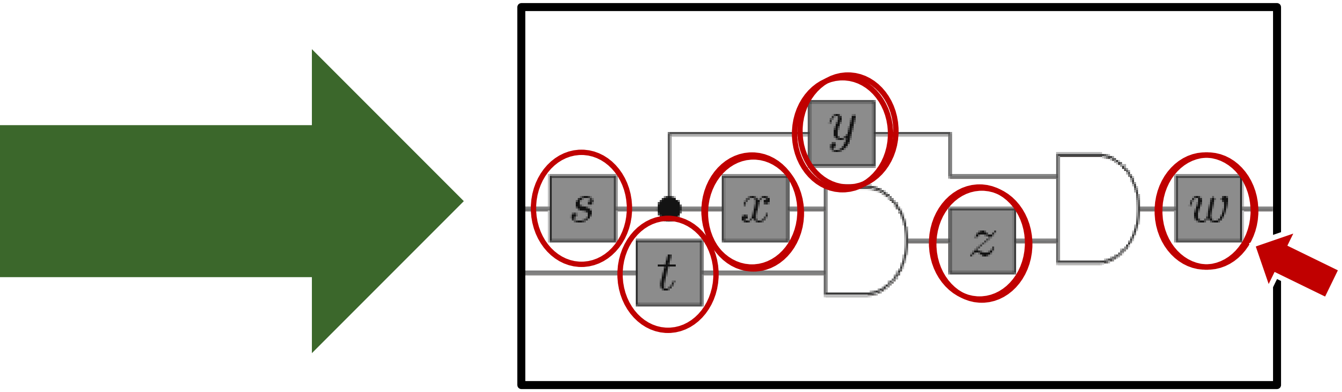
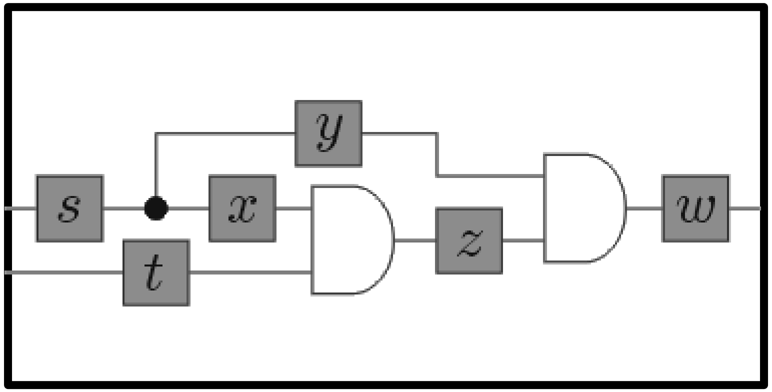


3-local conditions

$$(\cdots \vee \cdots \vee \cdots) \wedge (\cdots \vee \cdots \vee \cdots) \wedge \cdots$$

- 3-SAT is NP-hard.
- 3-SAT is in NP.

- An NP-hard problem solver solves anything in NP.

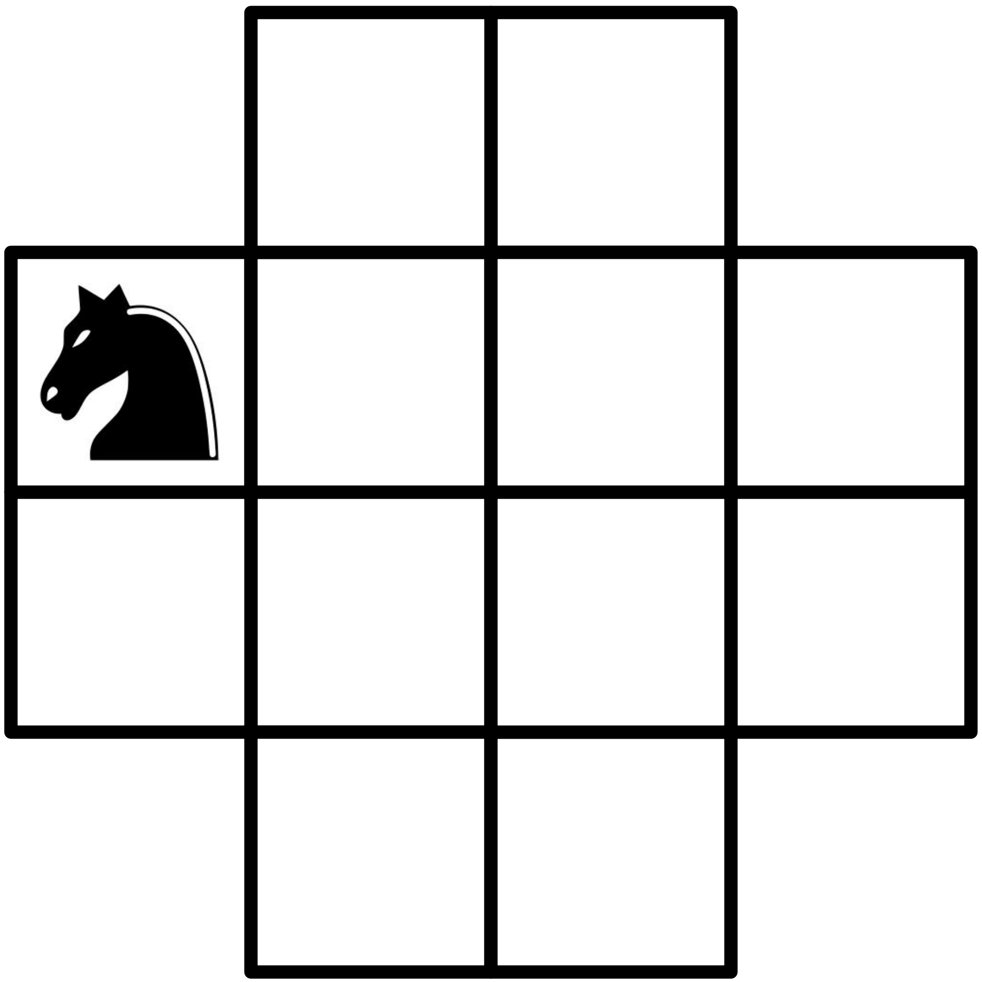


Could this circuit ever output 1?

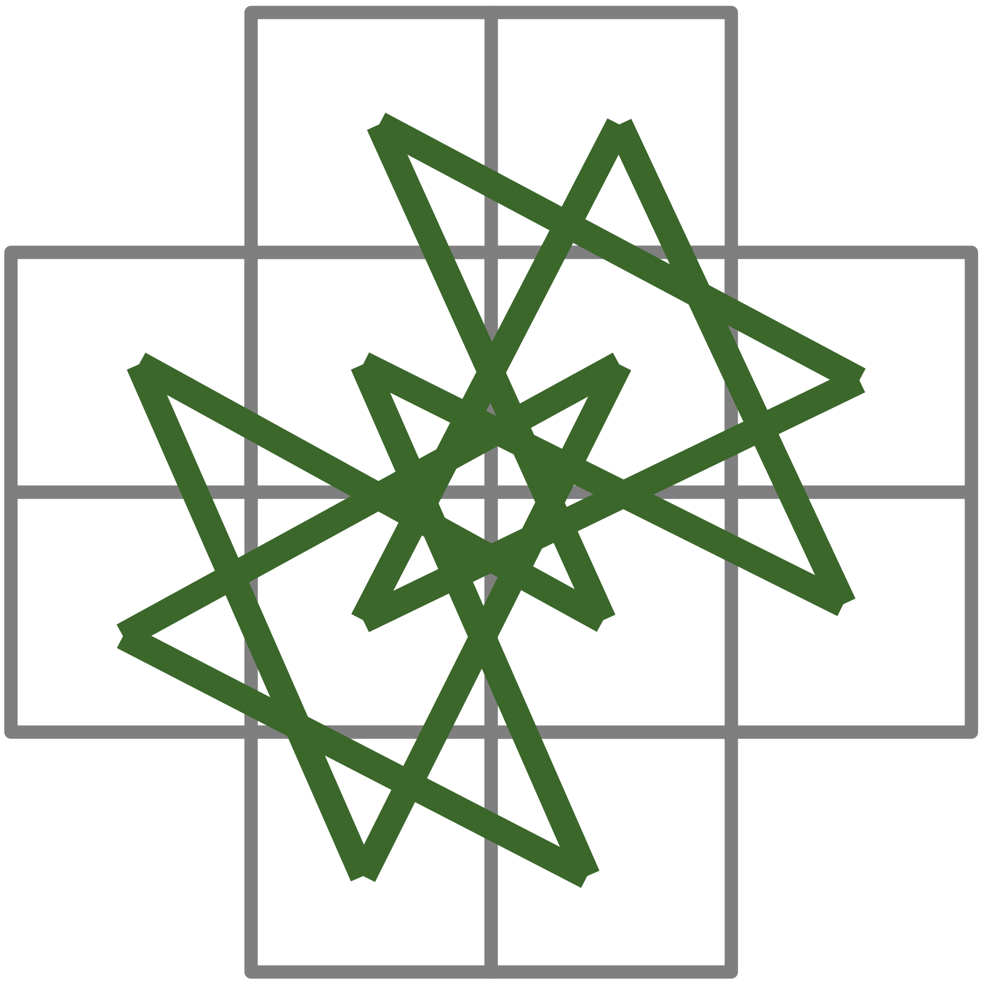
3-local conditions

- 3-SAT is NP-complete. [Cook, Levin]

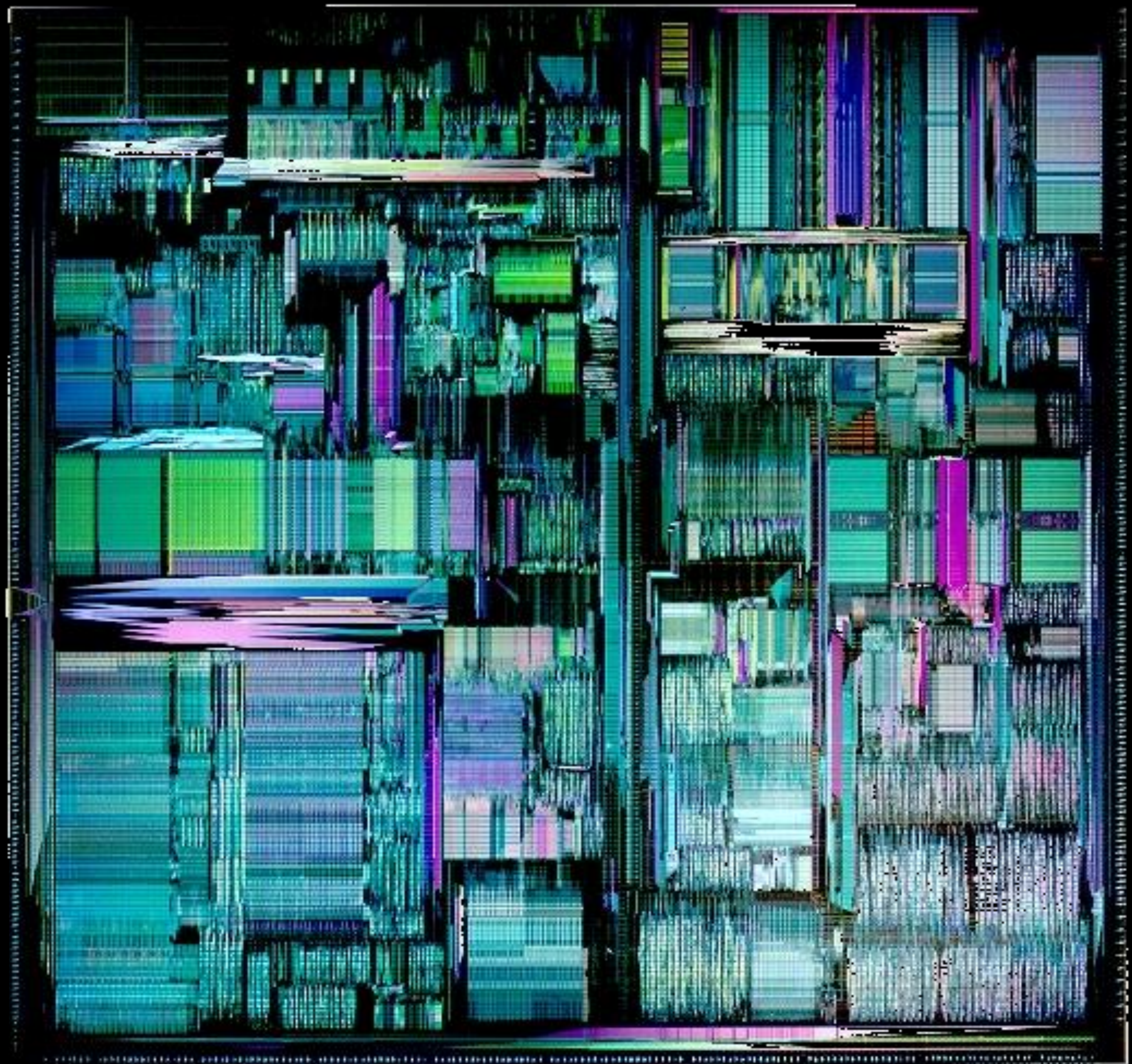
1 NP-complete problems: Hamiltonian cycle



1 NP-complete problems: Hamiltonian cycle



the puzzles of QMA



[1995 Pentium Pro

[wire

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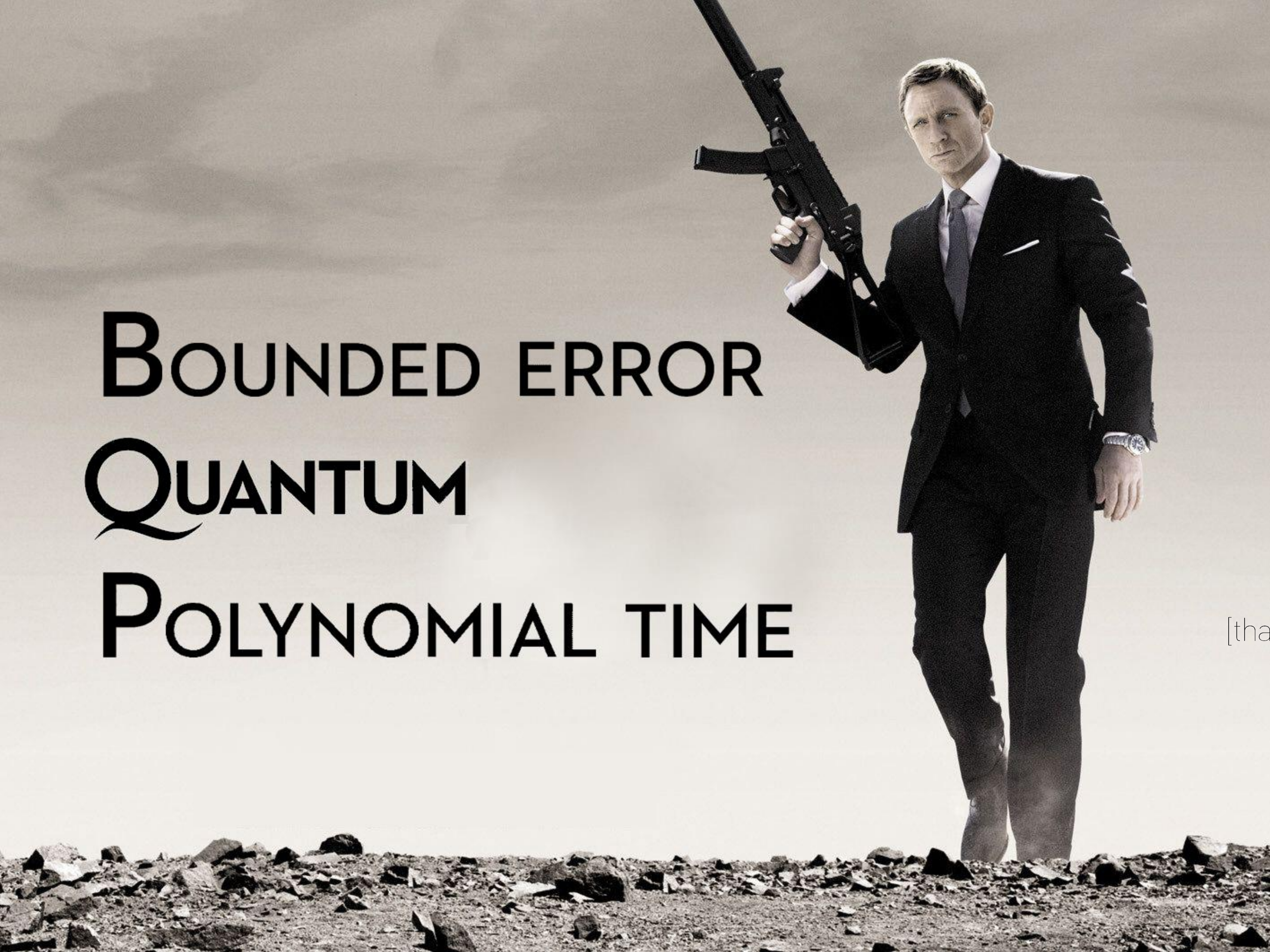
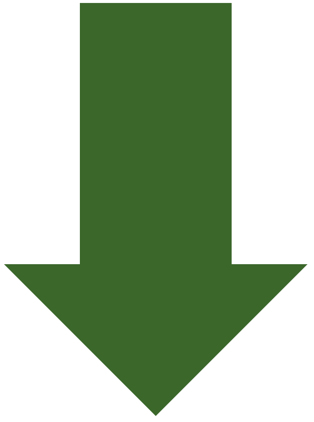
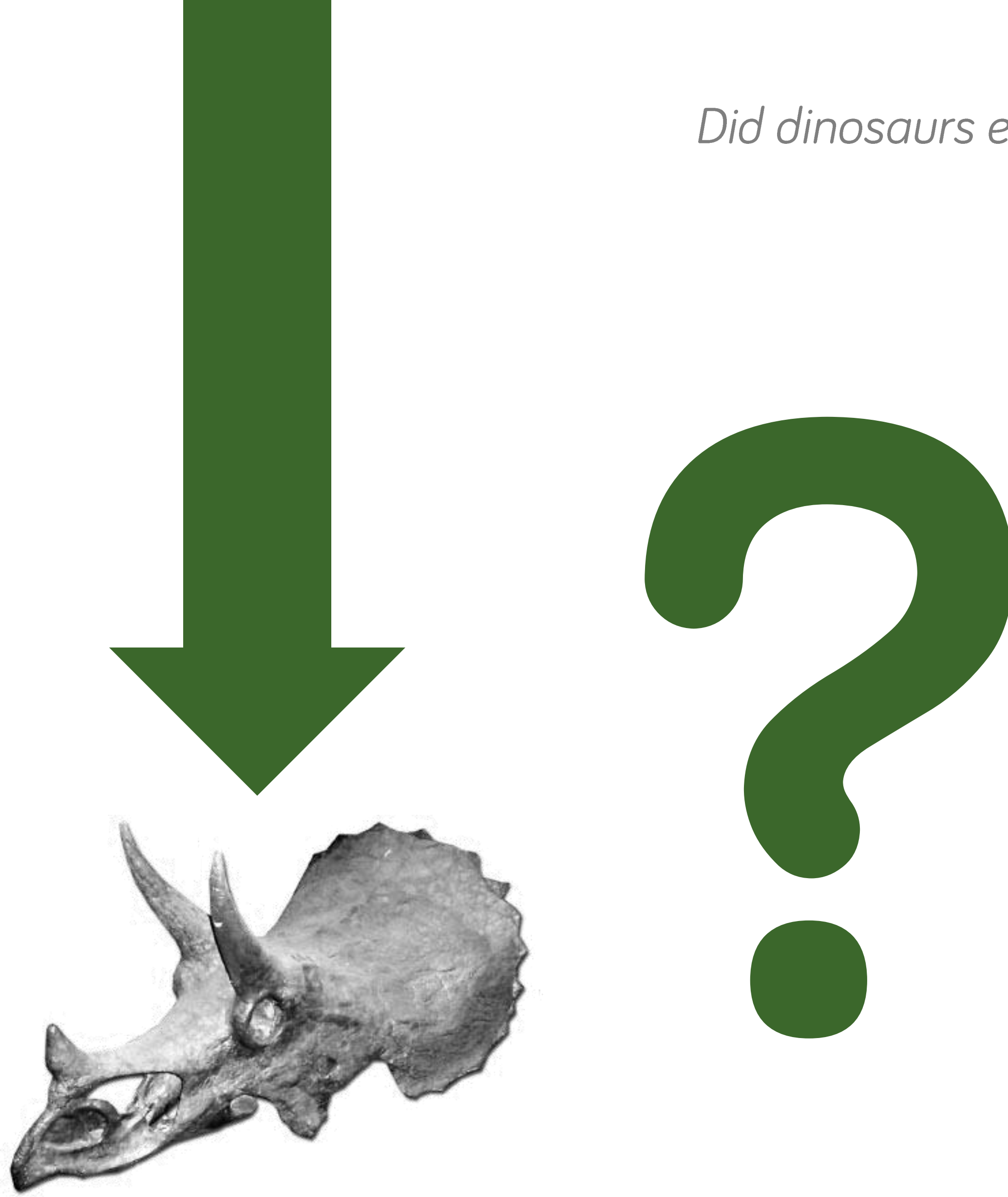

Bounded error
Quantum
Polynomial time
[tha

NP



2 The MA protocol

Did dinosaurs exist?



2 The MA protocol

Did dinosaurs exist?

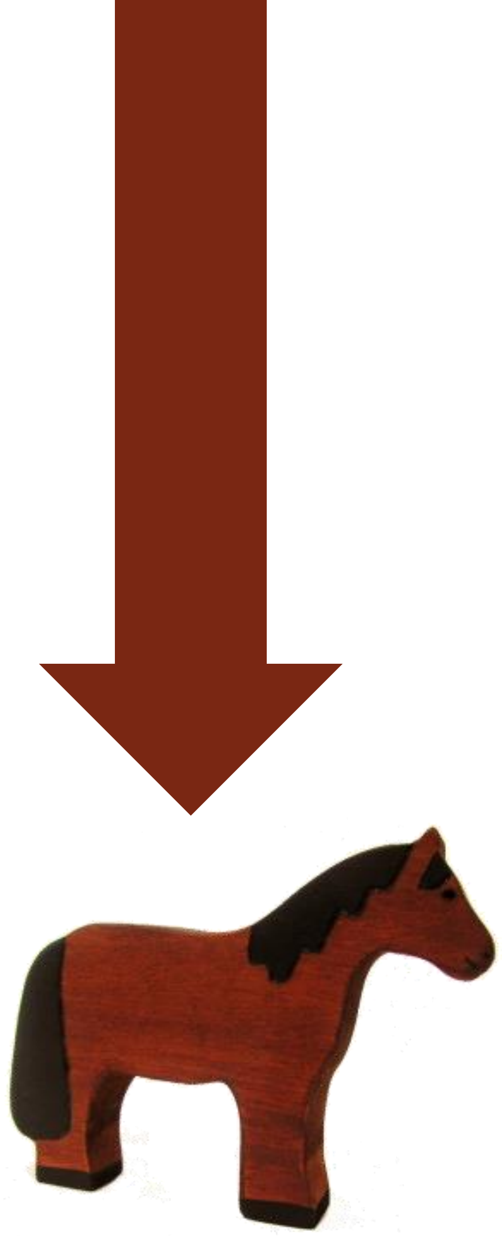
[wooden animals: Imagination Kids Toys]

2 The MA protocol

Did dinosaurs exist?

YES?
Eager to be
convinced.

[magnifying glass: hillllall]



NO?
Don't be
fooled
easily.

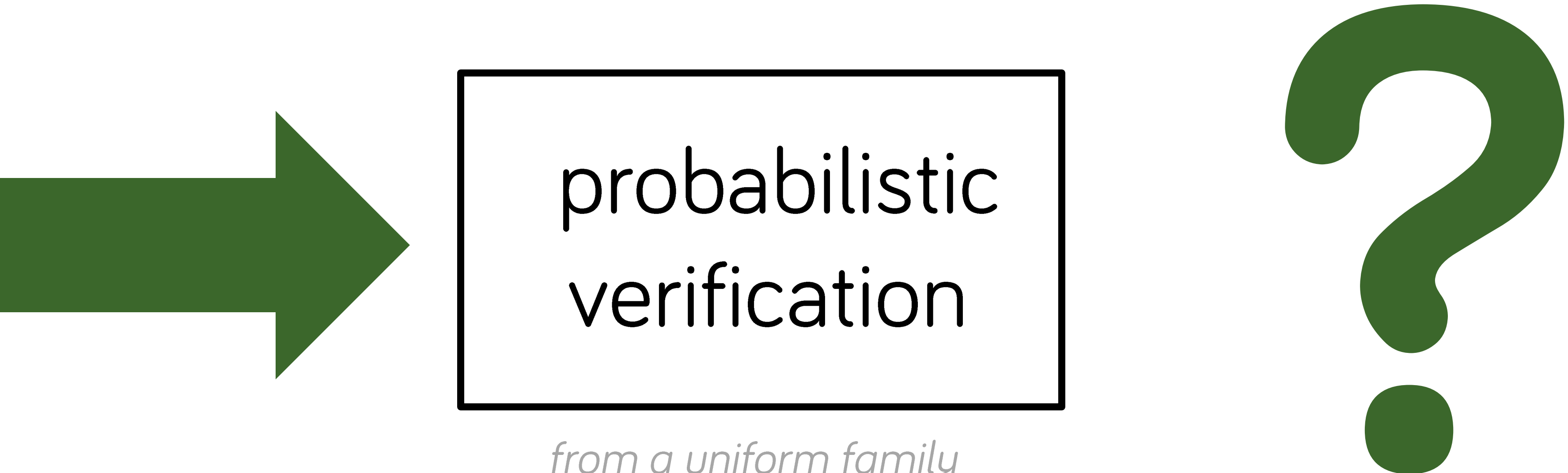
2 Probabilistic checks

Sometimes reject
a genuine proof?

Accept
a fake?

1 The MA protocol

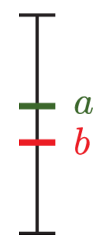
Probabilistic checks.



from a uniform family

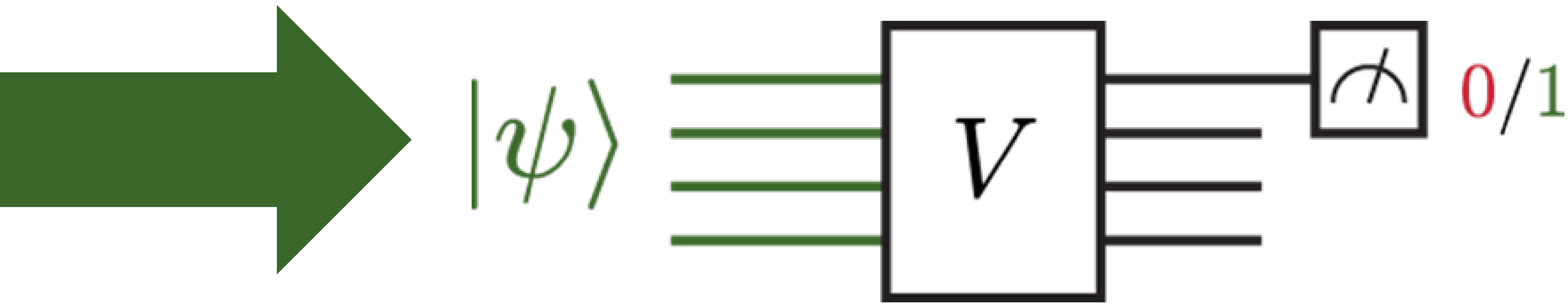
YES? Accept a good proof with p > a.

NO? Probability of accepting p < b.



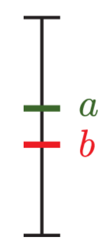
2 The QMA protocol

a quantum verifier



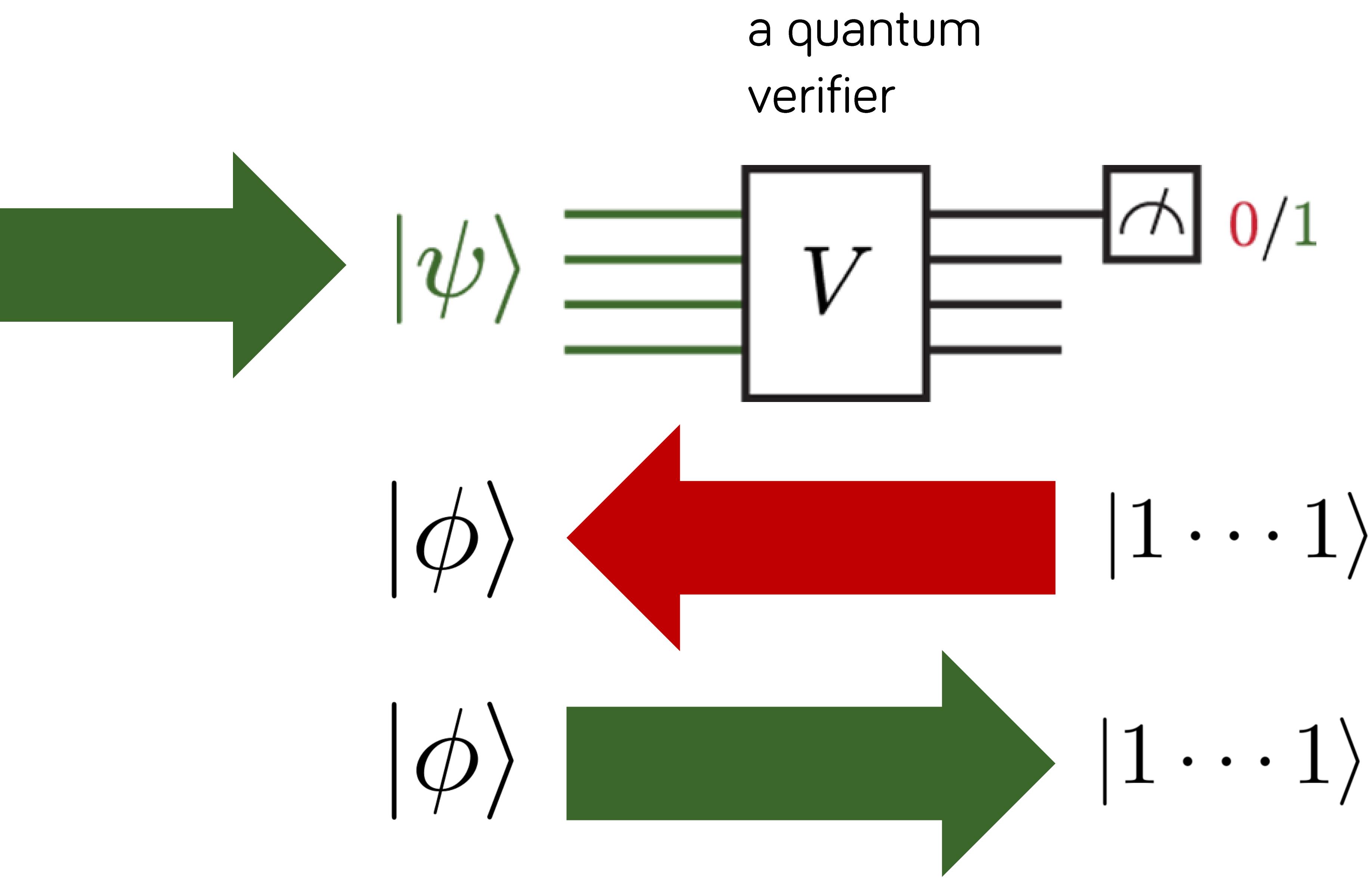
YES? Accept a good proof with $p > a$.

NO? Probability of accepting $p < b$.



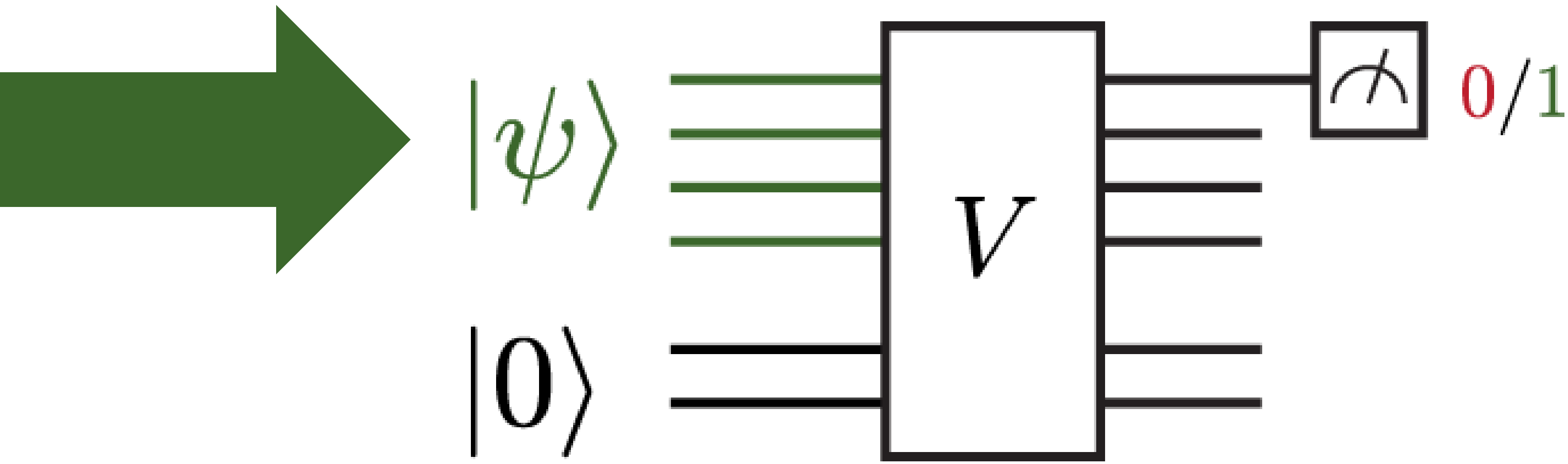
2 The QMA protocol

This is too simple.



2 The QMA protocol

Ancillas are necessary.



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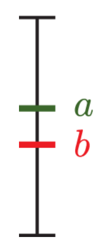
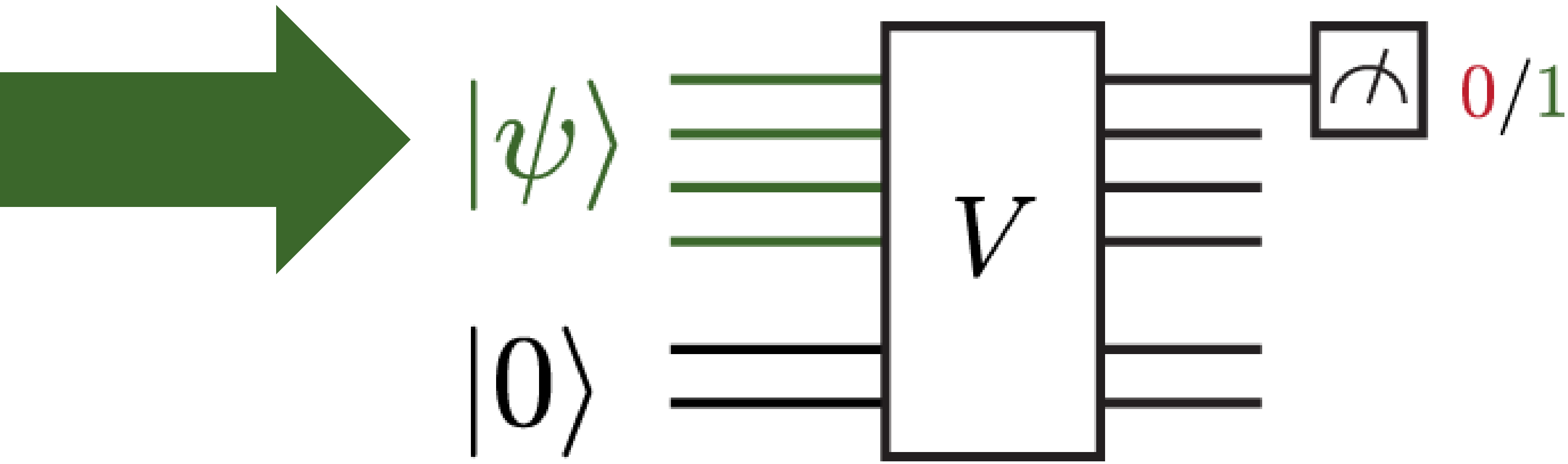


2 A QMA-hard question



Could we feed this quantum verifier something that likely outputs 1?

2 Implementing reversible quantum circuits

- The Schrödinger equation

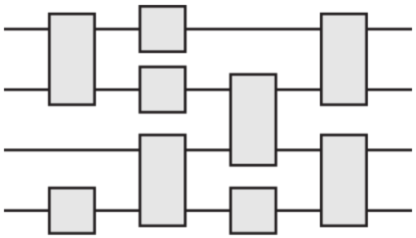
  a Hamiltonian *H*
  $$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

  made of local terms
  $$H(t) = \sum_j H_j(t)$$

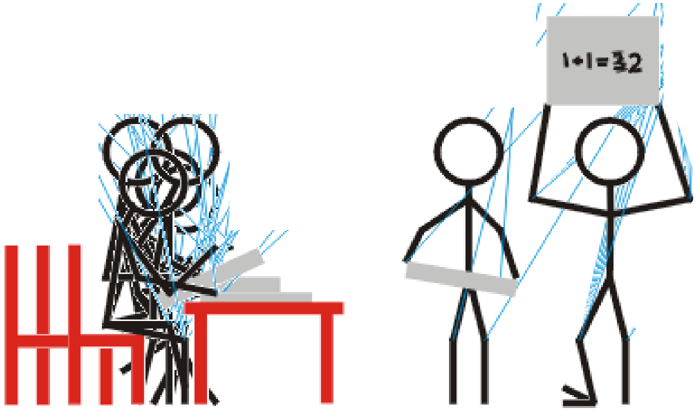
  unitary time evolution
  $$|\psi(t)\rangle = U_{t,0}|\psi(0)\rangle$$

- Implementing a circuit U

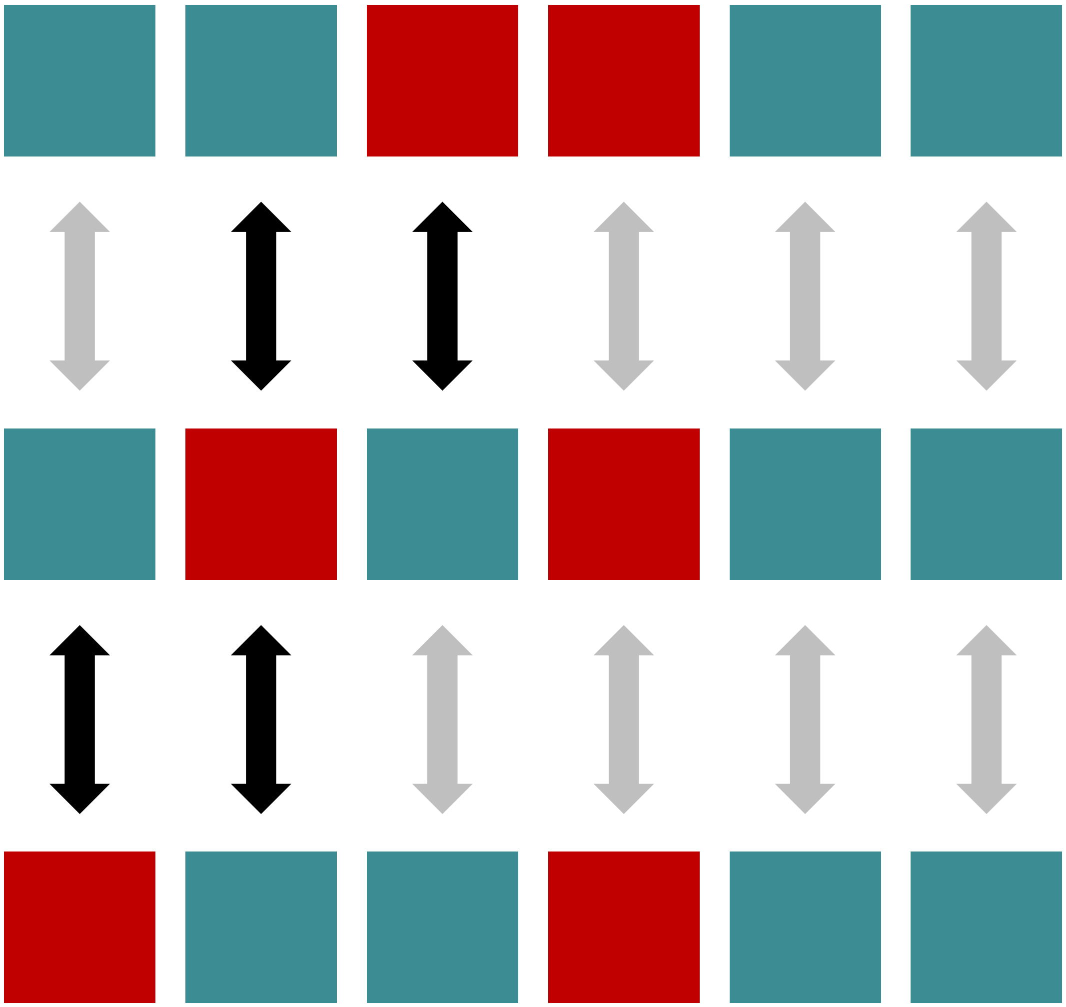
  how to make a system compute?



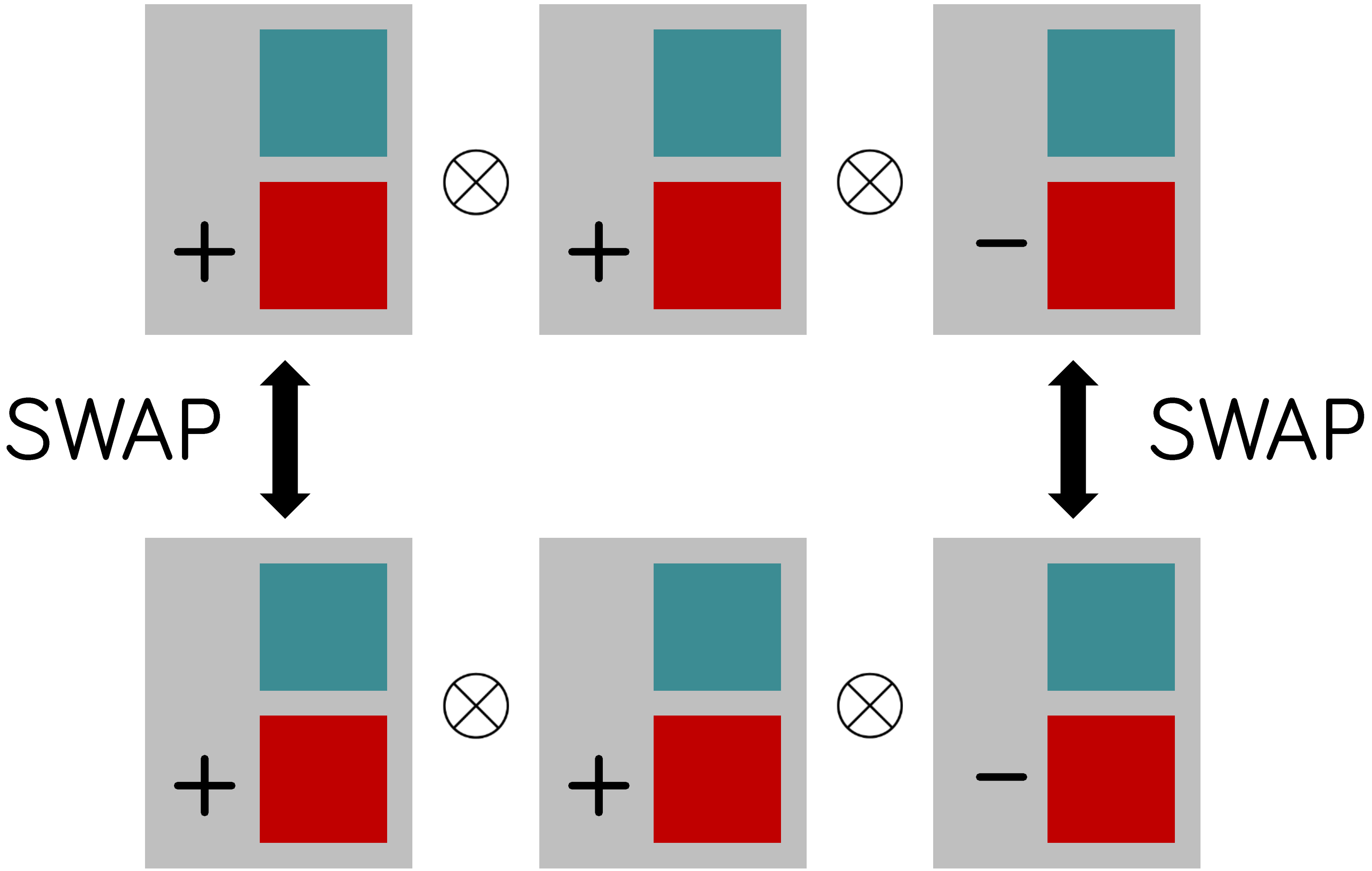
2 Snapshots of a computation



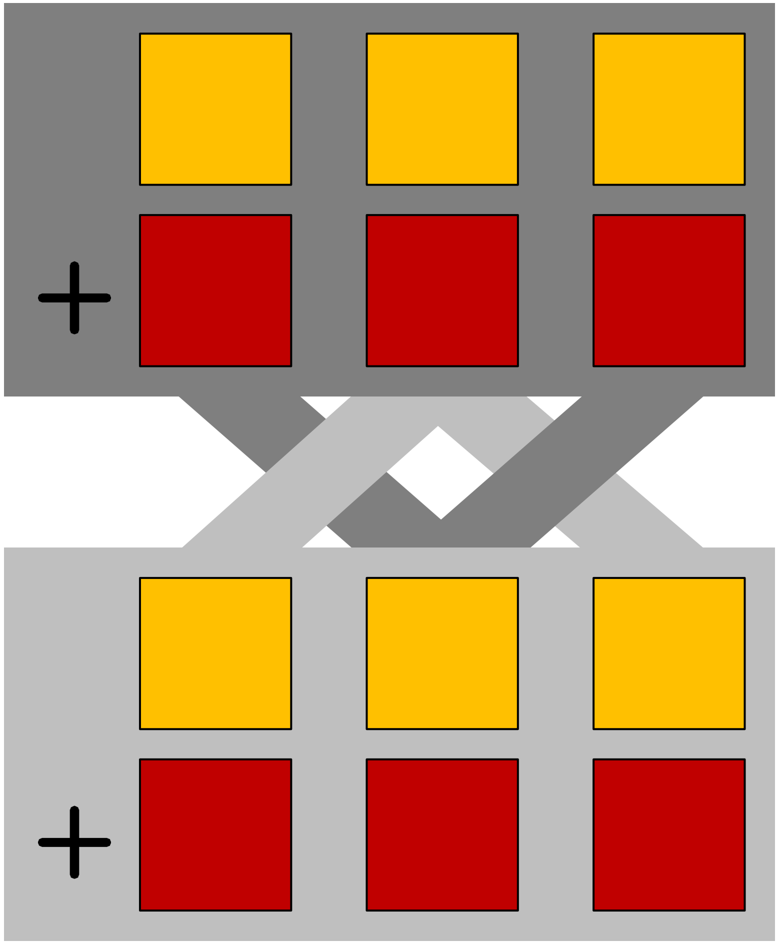
Locally comparing strings.



Locally comparing product states: SWAP.

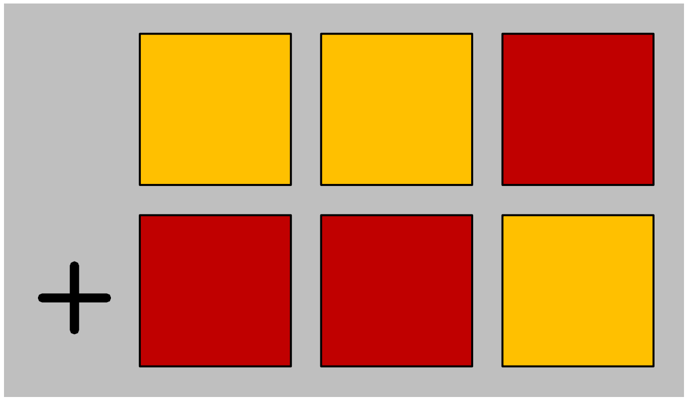


Locally comparing entangled states?

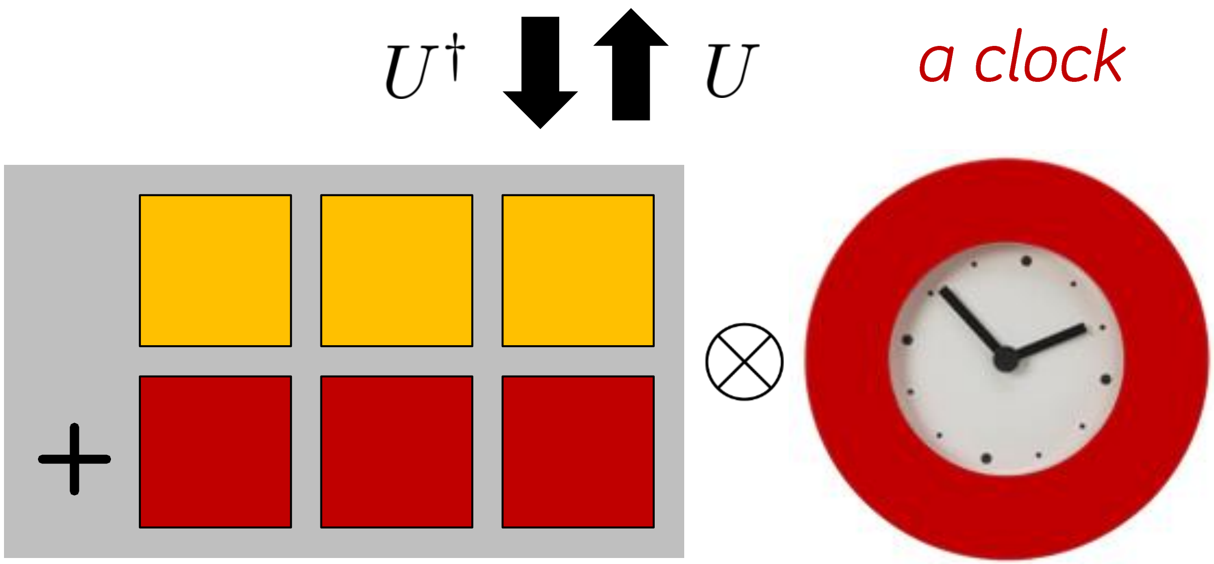


UGH!

2 The data & the clock

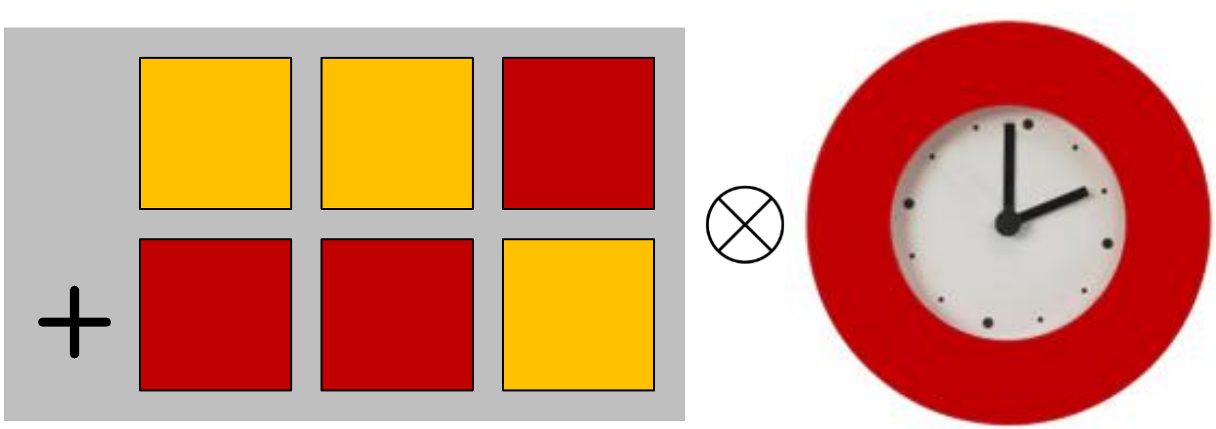


Hard to compare directly (locally).



2 The data & the clock

a clock



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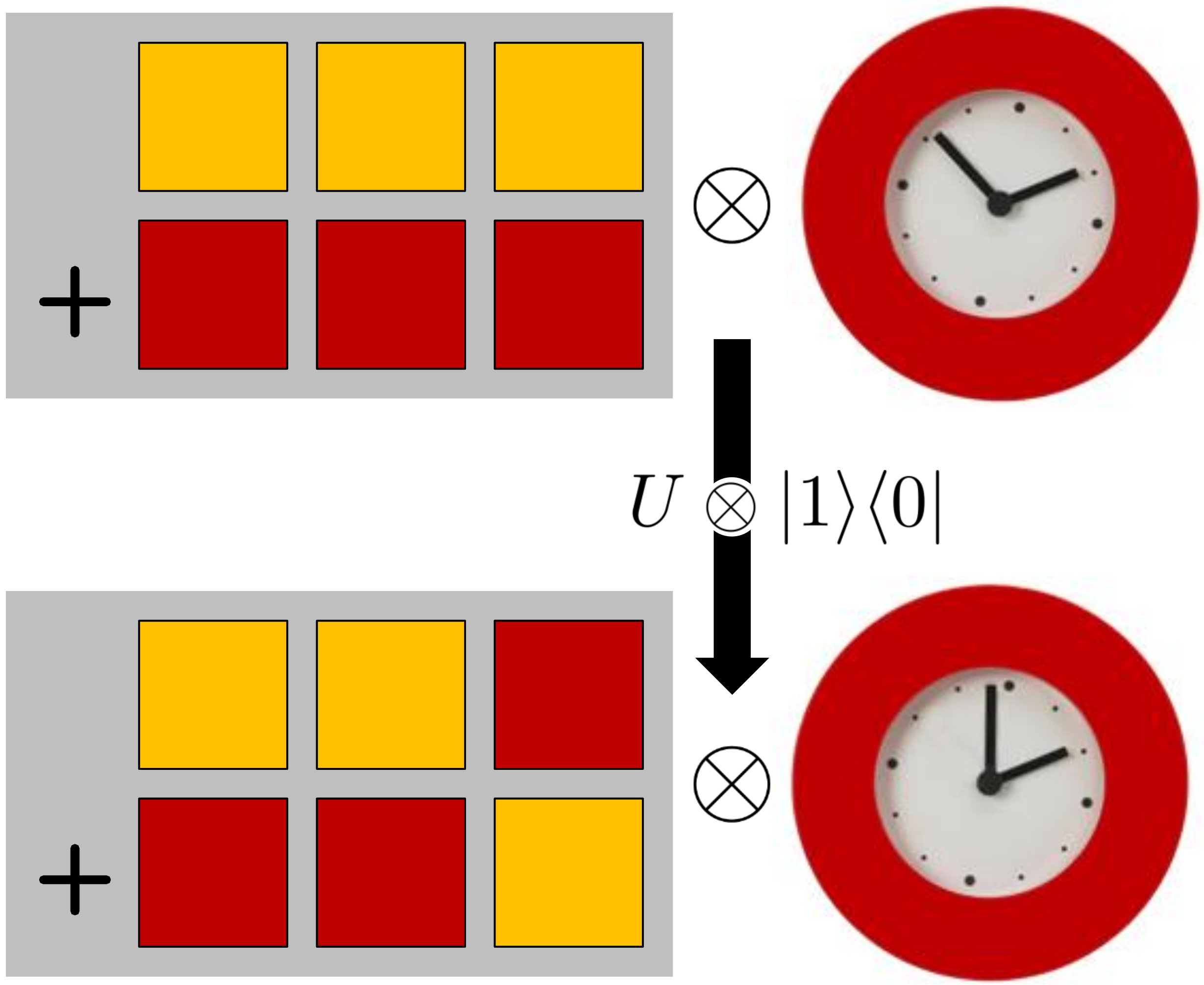

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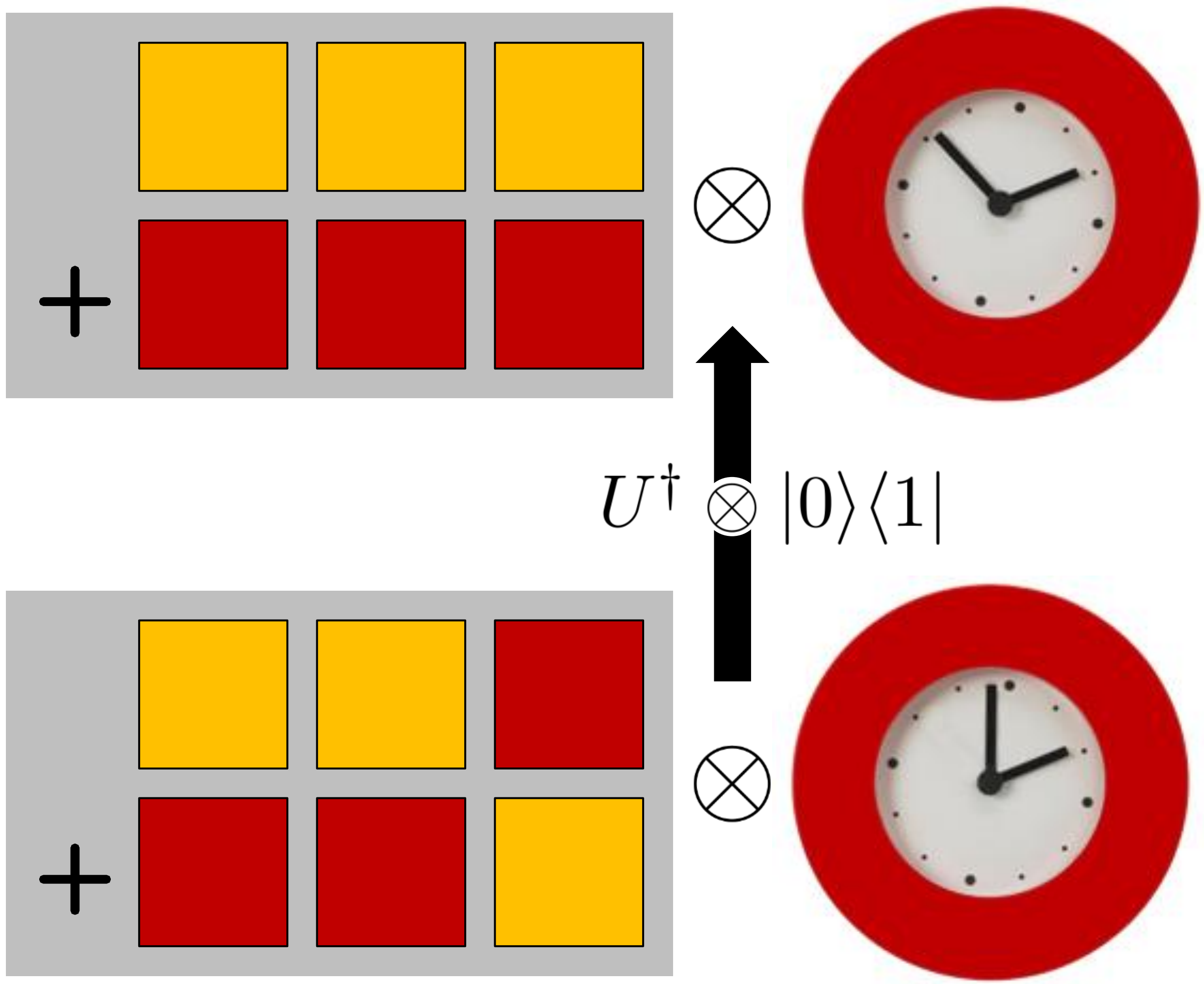

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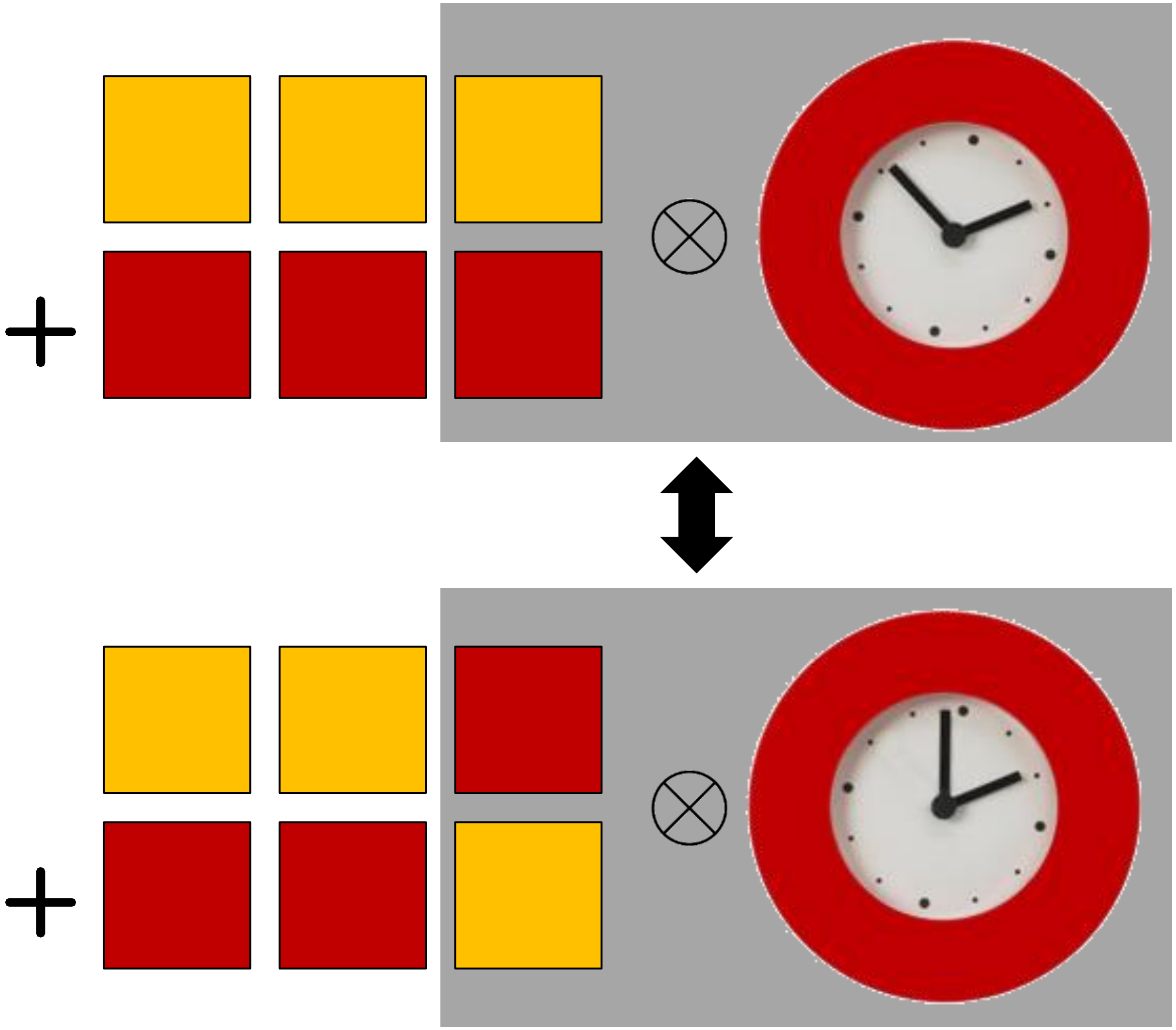
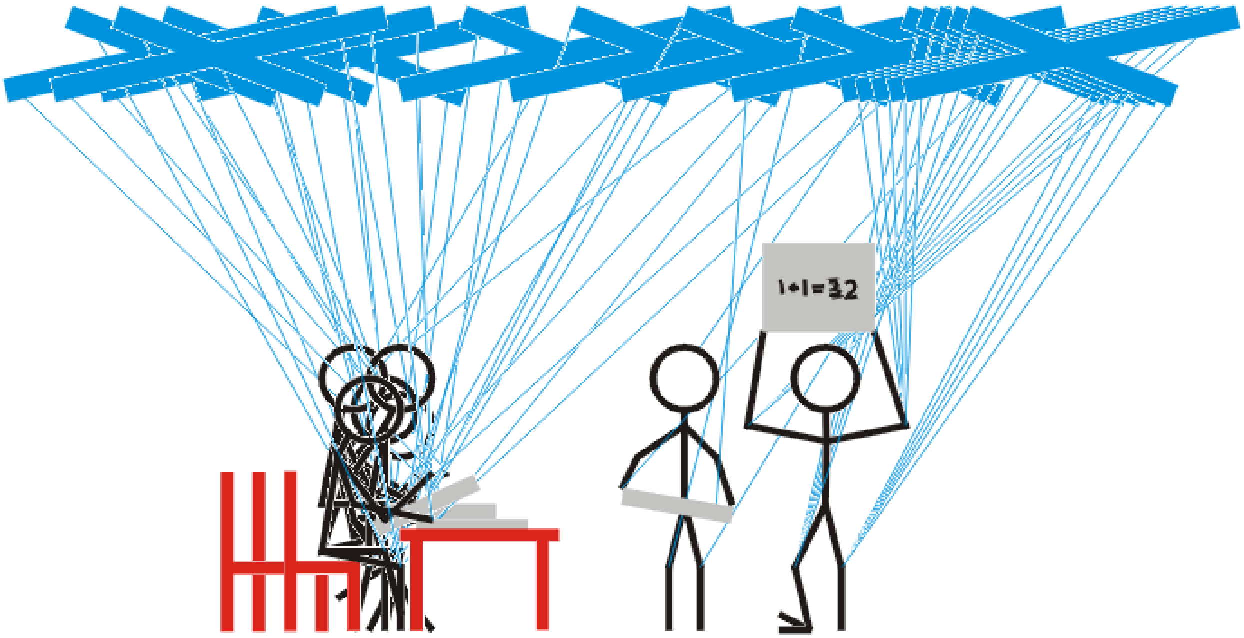
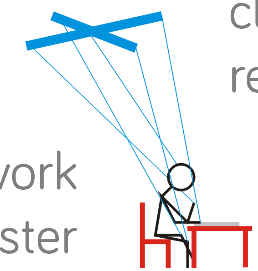


2 Snapshots of a computation & a "clock"



2 Feynman's (Hamiltonian) computer

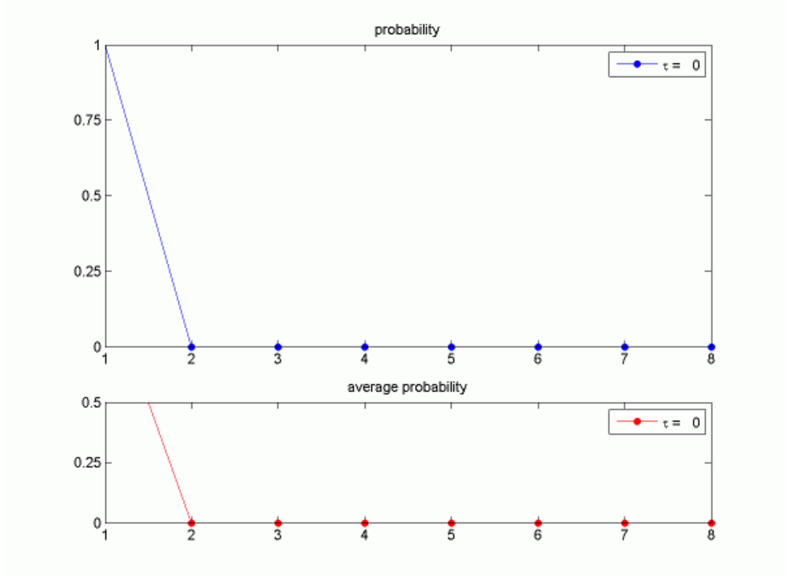


- The Hamiltonian

$$H_F = -\sum_{t=1}^{L} \left(U_t \otimes |t\rangle \langle t-1| + U_t^\dagger \otimes |t-1\rangle \langle t| \right)$$

- A quantum walk on a "line"

$$|\varphi_0\rangle \otimes |0\rangle_c$$
$$U_1 |\varphi_0\rangle \otimes |1\rangle_c$$
$$U_2 U_1 |\varphi_0\rangle \otimes |2\rangle_c$$
$$U_3 U_2 U_1 |\varphi_0\rangle \otimes |3\rangle_c$$



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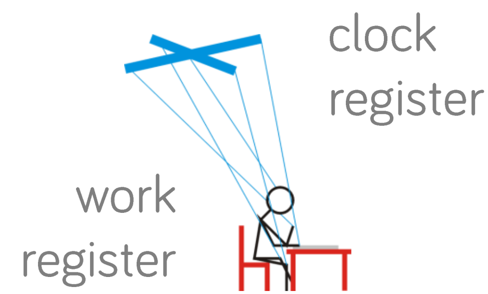

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$$|\varphi_0\rangle \otimes |0\rangle_c$$
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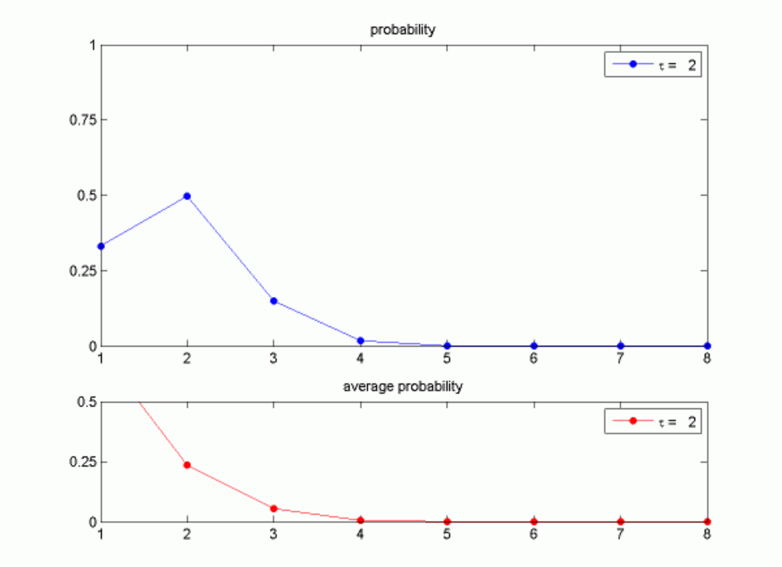
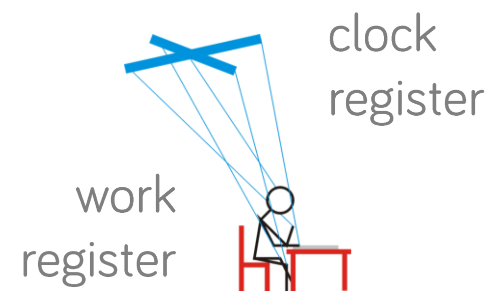
$$U_2 U_1 |\varphi_0\rangle \otimes |2\rangle_c$$
$$U_3 U_2 U_1 |\varphi_0\rangle \otimes |3\rangle_c$$



2 Feynman's (Hamiltonian) computer



- The Hamiltonian

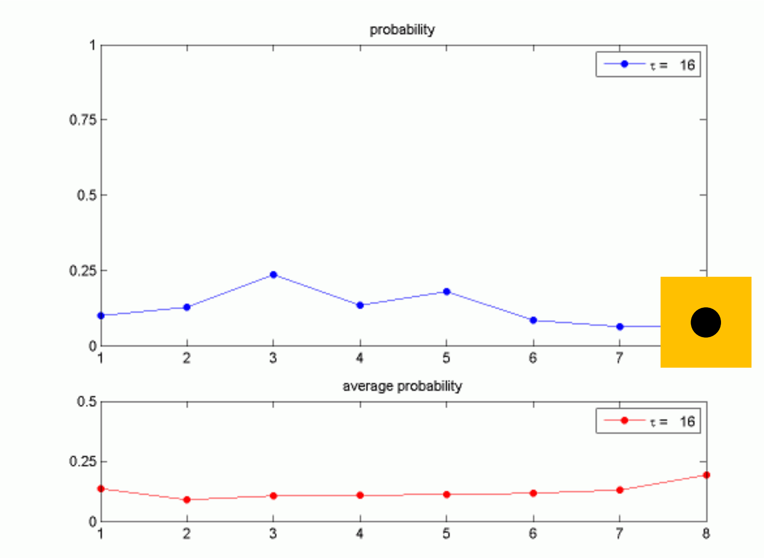
$$H_F = -\sum_{t=1}^{L} \left(U_t \otimes |t\rangle \langle t-1| + U_t^\dagger \otimes |t-1\rangle \langle t| \right)$$

- A quantum walk on a "line"

$$|\varphi_0\rangle \otimes |0\rangle_c$$

the output

$$U_L \cdots U_1 |\varphi_0\rangle \otimes |L\rangle$$



- Feynman's Hamiltonian

$$H_F = -\sum_{t=1}^{L} \left(U_t \otimes |t\rangle\langle t-1| + U_t^\dagger \otimes |t-1\rangle\langle t| \right)$$

- The "line" of states

$$\begin{array}{r} |\varphi_0\rangle \otimes |0\rangle_c \\ U_1 |\varphi_0\rangle \otimes |1\rangle_c \\ U_2 U_1 |\varphi_0\rangle \otimes |2\rangle_c \\ U_3 U_2 U_1 |\varphi_0\rangle \otimes |3\rangle_c \\ U_4 U_3 U_2 U_1 |\varphi_0\rangle \otimes |4\rangle_c \end{array}$$

$$H_F = - \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- The eigenvectors: combinations of plane waves

2 Hamiltonian QC

- Feynman's Hamiltonian

$$H_F = -\sum_{t=1}^{L} \left(U_t \otimes |t\rangle \langle t-1| + U_t^\dagger \otimes |t-1\rangle \langle t| \right)$$

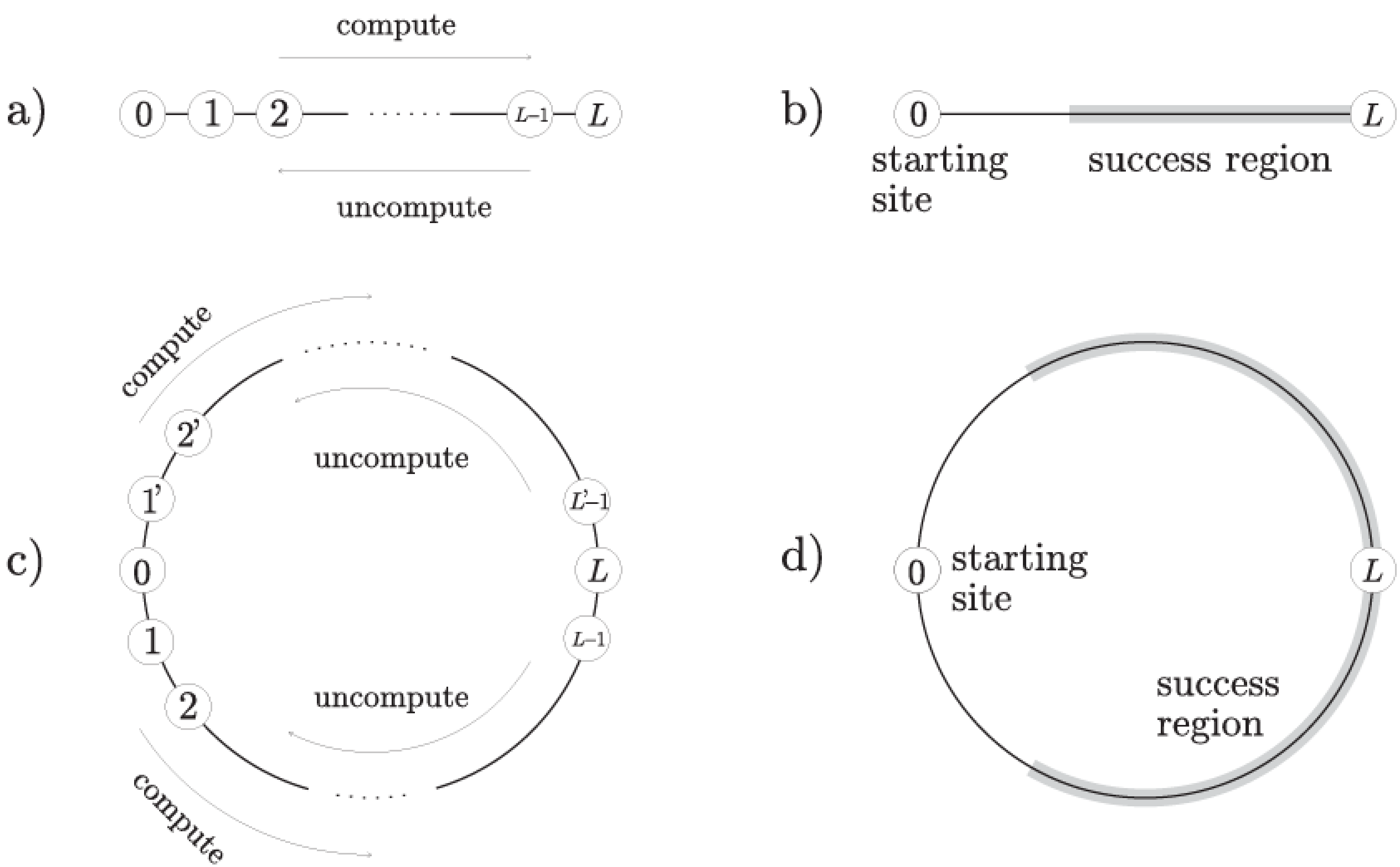
- The "line" of states

a possibility: wrap around a circle

$$\begin{aligned} |\varphi_0\rangle &\otimes |0\rangle_c \\ U_1 |\varphi_0\rangle &\otimes |1\rangle_c \\ U_2 U_1 |\varphi_0\rangle &\otimes |2\rangle_c \\ U_3 U_2 U_1 |\varphi_0\rangle &\otimes |3\rangle_c \\ U_4 U_3 U_2 U_1 |\varphi_0\rangle &\otimes |4\rangle_c \end{aligned}$$

$$H_F = -\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

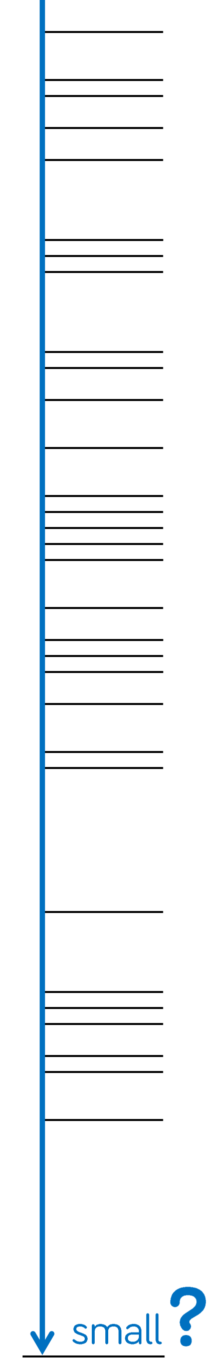
- The eigenvectors: combinations of plane waves


a)
compute
0
1
2
L–1
L
uncompute
b)
0
L
starting site
success region
c)
compute
uncompute
2'
1'
0
1
2
L'–1
L
L–1
uncompute
compute
d)
0
starting site
L
success region

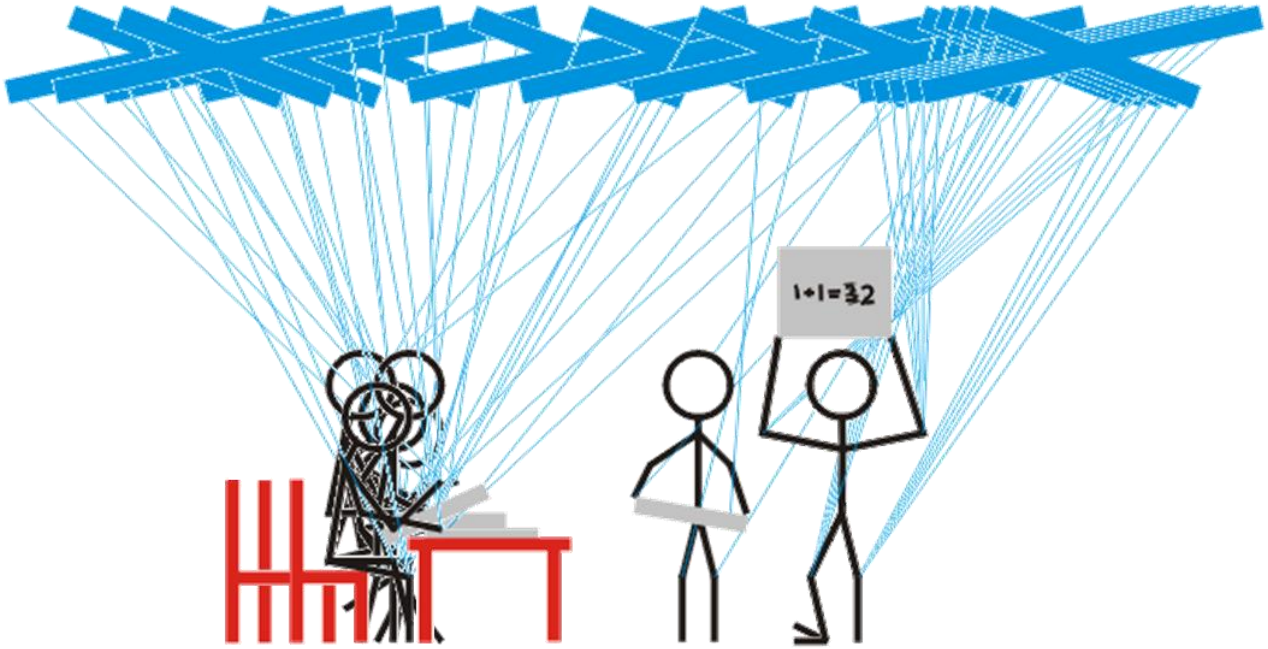
2 The Local Hamiltonian problem

Is
the
ground
state
energy
of a

LH



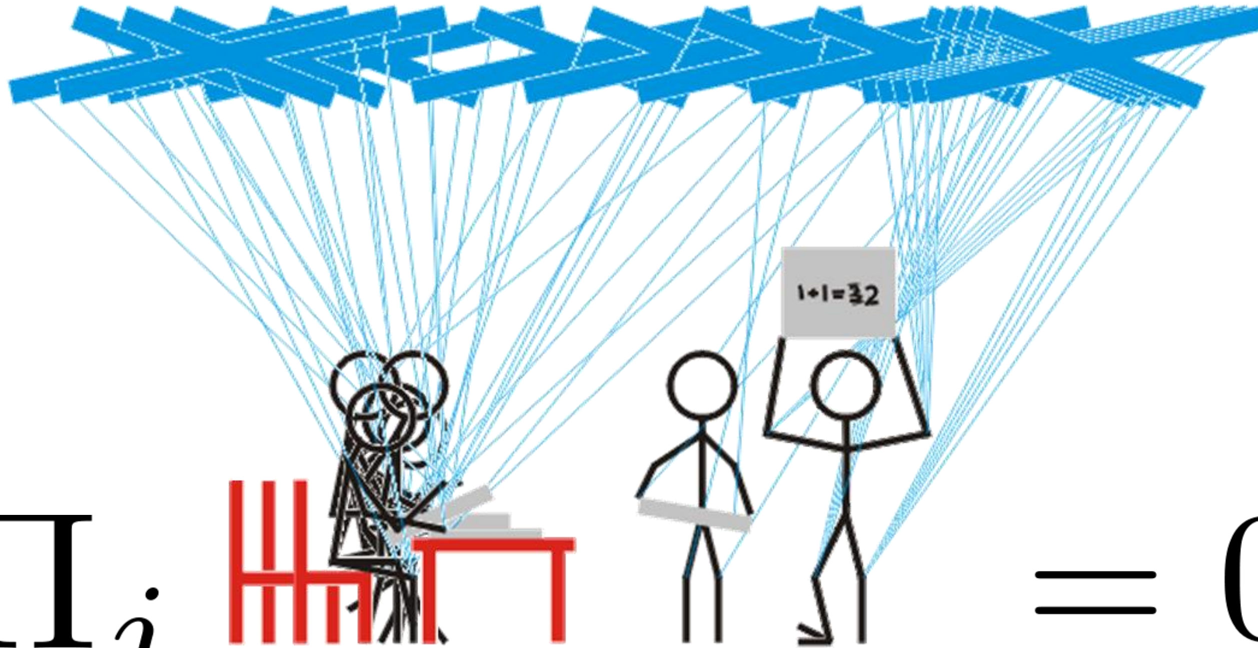
2 The history state



$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_t\rangle = U_t \cdots U_1 |\varphi_0\rangle$$

2 The history state: a ground state

Π_j  $= 0$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_t\rangle = U_t \cdots U_1 |\varphi_0\rangle$$

2 Do we have a history state?

k-local

c-o-n-d-i-t-i-o-n-s

clock encoding

state progression

$$|\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

2 Checking if a state is an accepted history

k-local

c-o-n-d-i-t-i-o-n-s

clock encoding

state progression

initialization

$$|\cdots 0\rangle \otimes |0\rangle$$

$$|\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

output

$$|\cdots 1\rangle \otimes |T\rangle$$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

a clock workshop

3 Run the clock, apply gates ...

$$|\varphi_{t-2}\rangle \otimes |t-2\rangle$$

$$|\varphi_{t-1}\rangle \otimes |t-1\rangle$$

$$|\varphi_{t}\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

$$|\varphi_{t+2}\rangle \otimes |t+2\rangle$$

$$|\varphi_{t+3}\rangle \otimes |t+3\rangle$$

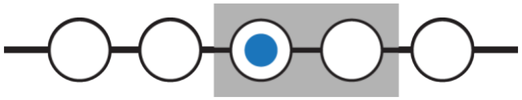
? ? ?

3 Constructing local clocks

- the pulse

3 Constructing local clocks

- the pulse



transitions: 2-local

?

- joining the states by projectors

3 Constructing local clocks

- the pulse

transitions: 2-local

00100

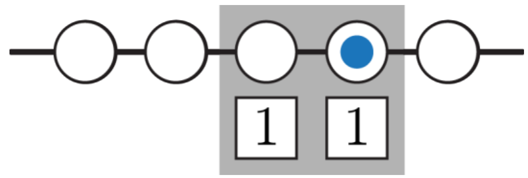
⬇

+00010

- joining the states
  by projectors $|01-10\rangle\langle 01-10|$

3 Constructing local clocks

- the pulse



transitions: 2-local
2-qubit gates: 4-local

interaction
with the data

- joining the states
by projectors $|01-10\rangle\langle 01-10|$

3 Constructing local clocks

- the pulse

transitions: 2-local
2-qubit gates: 4-local

00000 a “dead” state

Initialization!

- joining the states by projectors $|01 - 10\rangle\langle 01 - 10|$

3 Constructing local clocks

- the domain wall 

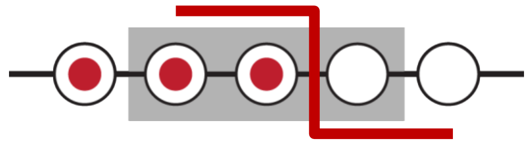
$$|t\rangle = |4\rangle$$
$$= |11110\rangle$$

- 2-local terms "compatible" with **11...1100...00**

?

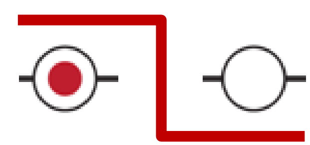
$$|01\rangle\langle 01|$$

3 Constructing local clocks

- the domain wall  transitions: 3-local

$$|t\rangle = |3\rangle = |11100\rangle$$

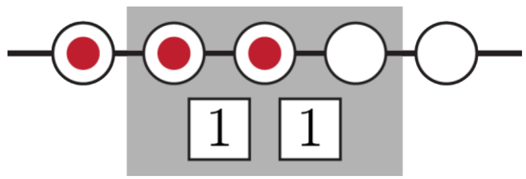
- joining states by transitions? $|100 - 110\rangle\langle 100 - 110|$

- enforce a domain wall: fix the ends 

- the ground state $\cdots + |2\rangle + |3\rangle + \ldots$

3 Constructing local clocks

- the domain wall



transitions: 3-local
2-qubit gates: 5-local

- interacting with work (data) qubits

$$H_t = \frac{1}{2}\left(|t+1\rangle\langle t+1| + |t\rangle\langle t|\right) - \frac{1}{2}\left(U_{t+1} \otimes |t+1\rangle\langle t| + U_{t+1}^{\dagger} \otimes |t\rangle\langle t+1|\right)$$

5-local

3 Local Hamiltonian: putting it all together

- punish bad ancilla initialization

$$\sum_{a=1}^{n_a} |1\rangle\langle 1|_a \otimes |10\rangle\langle 10|_{c_1,c_2} \qquad\qquad |\varphi_0\rangle \otimes |0\rangle_c$$

- check the computation

$$H_t = \frac{1}{2}\left(|t+1\rangle\langle t+1| + |t\rangle\langle t|\right) \\ -\frac{1}{2}\left(U_{t+1} \otimes |t+1\rangle\langle t| + U_{t+1}^{\dagger} \otimes |t\rangle\langle t+1|\right)$$

- punish non-accepting computations

$$|0\rangle\langle 0|_{out} \otimes |1\rangle\langle 1|_{c_L} \qquad\qquad |\varphi_L\rangle \otimes |L\rangle_c$$

- YES

ground state

- NO

lower bound on the ground state energy

good
clock
states

…01…
bad
clock
states

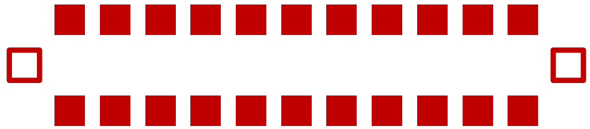
history states

non-uniform superpositions

history states

a polynomially small gap

$$\Delta = O\left(L^{-2}\right)$$



history states

well	badly

initialized history states

well

initialized histories

accepted states

well

initialized histories

$H_A + H_B$

$$\lambda_0 \geq \sin^2 \frac{\vartheta}{2} \times \min(\Delta_A, \Delta_B)$$

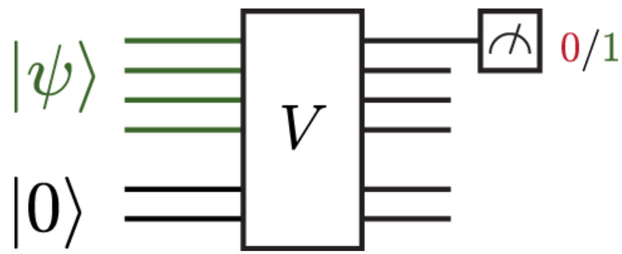
L^{-2}

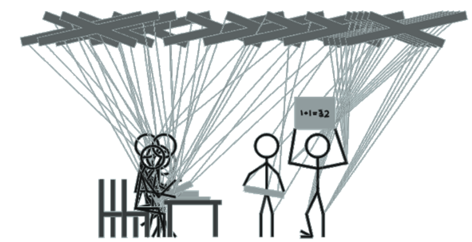
L^{-1}

accepted states

3 LH and QMA verification

$$H_{clock} + H_{init} + H_{prop} + H_{out}$$





NO no witness accepted by V more likely than ε

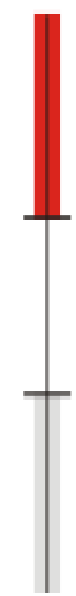
any state has energy

$$\langle \eta | H | \eta \rangle \geq \frac{c\left(1-\sqrt{\epsilon}\right)}{L^3}$$

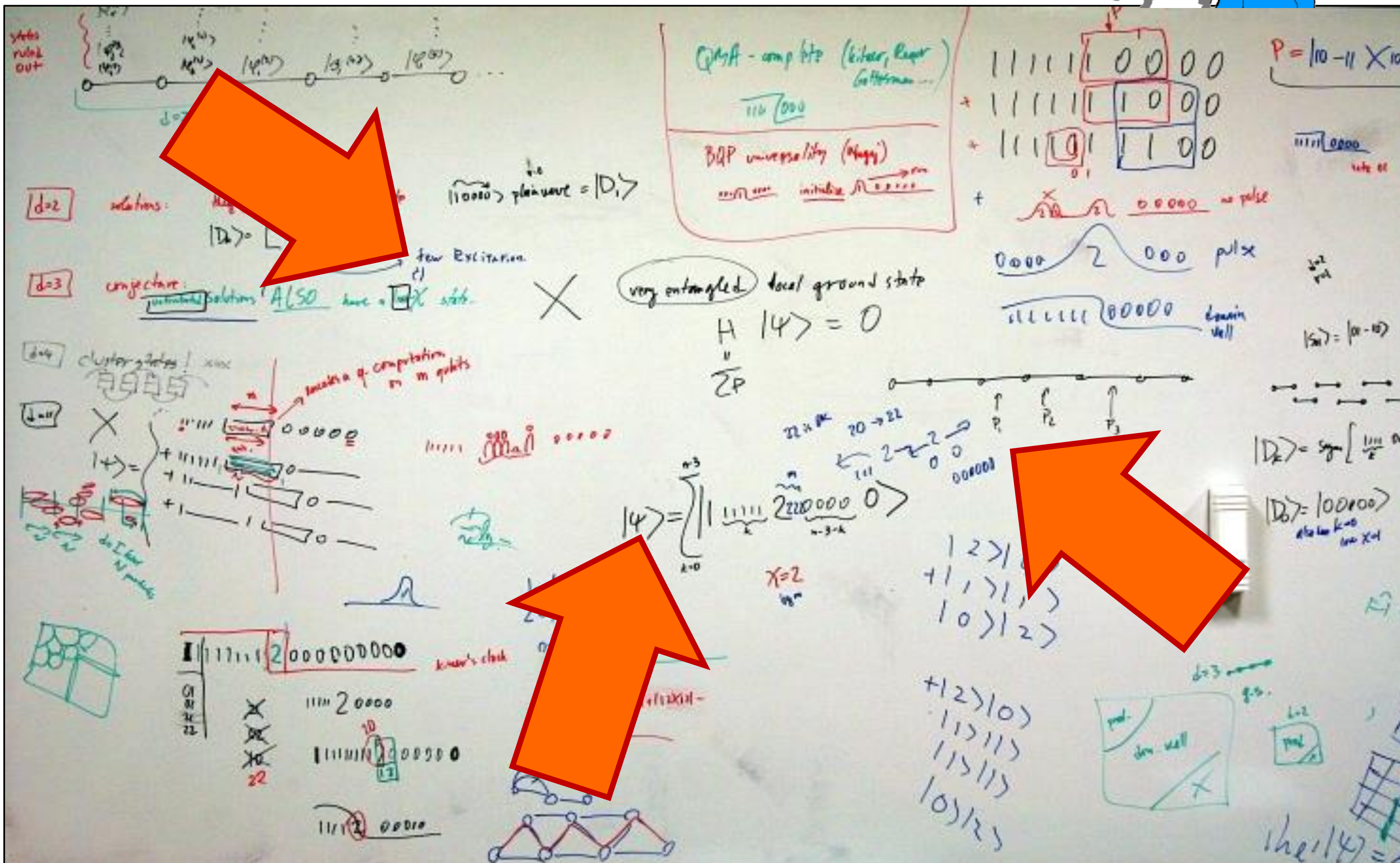
YES there is a proof accepted by V with probability 1-ε

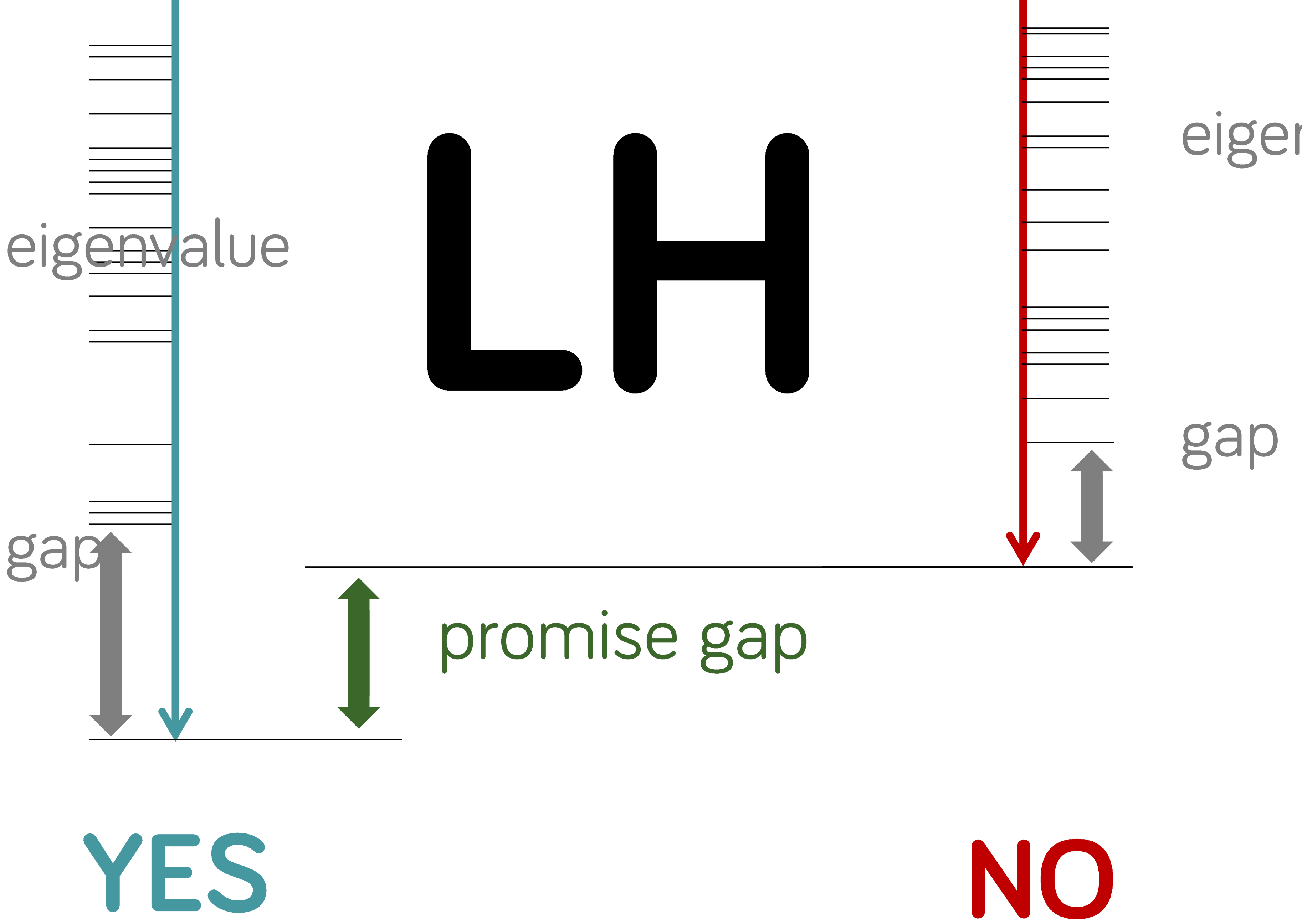
$$\langle \psi_{hist} | H | \psi_{hist} \rangle \leq 0 + 0 + \frac{\epsilon}{L+1}$$

the history for the proof

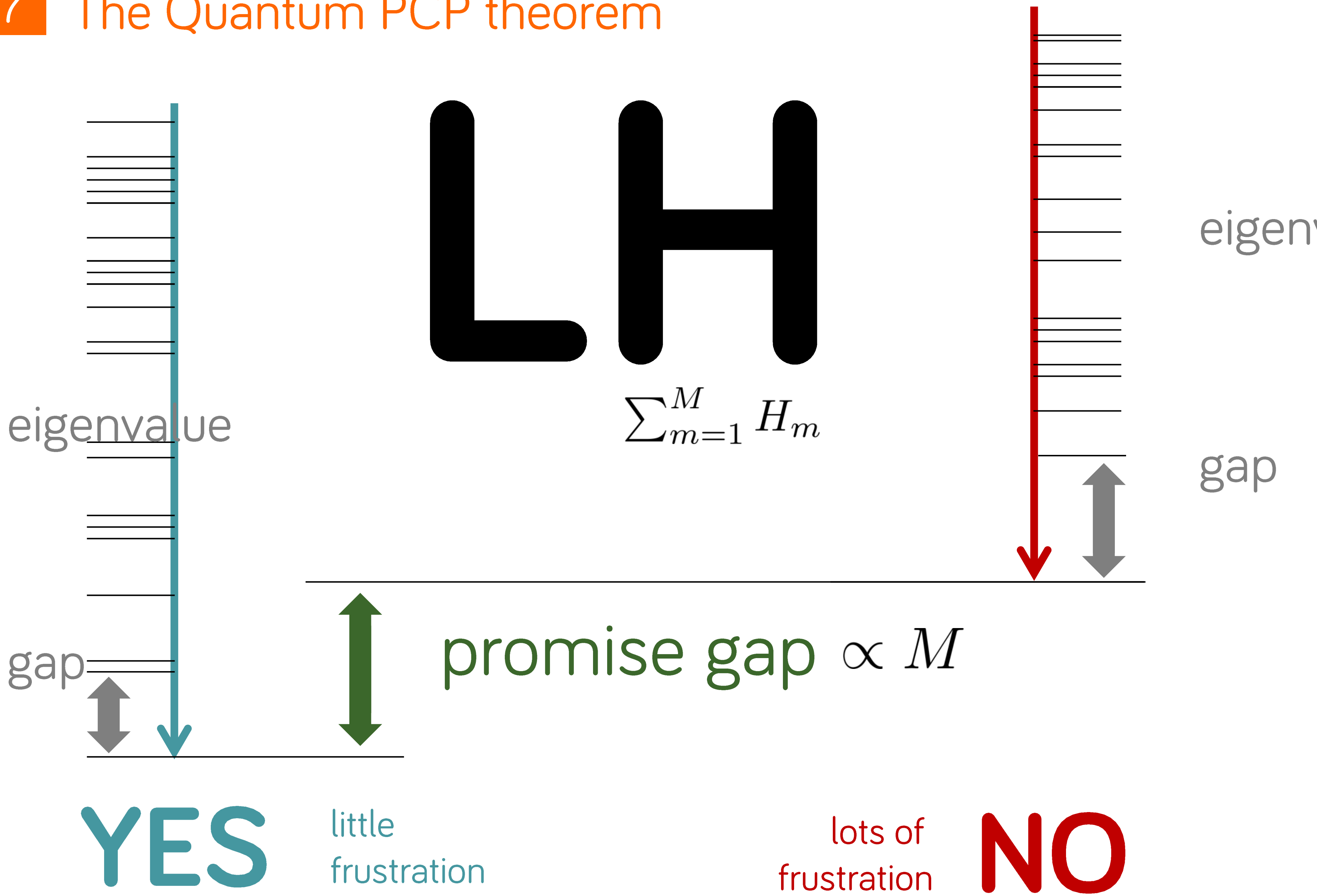


7 A randomly checked proof

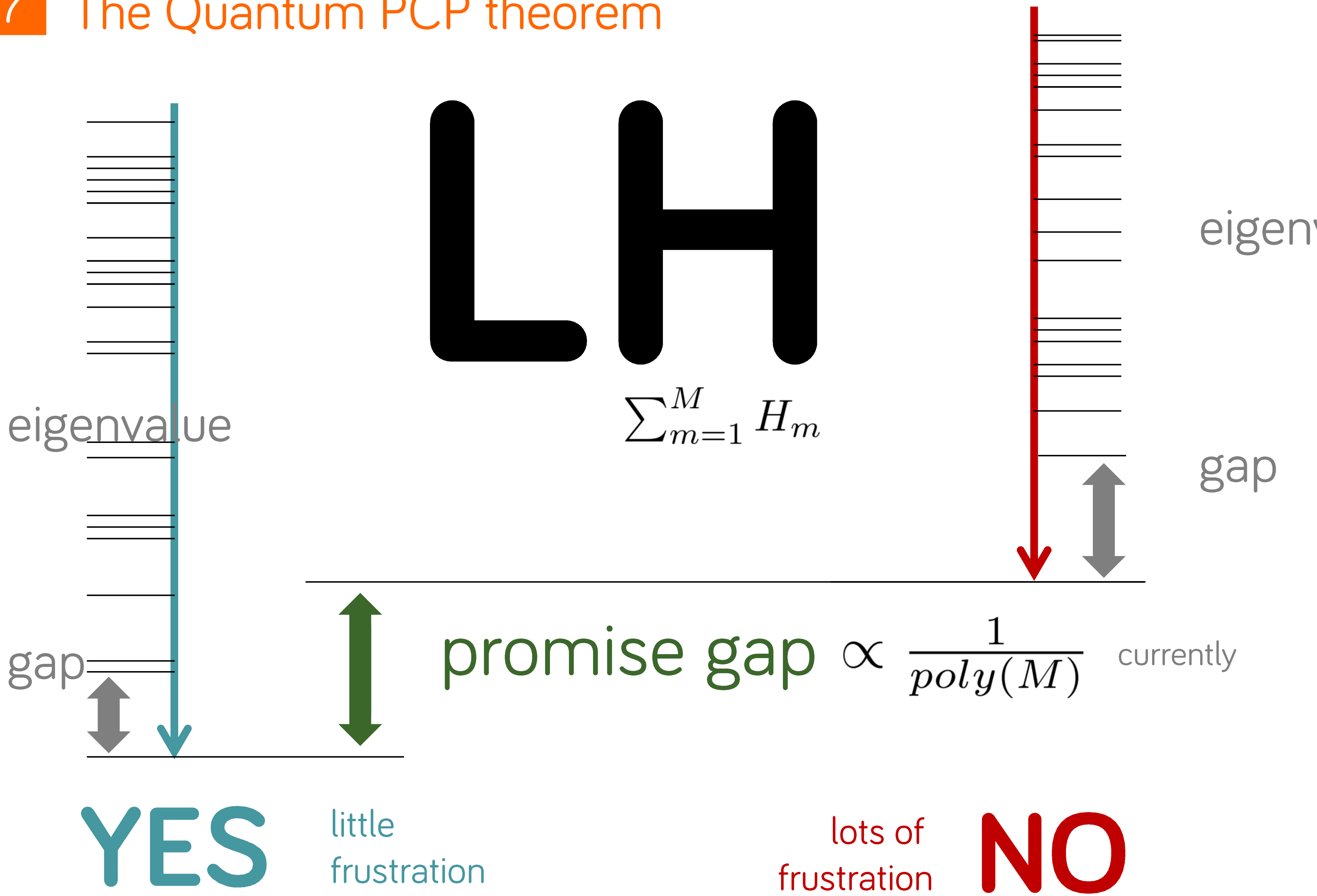



eigenvalue
gap
LH
eigen
gap
promise gap
YES
NO

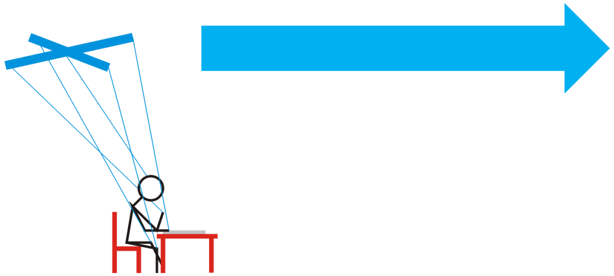
7 The Quantum PCP theorem



7 The Quantum PCP theorem

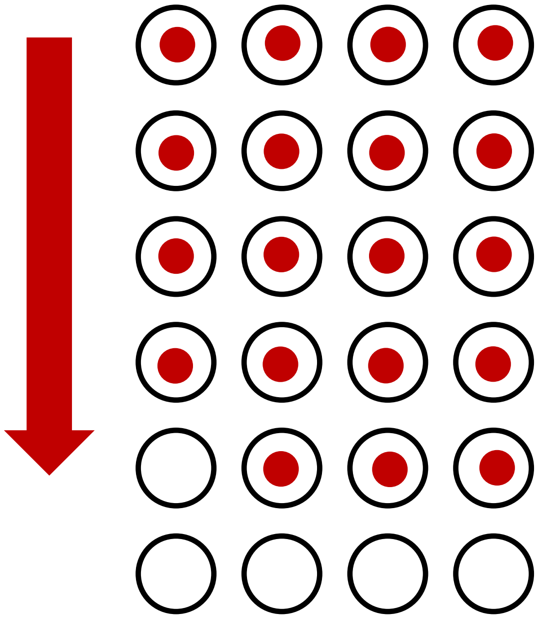


clock/work registers



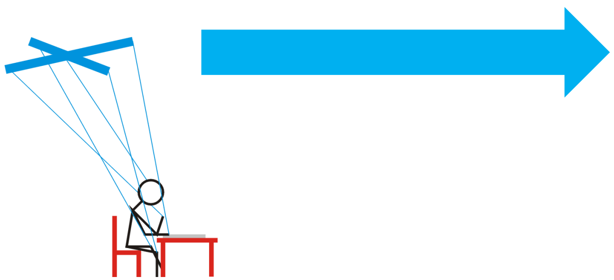
[Kempe, Kitaev, Regev]

a geometric clock

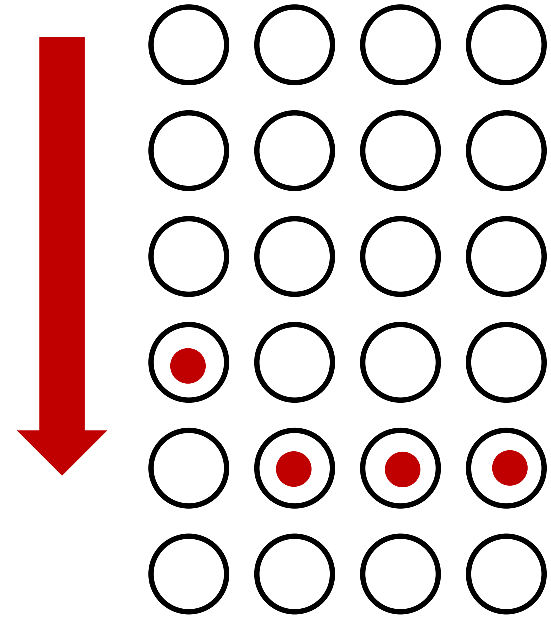


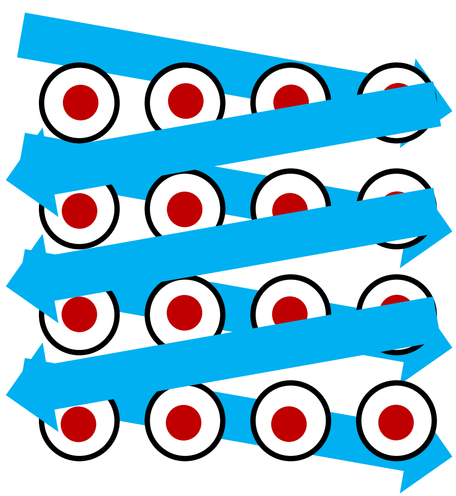
[Mizel] [Aharonov+]

clock/work registers

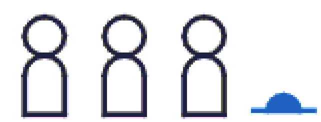


a geometric clock

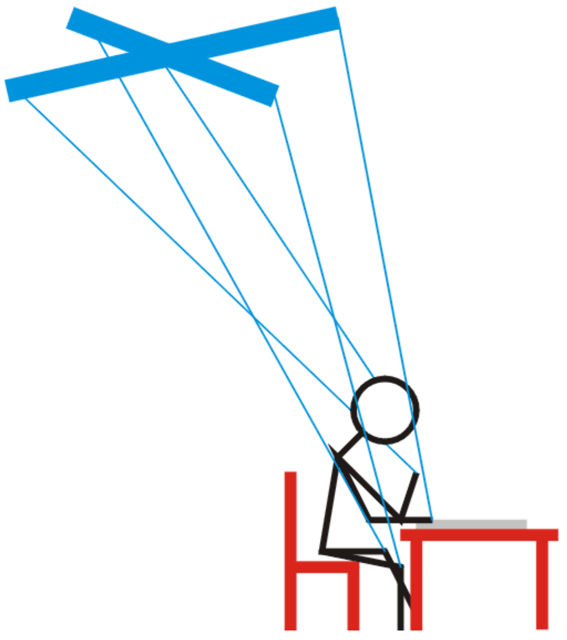




geometric locality



moving data on a line



An introduction to

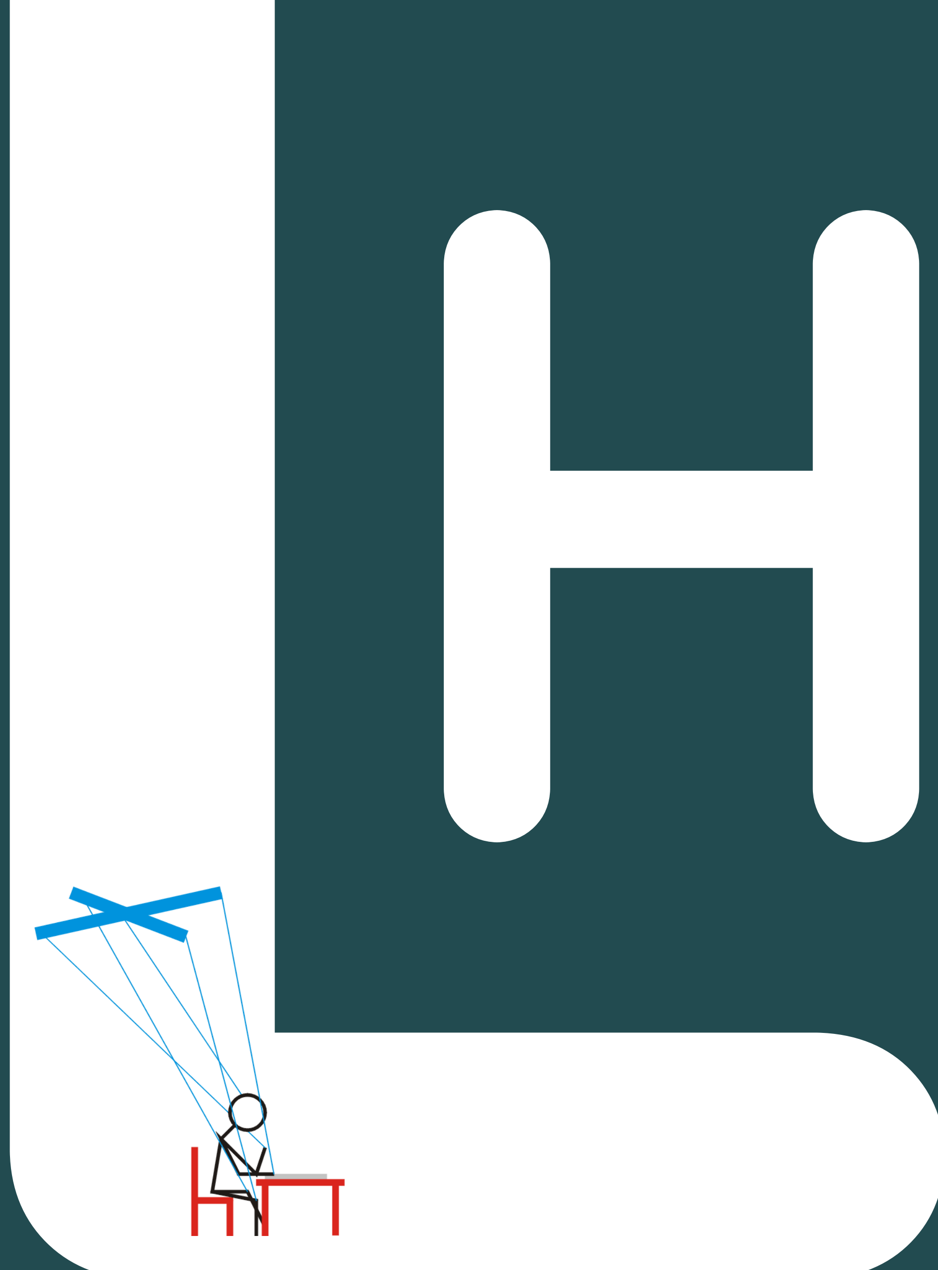
Local Hamiltonians & Quantum Complexity

QMA complete

history state

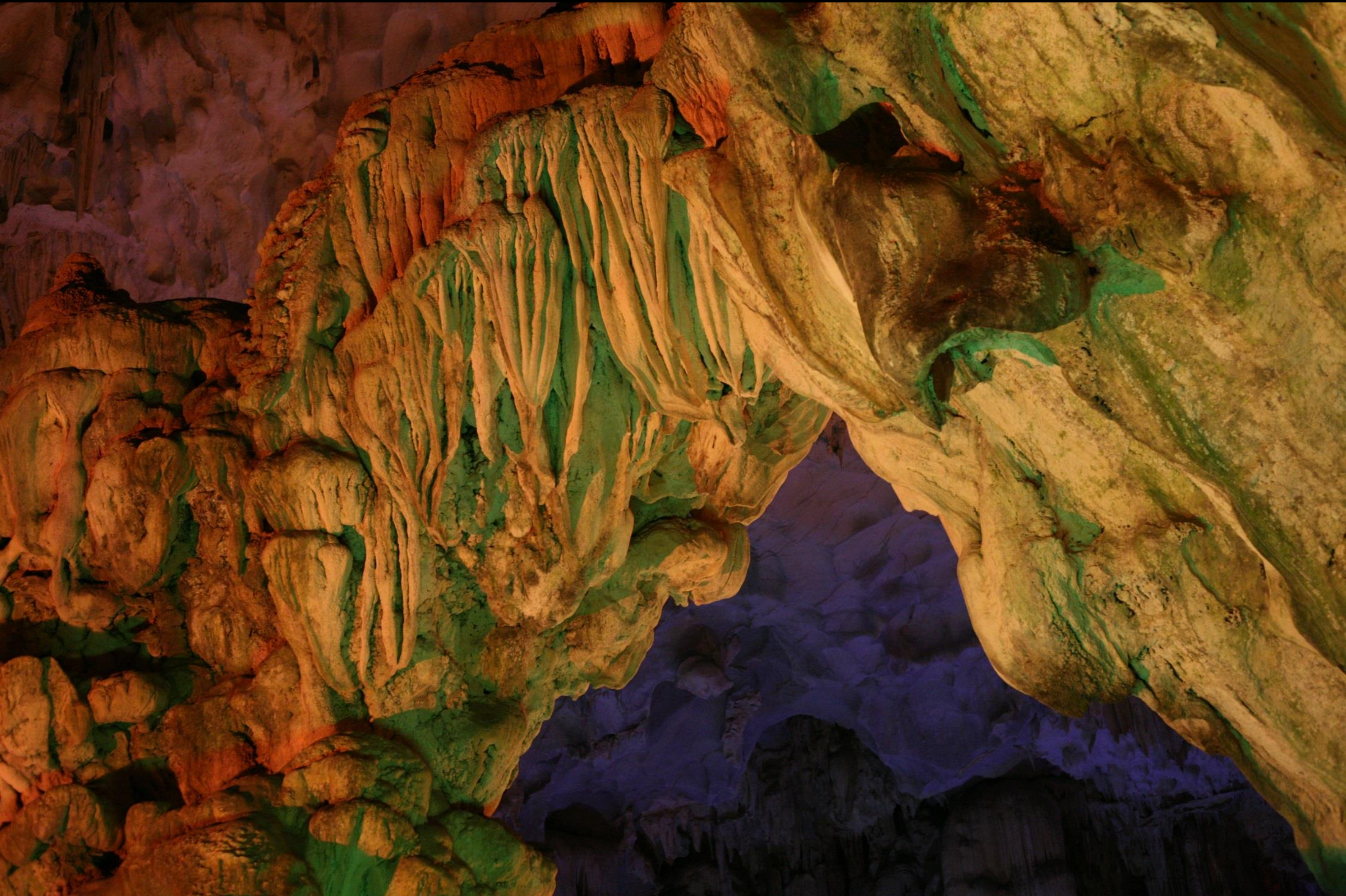
clock

constructions


LH

Quantum SAT

[wisegeek.com]



VẬT LÝ QUANG VINH MUÔN NĂM!