



Quantum 3-SAT

is QMA_1 -complete

FOCS '13

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WATERLOO

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wien

classical

quantum

2SAT

q2SAT

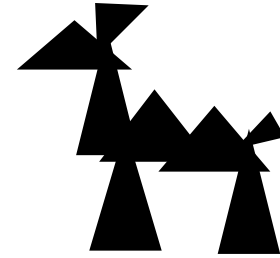
MAX 2-SAT

2-local Hamiltonian

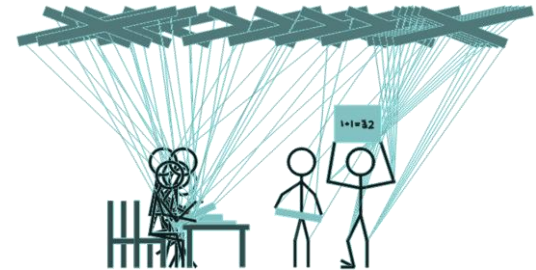
3SAT

q3SAT

1 **quantum sat**
without frustration



2 **the history**
of (a) quantum computation

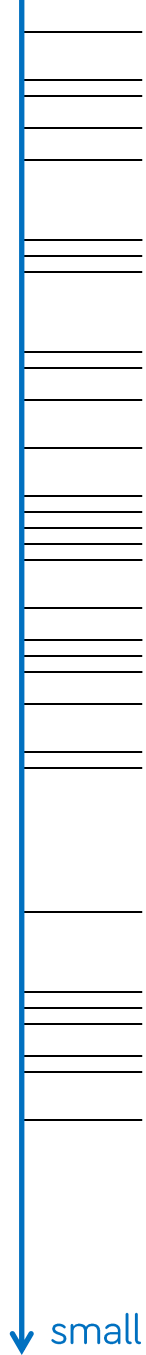
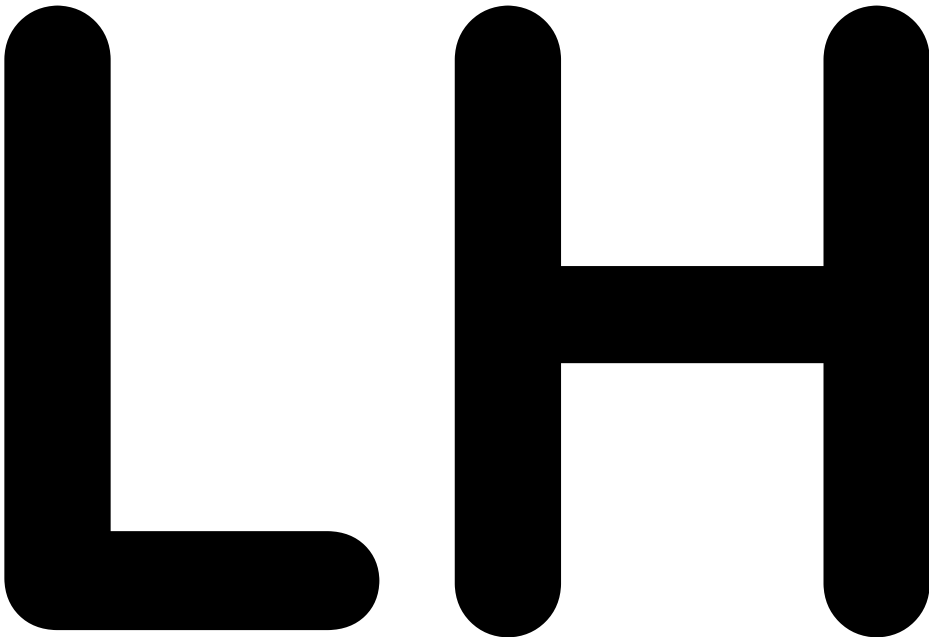


3 **a local clock**
superpositions & interference



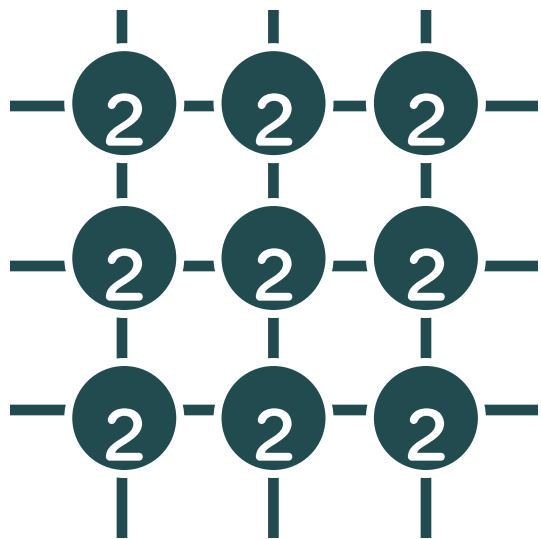
1 The Local Hamiltonian problem

Is
the
ground
state
energy
of a



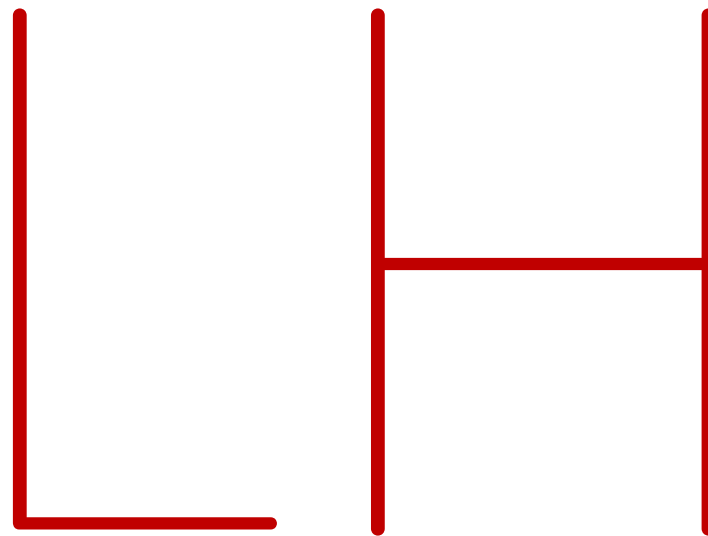
small?

1 2-local Hamiltonian: QMA-complete



[Oliveira, Terhal '04]

a global minimum



$$\sum H_{j k}$$



[Hallgren, N., Narayanaswami '13]

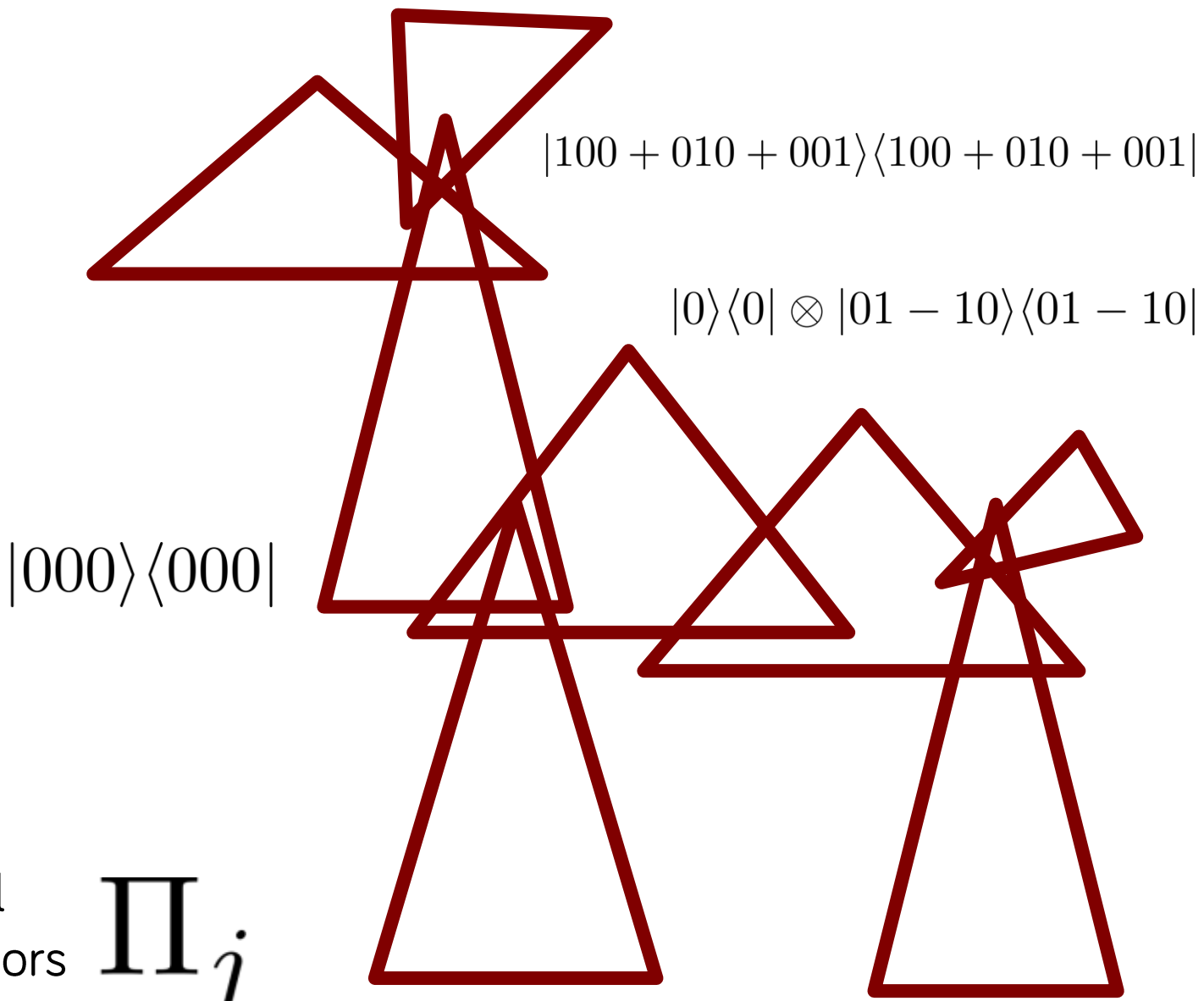
A young woman on the left and a young boy on the right are both wearing blue caps and dark t-shirts. The woman has a frustrated expression, looking down. The boy has a pained or frustrated expression, with his eyes closed and mouth open. The background is a wooded area with trees and a person in a yellow shirt sitting on the ground in the distance.

frustrated

FRUST
RATED

1

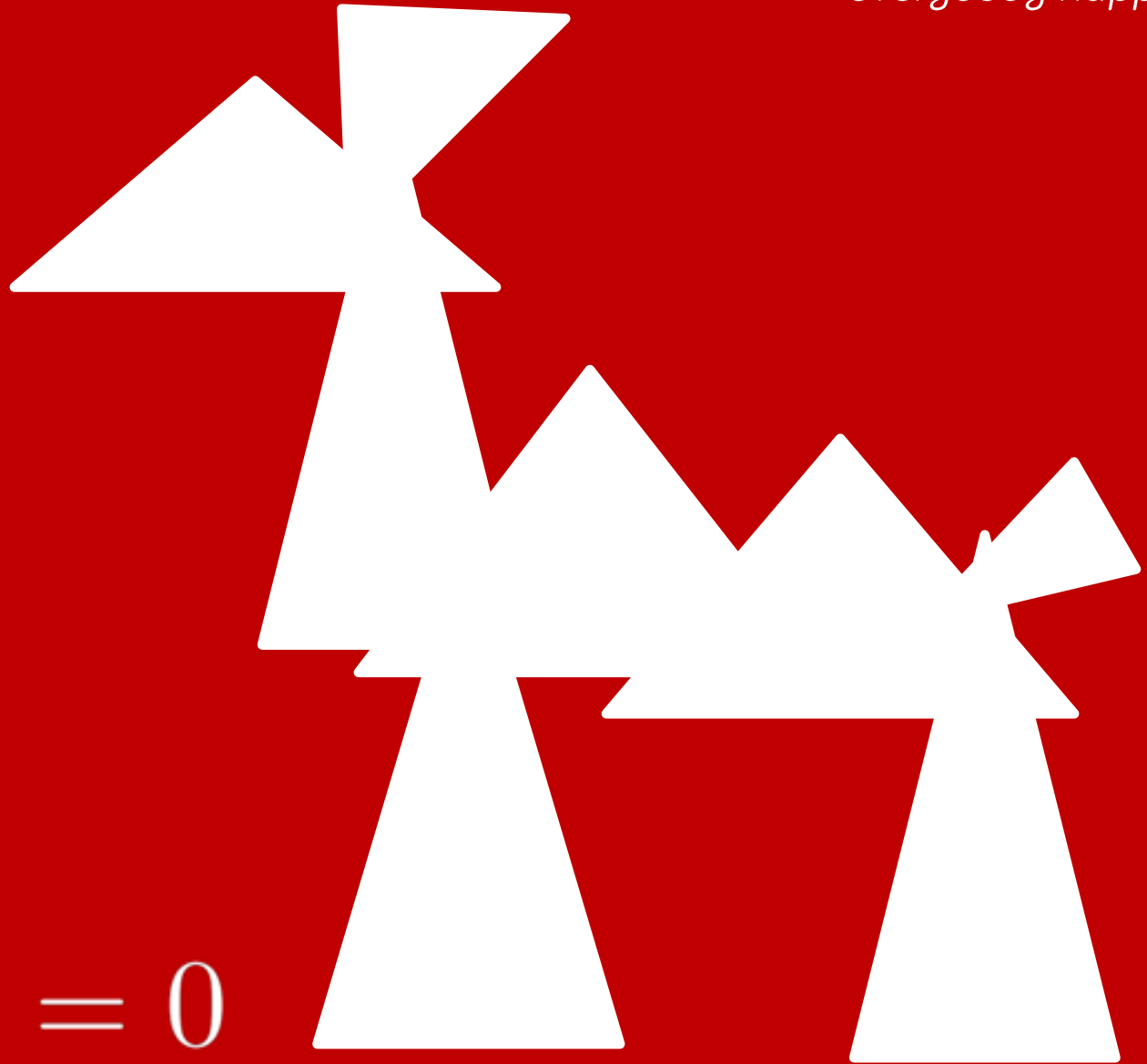
Quantum 3-SAT



■ 3-local projectors Π_j

1 Quantum 3-SAT

Can we make everybody happy?



$$\Pi_j |\psi\rangle = 0$$

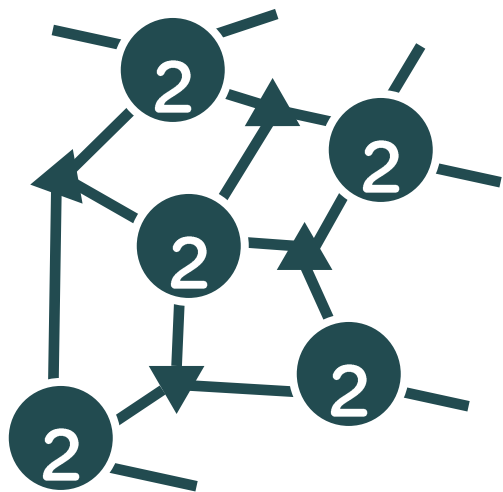
1 qSAT: QMA₁-complete



[N. '08]

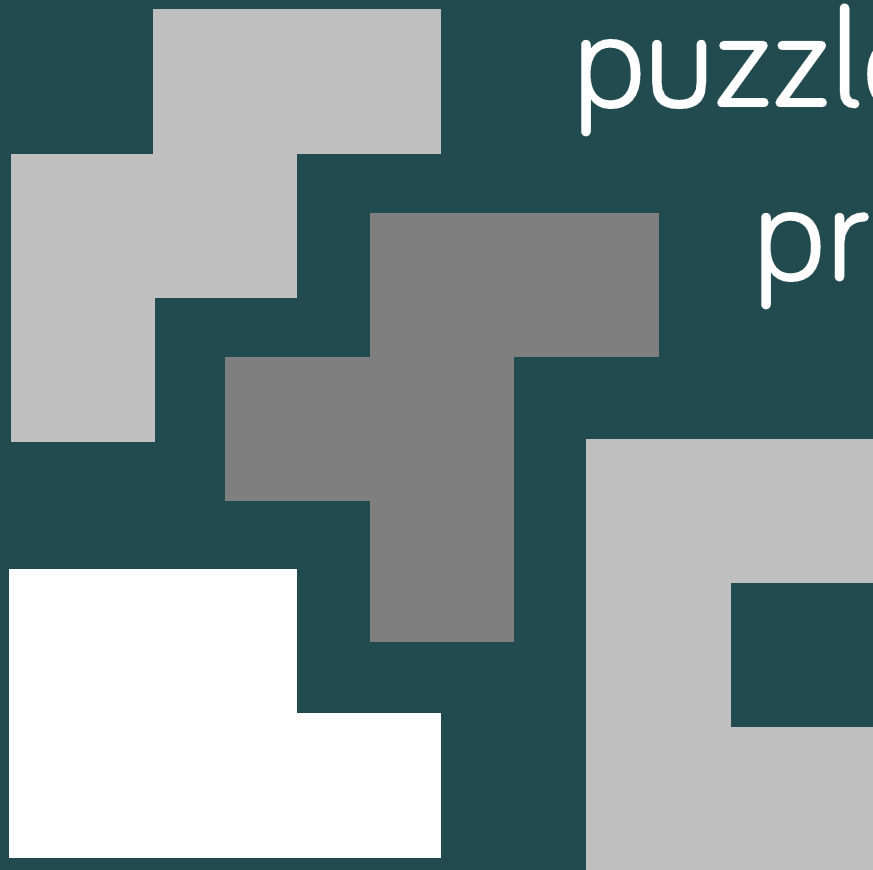
Π_j

[Gosset, N. '13]



unfrustrated

qSAT



puzzles &
proofs

Dinosaurs? Really?

NO?

Don't get
fooled
easily.

YES?

Accept
a genuine proof
without a doubt.





NO?

Don't get
fooled
easily.



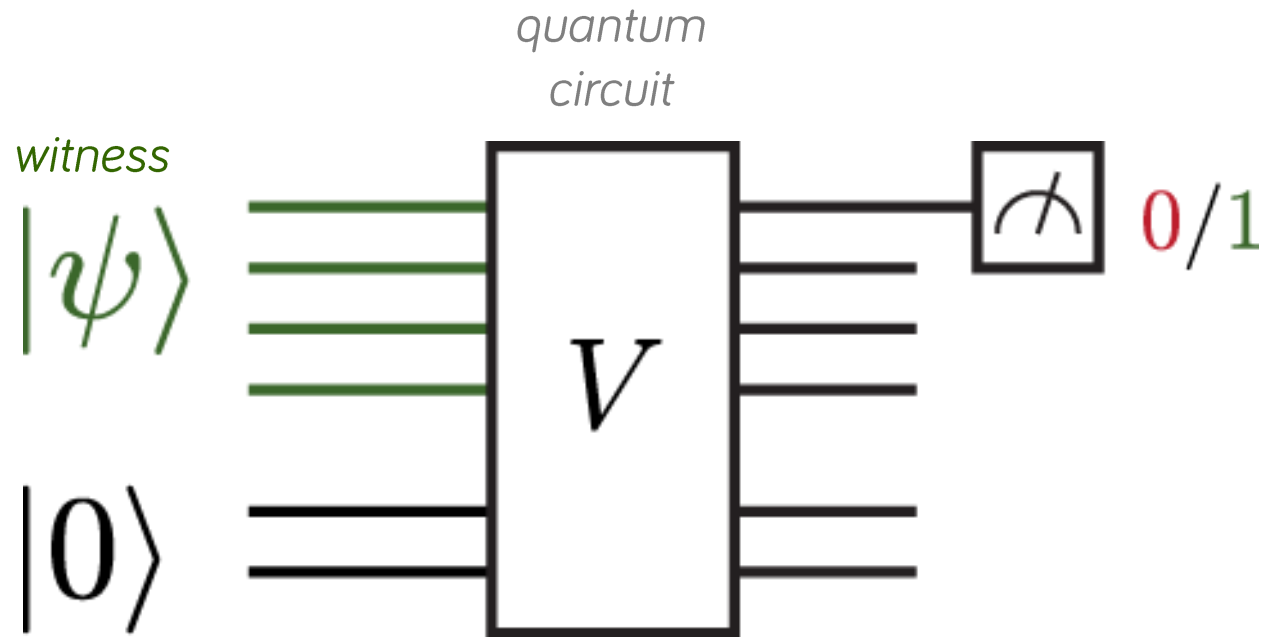
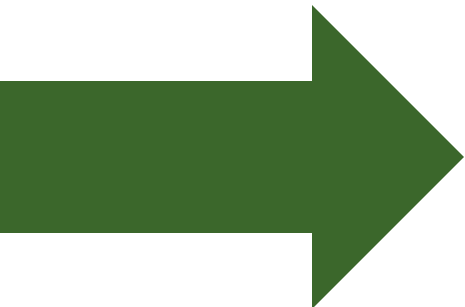
YES?

Accept
a genuine proof
without a doubt.



MA₁
perfect
completeness

1 A quantum analogue: QMA₁

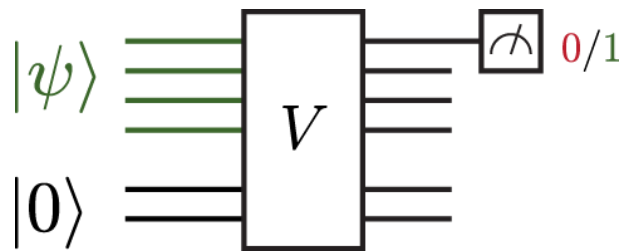
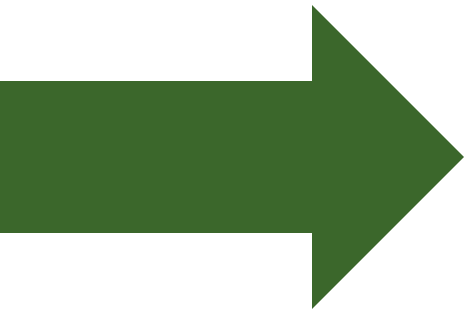


YES? Accept a good proof.

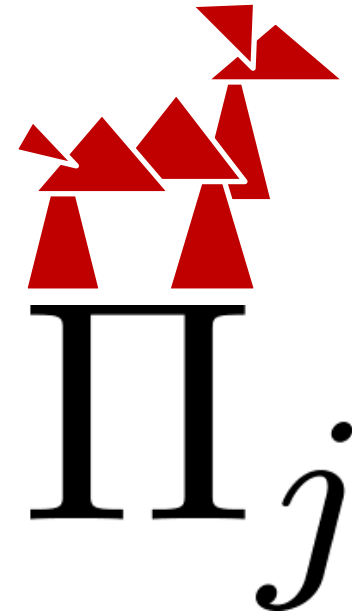
NO? Get fooled with small p .



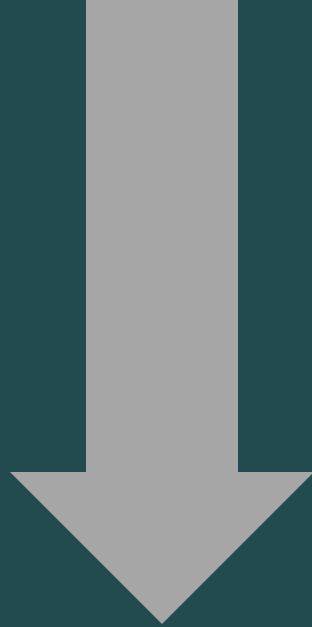
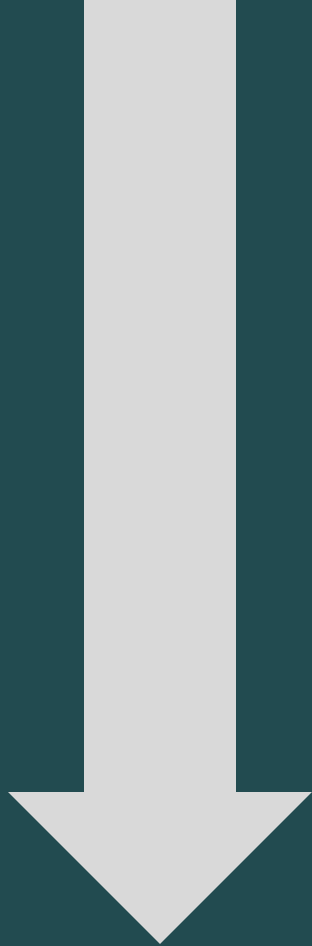
1 A candidate for a QMA_1 -hard problem



Does this circuit accept?



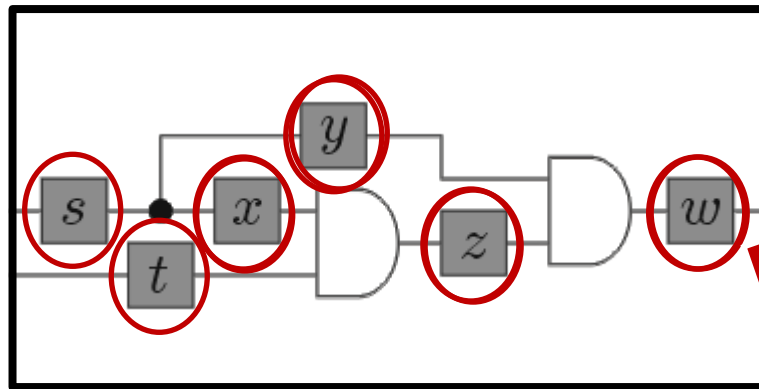
Is there a common ground state?



computation
ground states
& quantum sat

2 What is hard for NP?

- 3-SAT is NP-complete. [Cook, Levin]



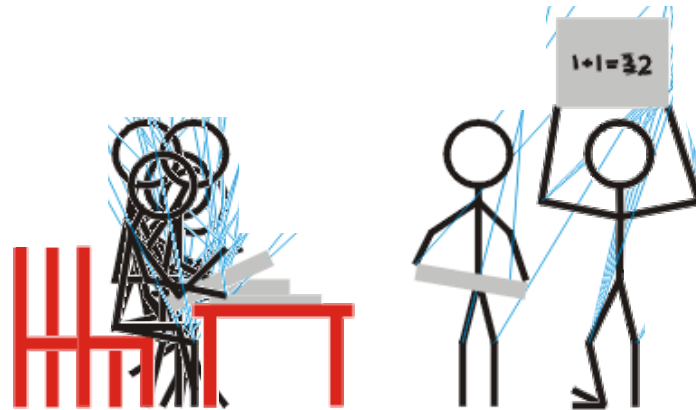
Could this circuit ever output 1?

3-local conditions

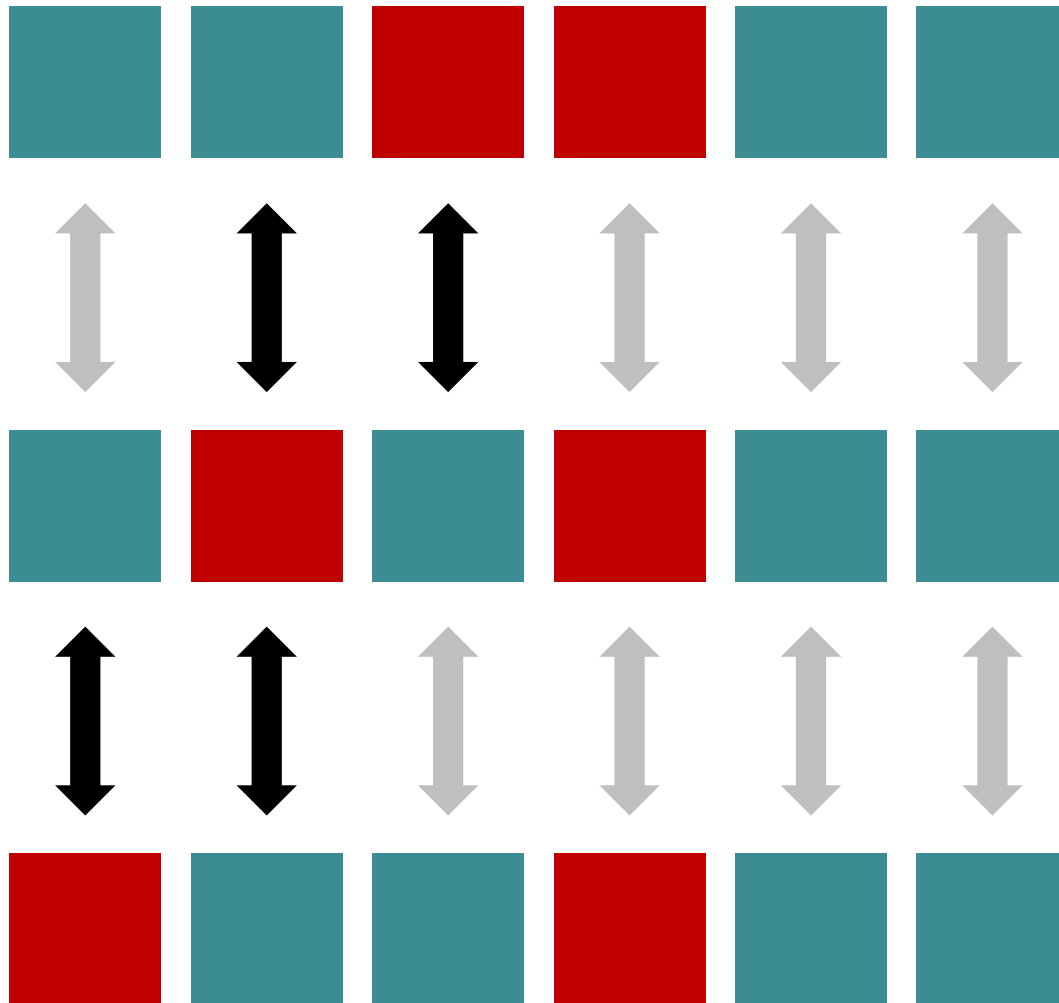
$$(\dots \vee \dots \vee \dots) \wedge (\dots \vee \dots \vee \dots) \wedge \dots$$

- Can we do the same for a quantum computation?

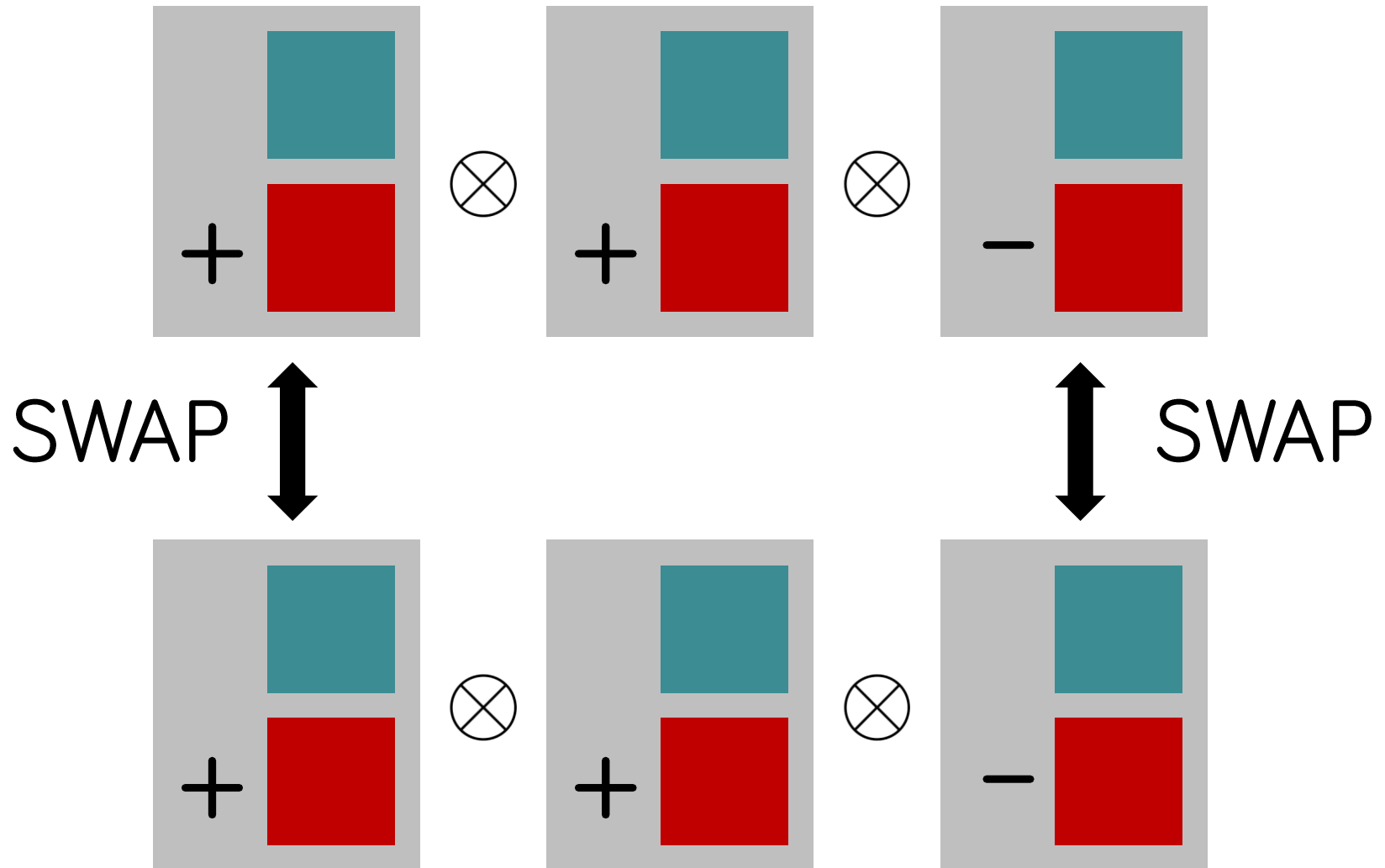
2 Snapshots of a computation



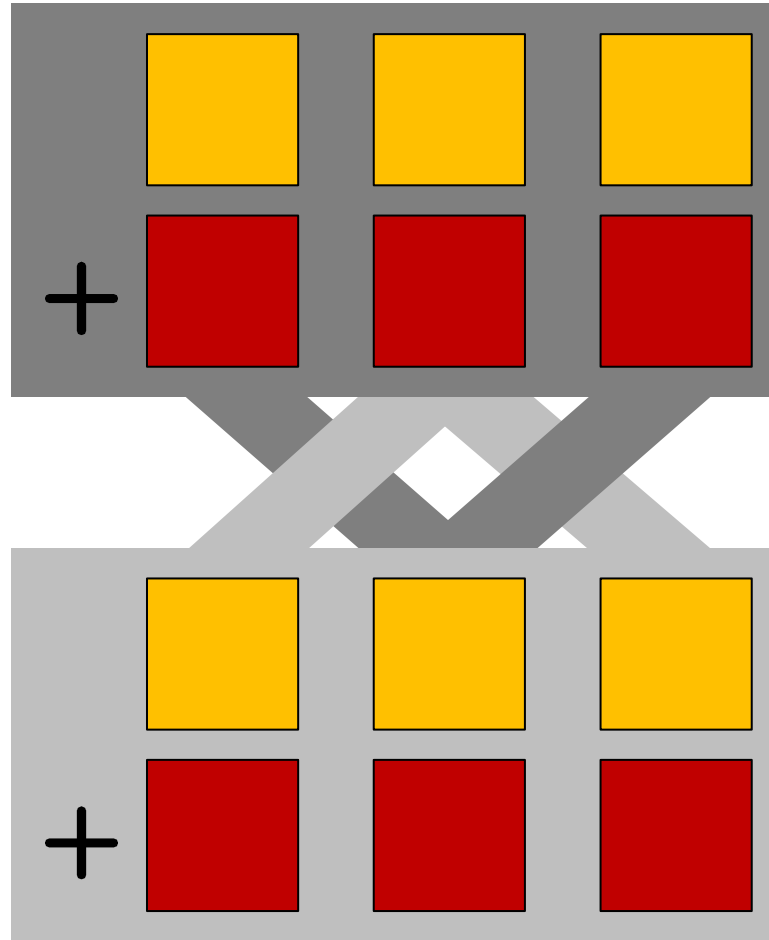
Locally comparing **strings**.



Locally comparing **product** states: SWAP.

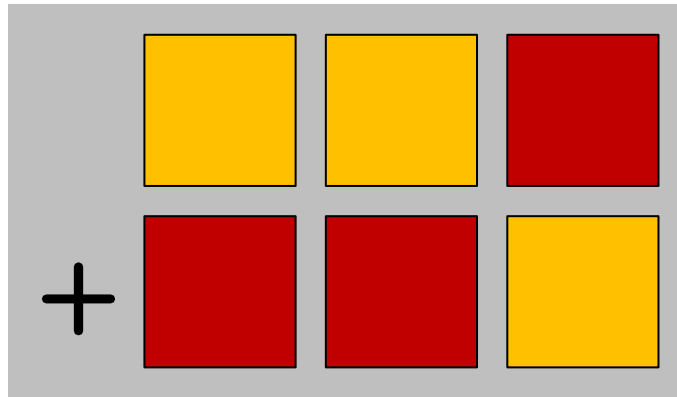


Locally comparing **entangled** states?



UGH!

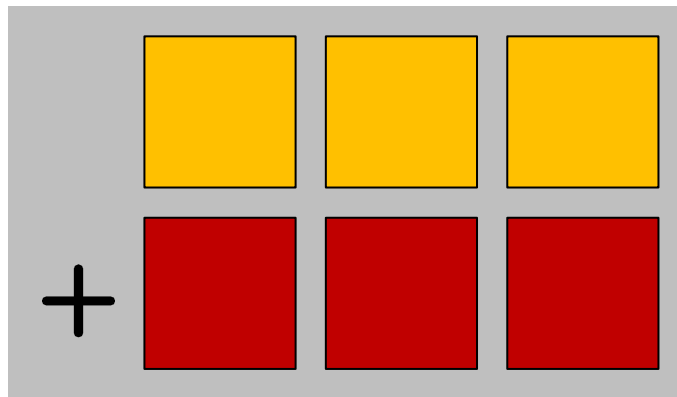
2 The data & the clock



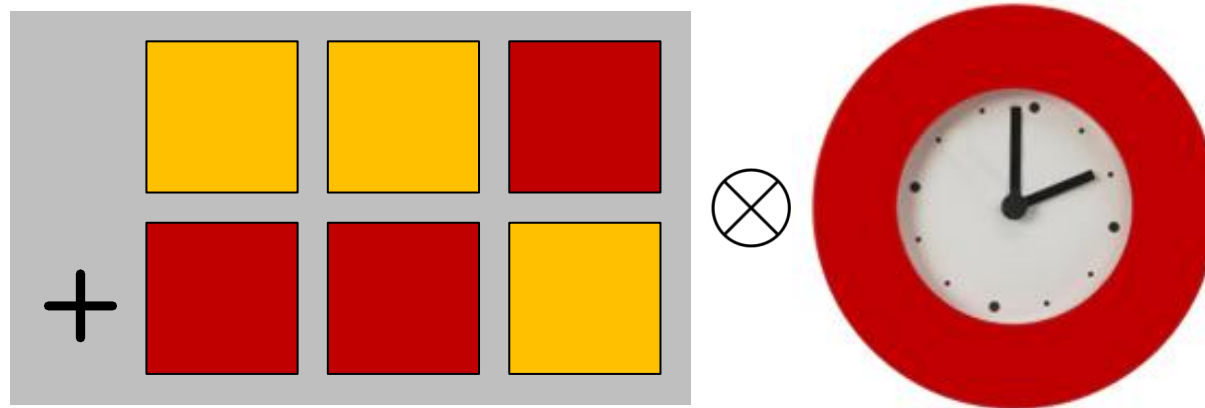
Hard to compare directly (locally).

U^\dagger   U

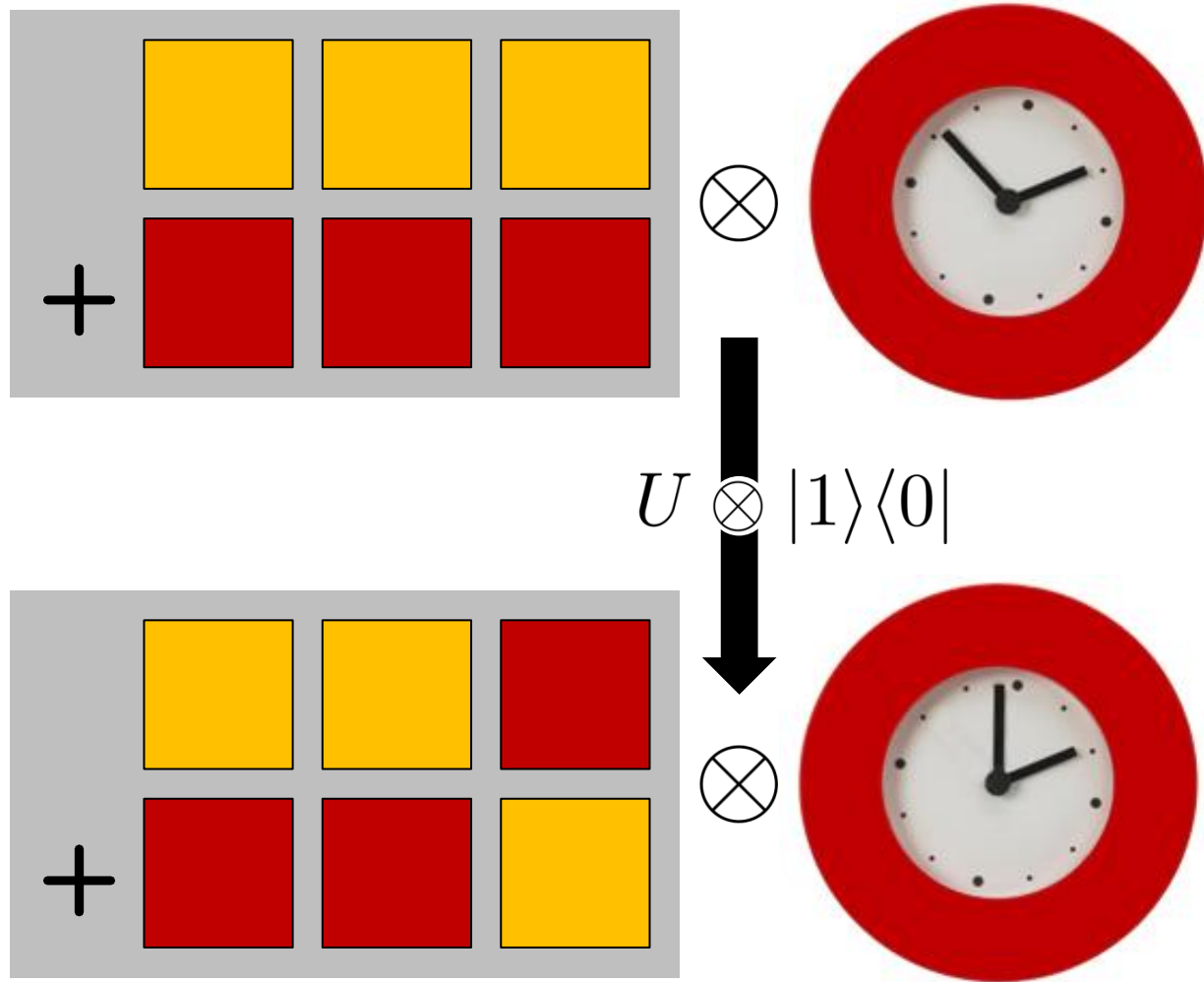
a clock



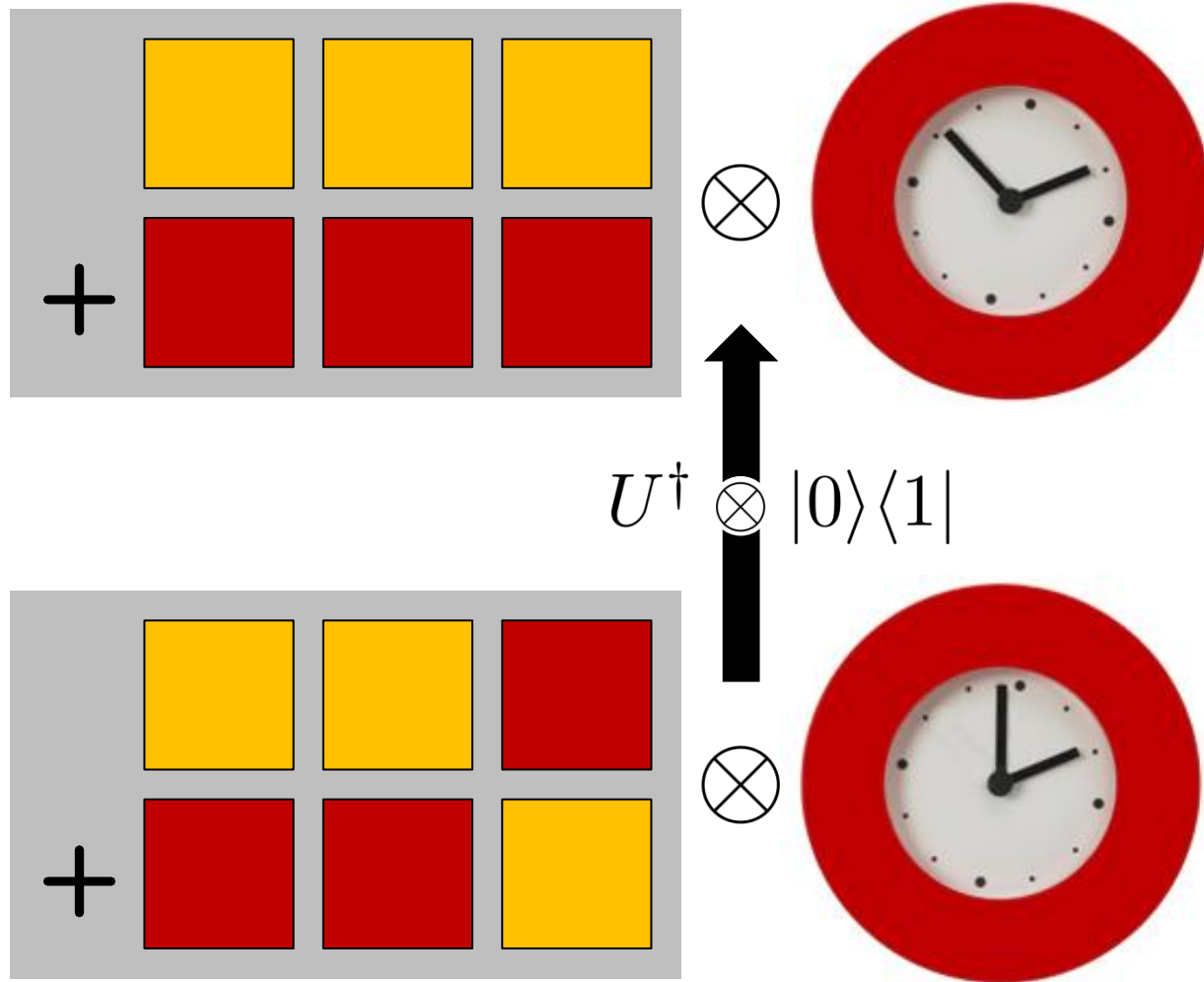
2 The data & the clock



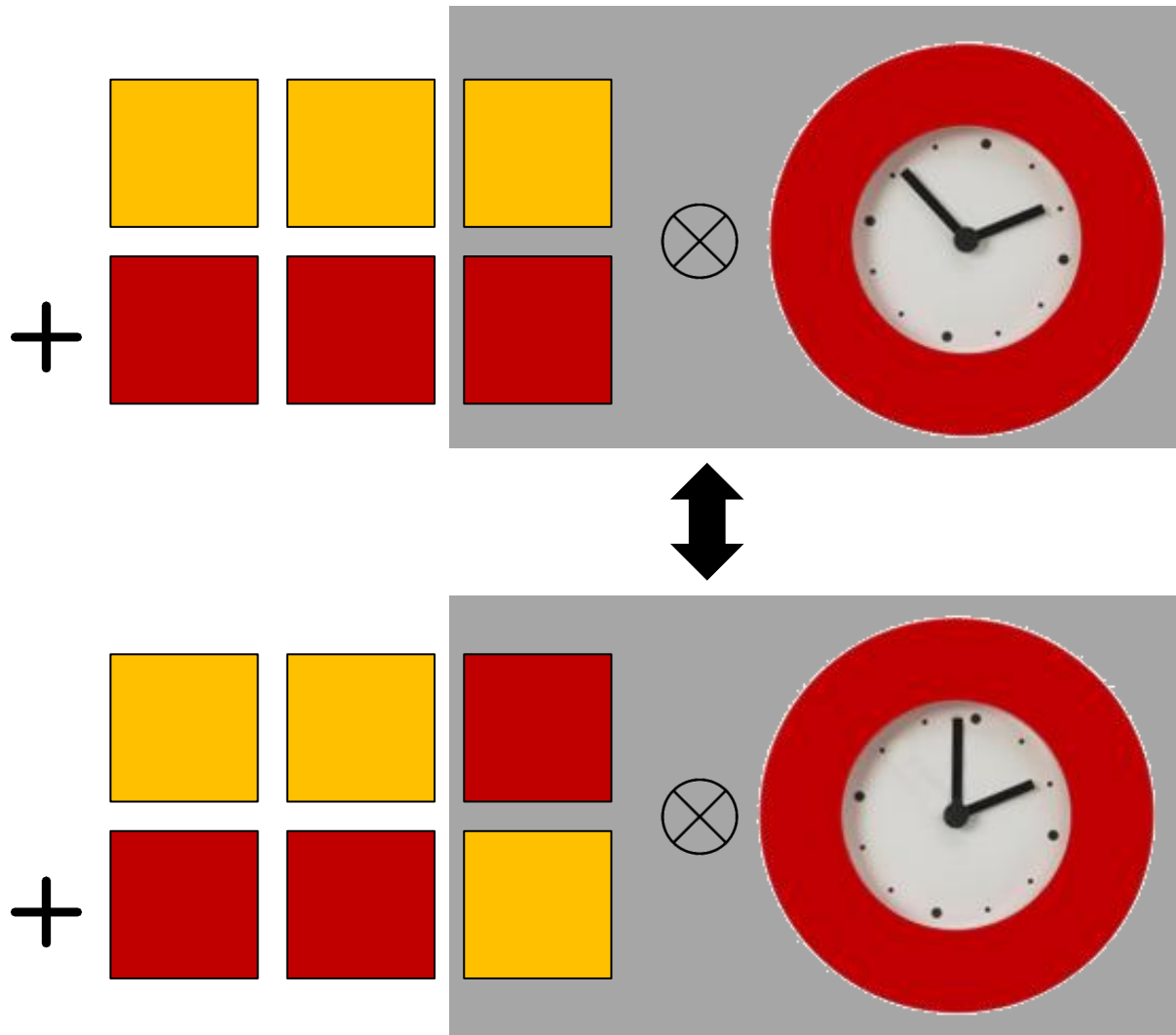
2 The data & the clock



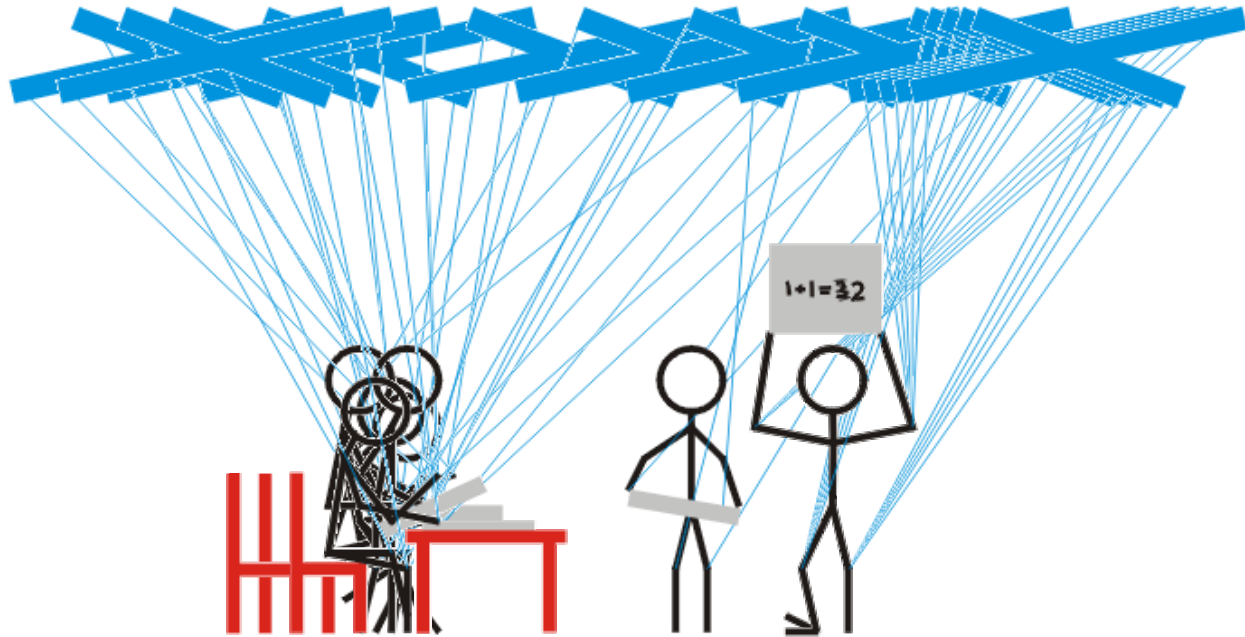
2 The data & the clock



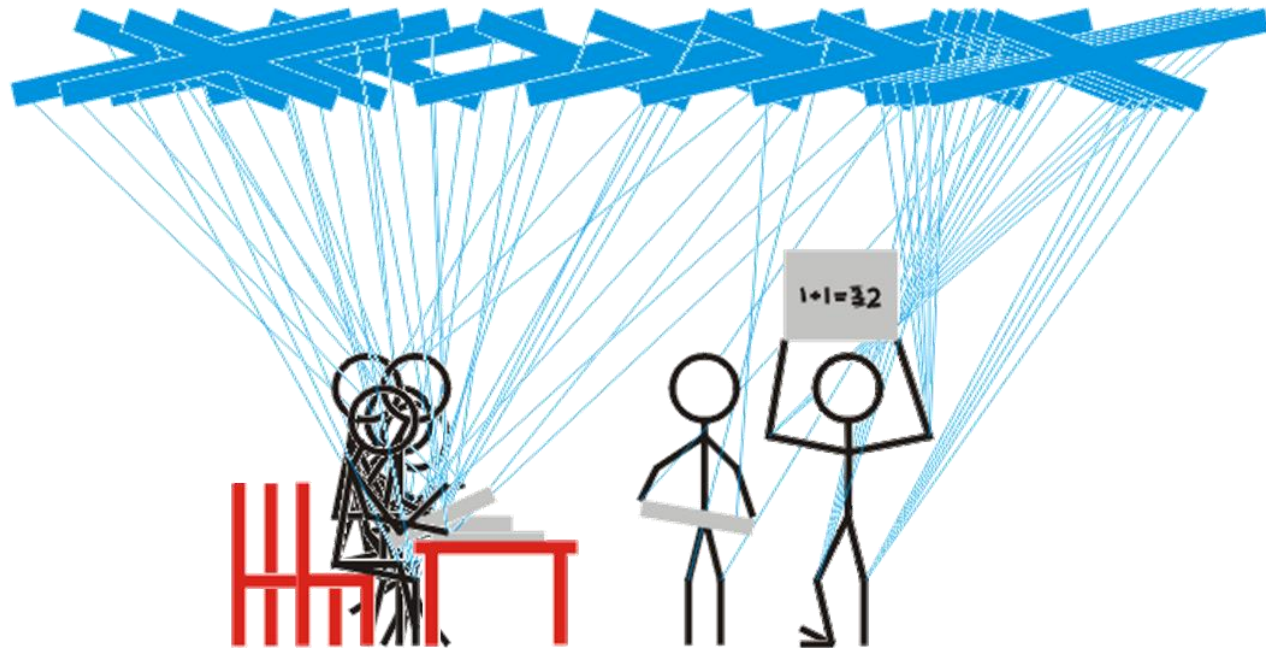
2 The data & the clock



2 The history state



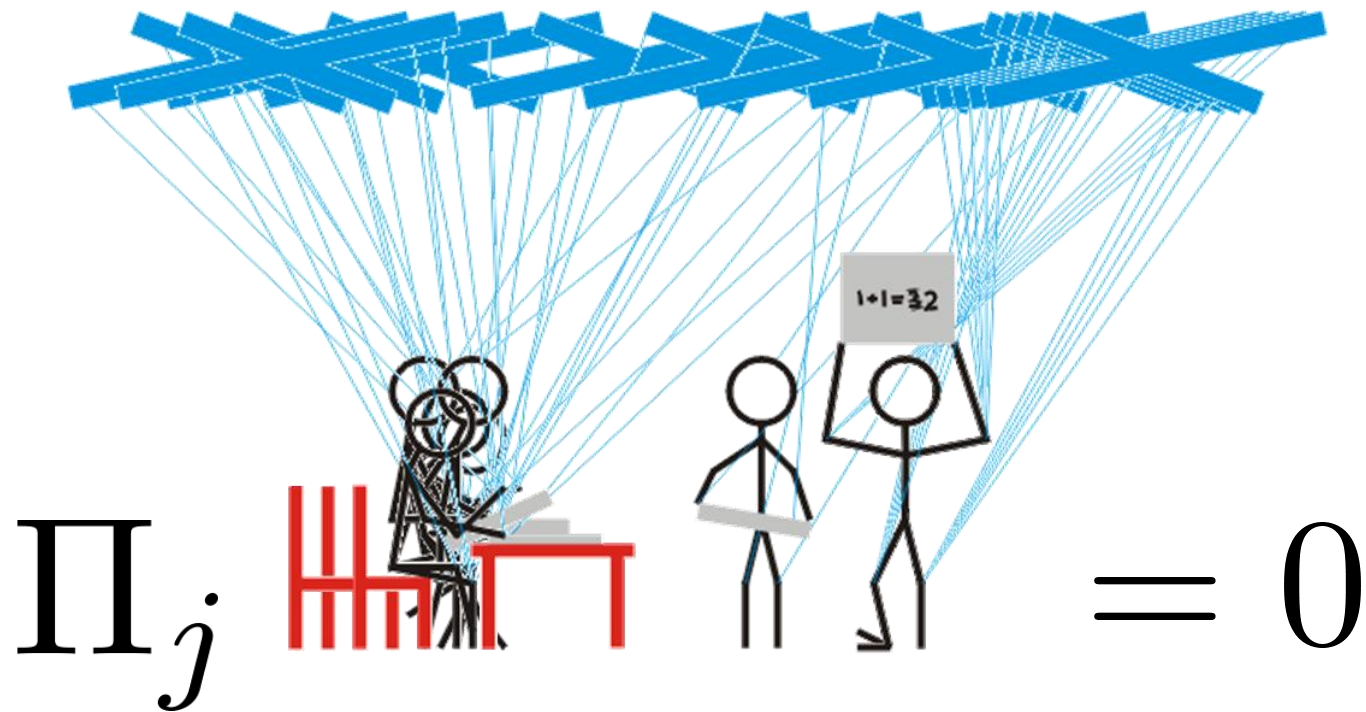
2 The history state



$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^T |\varphi_t\rangle \otimes |t\rangle$$

$\underbrace{U_t \cdots U_1 |\varphi_0\rangle}_{|t\rangle}$

2 The history state: a ground state



$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^T \underbrace{|\varphi_t\rangle \otimes |t\rangle}_{U_t \cdots U_1 |\varphi_0\rangle}$$

2 Do we have a history state?

k-local
c-o-n-d-i-t-i-o-n-s

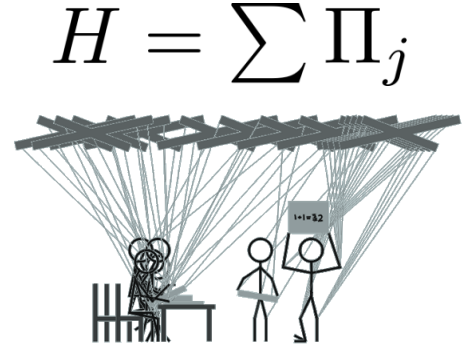
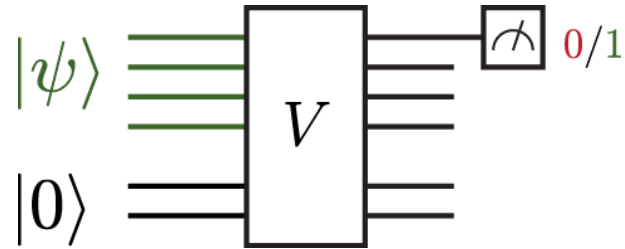
clock encoding
state progression

$$\begin{aligned} &|\varphi_t\rangle \otimes |t\rangle \\ &|\varphi_{t+1}\rangle \otimes |t+1\rangle \end{aligned}$$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^T |\varphi_t\rangle \otimes |t\rangle$$



2 Computation and Quantum SAT



NO

nothing is likely to be accepted

YES

there is a perfectly accepted proof



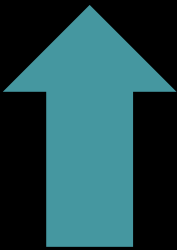
any state has a high energy

its history state has energy 0

● YES

ground state

• NO



lower bound on the
ground state energy

good
clock
states

bad
clock
states

history states

non-uniform
superpositions

well

badly

initialized history states

well

initialized histories

accepted
states

well

initialized histories

accepted
states

$$H_A + H_B$$

$$\lambda_0 \geq \sin^2 \frac{\vartheta}{2} \times \min(\Delta_A, \Delta_B)$$



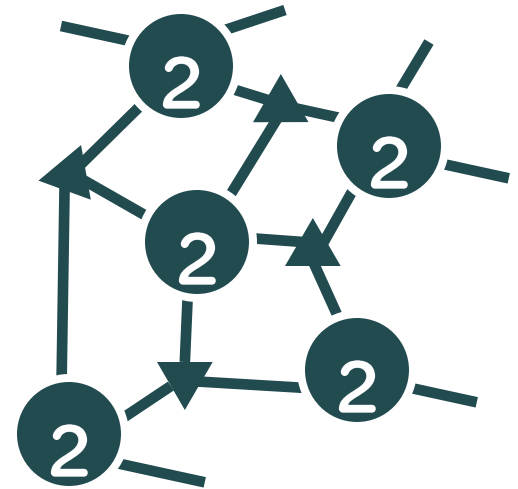
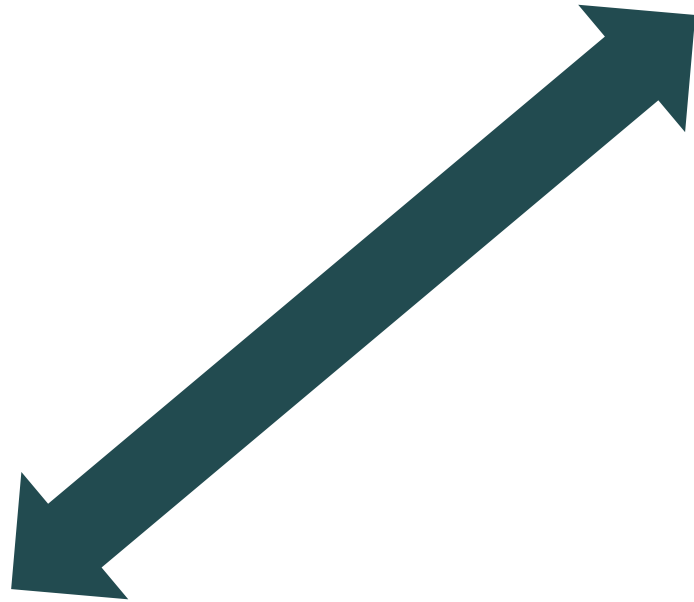
L^{-2}

L^{-1}

L^{-3}

promise
gap
(soundness)

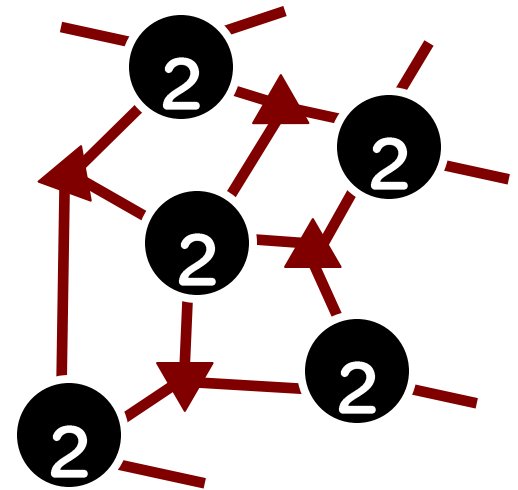
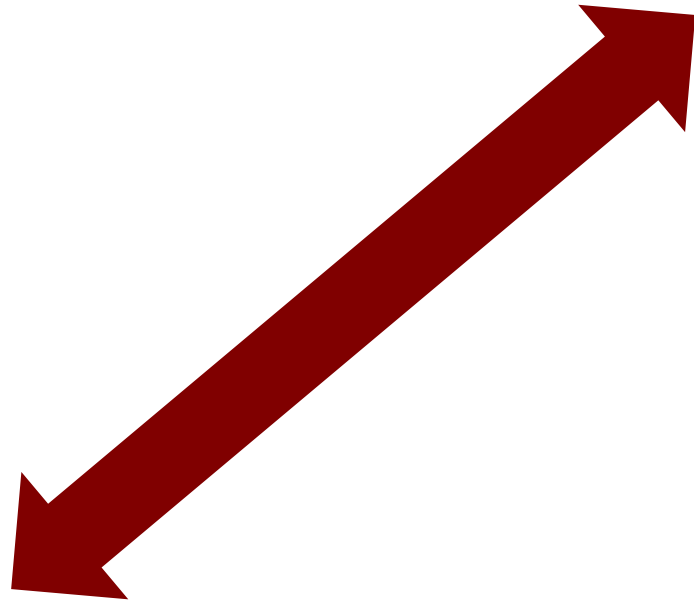
3 Computation & projectors



unfrustrated
quantum 3-SAT

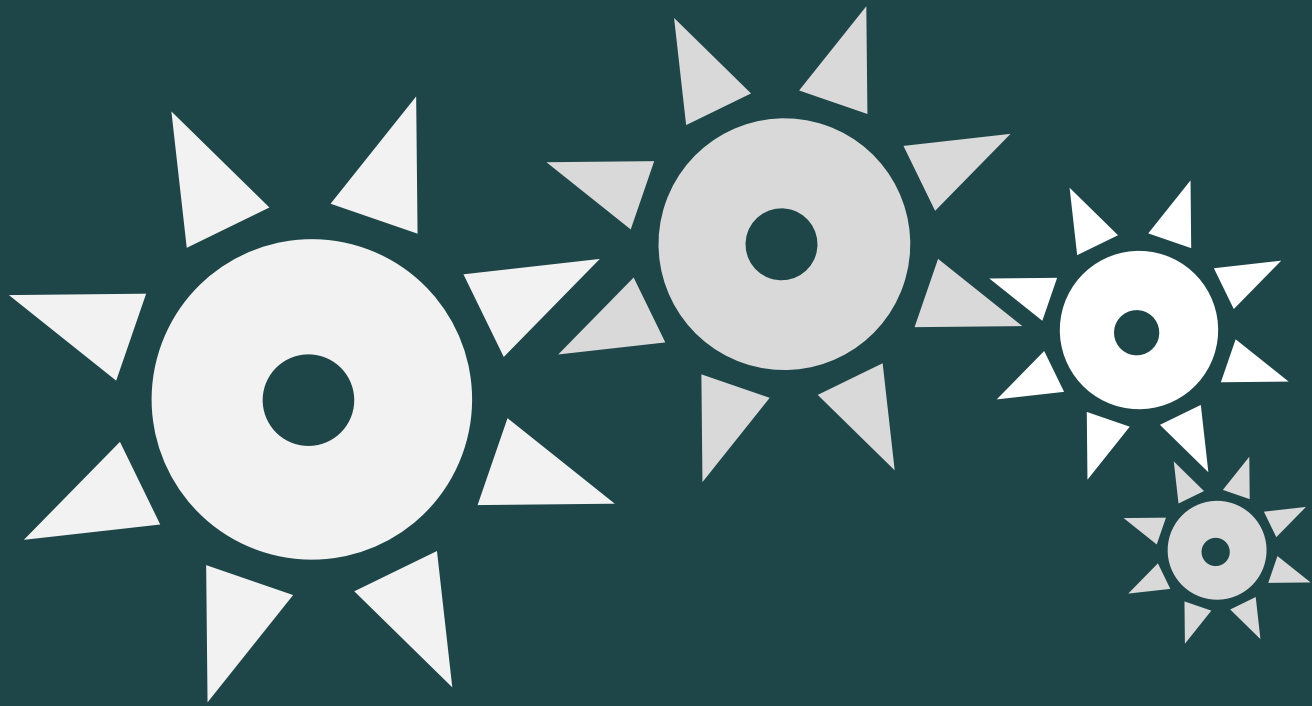
a verifier circuit with
a genuine witness

3 Computation & projectors



frustrated
quantum 3-SAT

a verifier circuit with
no likely witness



a clock workshop

3 Run the clock, apply 2-qubit gates ...

$$|\varphi_{t-2}\rangle \otimes |t-2\rangle$$

$$|\varphi_{t-1}\rangle \otimes |t-1\rangle$$

$$|\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

$$|\varphi_{t+2}\rangle \otimes |t+2\rangle$$

$$|\varphi_{t+3}\rangle \otimes |t+3\rangle$$



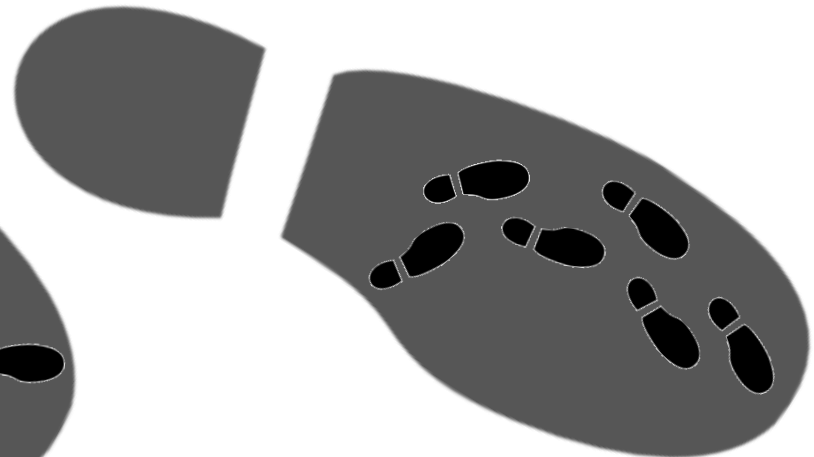
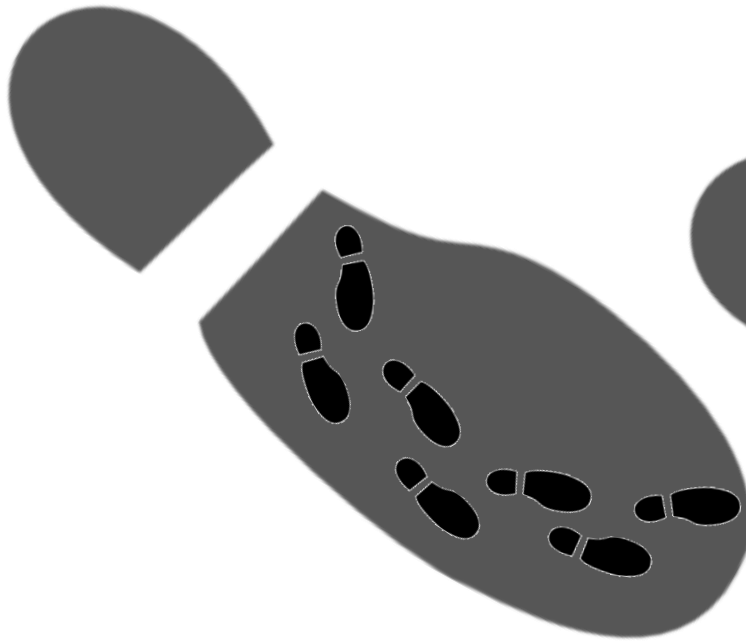
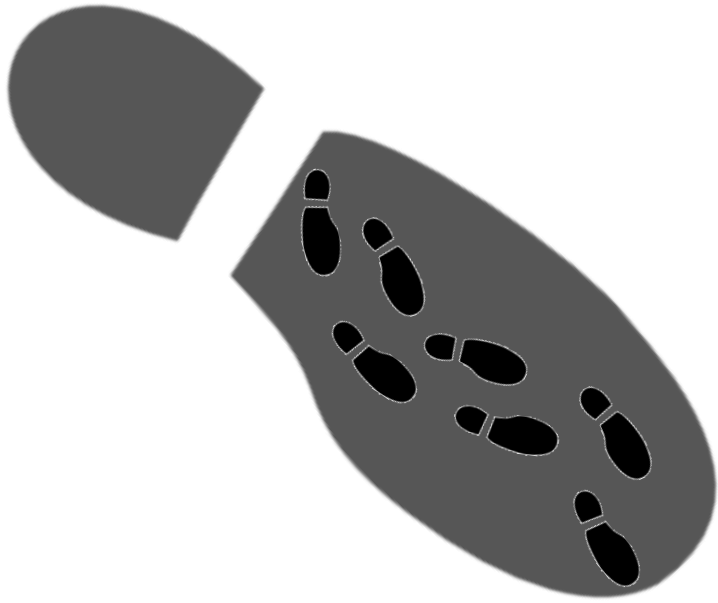


l-i-n-e-a-r

clock progression

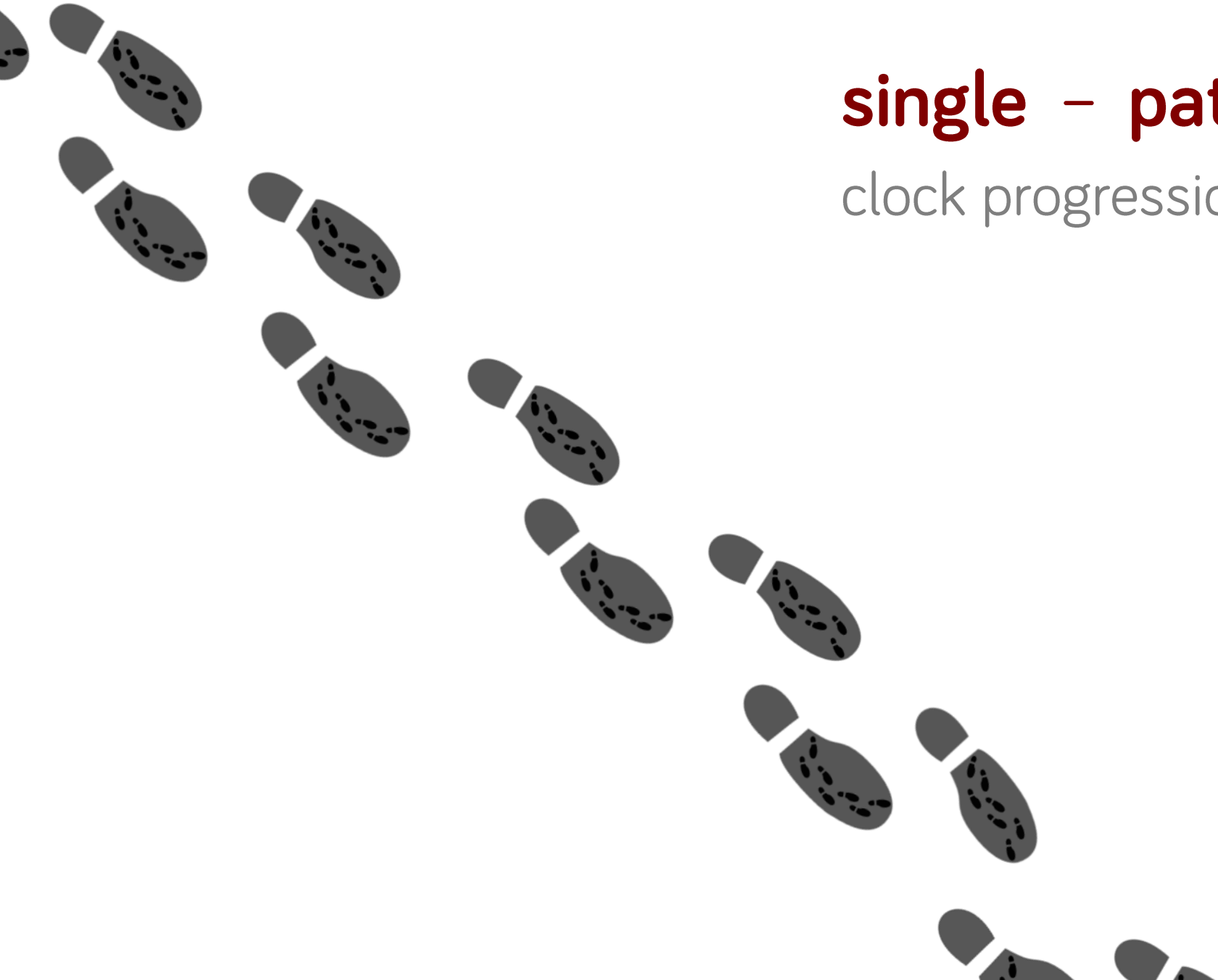
com-pos-ite

clock progression



single – path

clock progression






double = path

clock progression

double = path

clock progression





double = path

clock progression

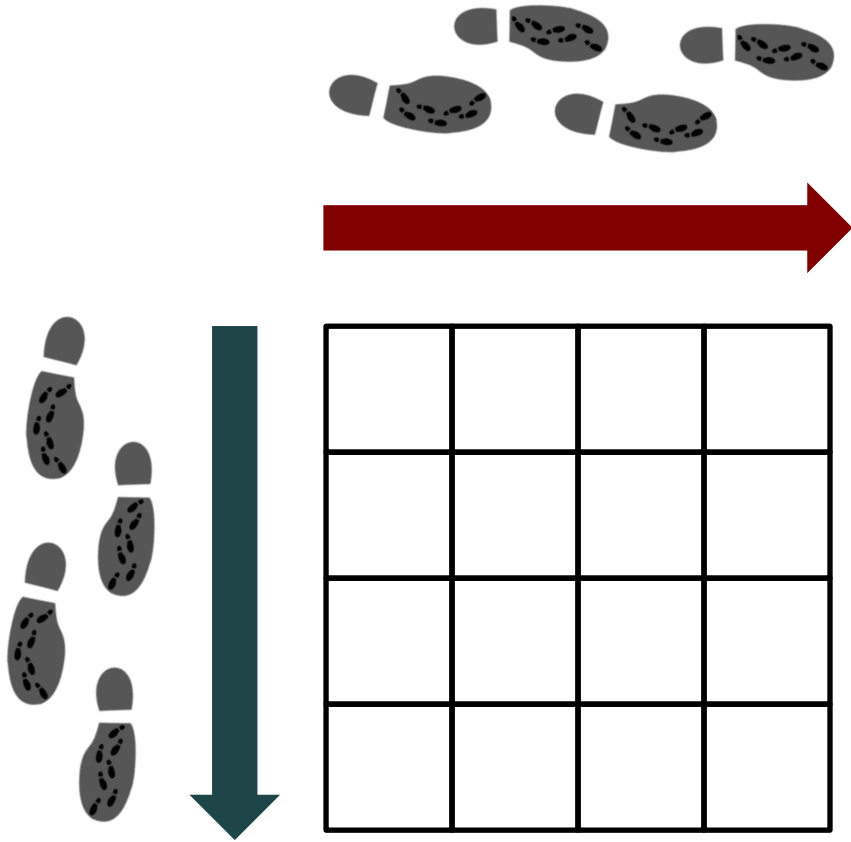


double = path

clock progression

2 clocks: 2D

clock progression



3 Constructing a clock: unary

- the domain wall 

$$\begin{aligned} |t\rangle &= |3\rangle \\ &= |10000\rangle \end{aligned}$$

3 Constructing a clock: tick, tock

- the domain wall 

$$\begin{aligned} |t\rangle &= |\mathbf{3}\rangle \\ &= |11000\rangle \end{aligned}$$

- unique transitions: 3-local

3 Constructing a clock: data-clock interaction

- the domain wall  5-local

- 2-qubit gates: 5-local
- unique transitions: 3-local

quantum 5-SAT [A. Kitaev]
quantum 4-SAT [S. Bravyi]

3 A “composite” domain wall clock

$$\begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} 110 \\ 000 \end{array} \begin{array}{r} 000 \\ 000 \end{array}$$

$$\begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} 101 \\ 000 \end{array} \begin{array}{r} 000 \\ 000 \end{array}$$

$$\begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} 110 \\ 000 \end{array}$$

$$\begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} 101 \\ 000 \end{array}$$

$$\begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} 110 \\ 000 \end{array}$$

$$\begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} + \\ \begin{array}{r} 100 \\ 011 \end{array} \end{array} \begin{array}{r} 101 \\ 000 \end{array}$$

3 Made of "legal" triplets

$$\begin{array}{r} +100 \\ 011 \end{array} \quad 110 \quad 000 \quad 000$$

$$\begin{array}{r} +100 \\ 011 \end{array} \quad 101 \quad 000 \quad 000$$

$$\begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad 110 \quad 000$$

$$\begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad 101 \quad 000$$

$$\begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad 110$$

$$\begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad 101$$

3 Enforce sequences of triplets

$$\begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} 110 \\ 110 \end{array} \quad \begin{array}{r} 000 \\ 000 \end{array} \quad \begin{array}{r} 000 \\ 000 \end{array}$$

$$\begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} 101 \\ 101 \end{array} \quad \begin{array}{r} 000 \\ 000 \end{array} \quad \begin{array}{r} 000 \\ 000 \end{array}$$

$$\begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} 110 \\ 110 \end{array} \quad \begin{array}{r} 000 \\ 000 \end{array}$$

$$\begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} 101 \\ 101 \end{array} \quad \begin{array}{r} 000 \\ 000 \end{array}$$

$$\begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} 110 \\ 110 \end{array}$$

$$\begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} +100 \\ 011 \end{array} \quad \begin{array}{r} 101 \\ 101 \end{array}$$

3 Look at the differences between lines

$\begin{matrix} +100 \\ 011 \end{matrix}$	110	000	000
$\begin{matrix} +100 \\ 011 \end{matrix}$	101	000	000
$\begin{matrix} +100 \\ 011 \end{matrix}$	$\begin{matrix} +100 \\ 011 \end{matrix}$	110	000
$\begin{matrix} +100 \\ 011 \end{matrix}$	$\begin{matrix} +100 \\ 011 \end{matrix}$	101	000
$\begin{matrix} +100 \\ 011 \end{matrix}$	$\begin{matrix} +100 \\ 011 \end{matrix}$	$\begin{matrix} +100 \\ 011 \end{matrix}$	110
$\begin{matrix} +100 \\ 011 \end{matrix}$	$\begin{matrix} +100 \\ 011 \end{matrix}$	$\begin{matrix} +100 \\ 011 \end{matrix}$	101

3 Look at the differences between lines

$$\begin{array}{r}
 + \begin{array}{l} 100 \\ 011 \end{array} \quad 110 \quad 000 \quad 000 \\
 + \begin{array}{l} 100 \\ 011 \end{array} \quad 101 \quad 000 \quad 000 \\
 + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad 110 \quad 000 \\
 + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad 101 \quad 000 \\
 + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad 110 \\
 + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad 101
 \end{array}$$

a projector on

$$\left| \begin{array}{l} 100 \\ -011 \end{array} \right\rangle$$

3 Look at the differences, combine lines

$$\begin{array}{r}
 + \begin{array}{l} 100 \\ 011 \end{array} \quad 110 \quad 000 \quad 000 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 + \begin{array}{l} 100 \\ 011 \end{array} \quad 101 \quad 000 \quad 000 \\
 + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad 110 \quad 000 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad 101 \quad 000 \\
 + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad 110 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad + \begin{array}{l} 100 \\ 011 \end{array} \quad 101 \\
 \hline
 \end{array}$$

a projector on

$$\left| \begin{array}{l} 100 \\ -011 \end{array} \right\rangle$$

ensures they
appear together

3 A clock made from superpositions

$$|1\rangle_c \quad + \begin{array}{c} 100 \\ 011 \end{array} \quad 110 \quad 000 \quad 000$$


A diagram of a 4-qubit chain represented by four circles connected by a horizontal line. The first circle from the left contains a red dot, while the other three are empty.

$$|2\rangle_c \quad + \begin{array}{c} 100 \\ 011 \end{array} \quad 101 \quad 000 \quad 000 \\ + \begin{array}{c} 100 \\ 011 \end{array} \quad + \begin{array}{c} 100 \\ 011 \end{array} \quad 110 \quad 000$$


A diagram of a 4-qubit chain represented by four circles connected by a horizontal line. The first and second circles from the left contain red dots, while the other two are empty.

$$|3\rangle_c \quad + \begin{array}{c} 100 \\ 011 \end{array} \quad + \begin{array}{c} 100 \\ 011 \end{array} \quad 101 \quad 000 \\ + \begin{array}{c} 100 \\ 011 \end{array} \quad + \begin{array}{c} 100 \\ 011 \end{array} \quad + \begin{array}{c} 100 \\ 011 \end{array} \quad 110$$


A diagram of a 4-qubit chain represented by four circles connected by a horizontal line. The first, second, and third circles from the left contain red dots, while the fourth is empty.

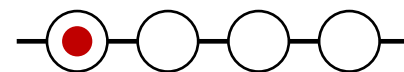
$$|4\rangle_c \quad + \begin{array}{c} 100 \\ 011 \end{array} \quad + \begin{array}{c} 100 \\ 011 \end{array} \quad + \begin{array}{c} 100 \\ 011 \end{array} \quad 101$$


A diagram of a 4-qubit chain represented by four circles connected by a horizontal line. All four circles from the left contain red dots.

3 A clock with 2-local progress

$|1\rangle_c$

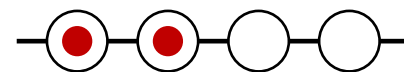
$$+ \begin{array}{cc} 100 & 110 \\ 011 & \end{array} \quad 000 \quad 000$$



$|2\rangle_c$

$$+ \begin{array}{cc} 100 & 101 \\ 011 & \end{array} \quad 000 \quad 000$$

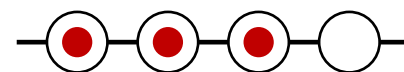
$$+ \begin{array}{cc} 100 & 110 \\ 011 & \end{array} \quad 000$$



$|3\rangle_c$

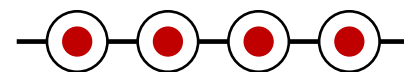
$$+ \begin{array}{cc} 100 & 101 \\ 011 & \end{array} \quad + \begin{array}{cc} 100 & 110 \\ 011 & \end{array} \quad 000$$

$$+ \begin{array}{cc} 100 & 110 \\ 011 & \end{array} \quad + \begin{array}{cc} 100 & 110 \\ 011 & \end{array} \quad + \begin{array}{cc} 100 & 110 \\ 011 & \end{array} \quad 110$$



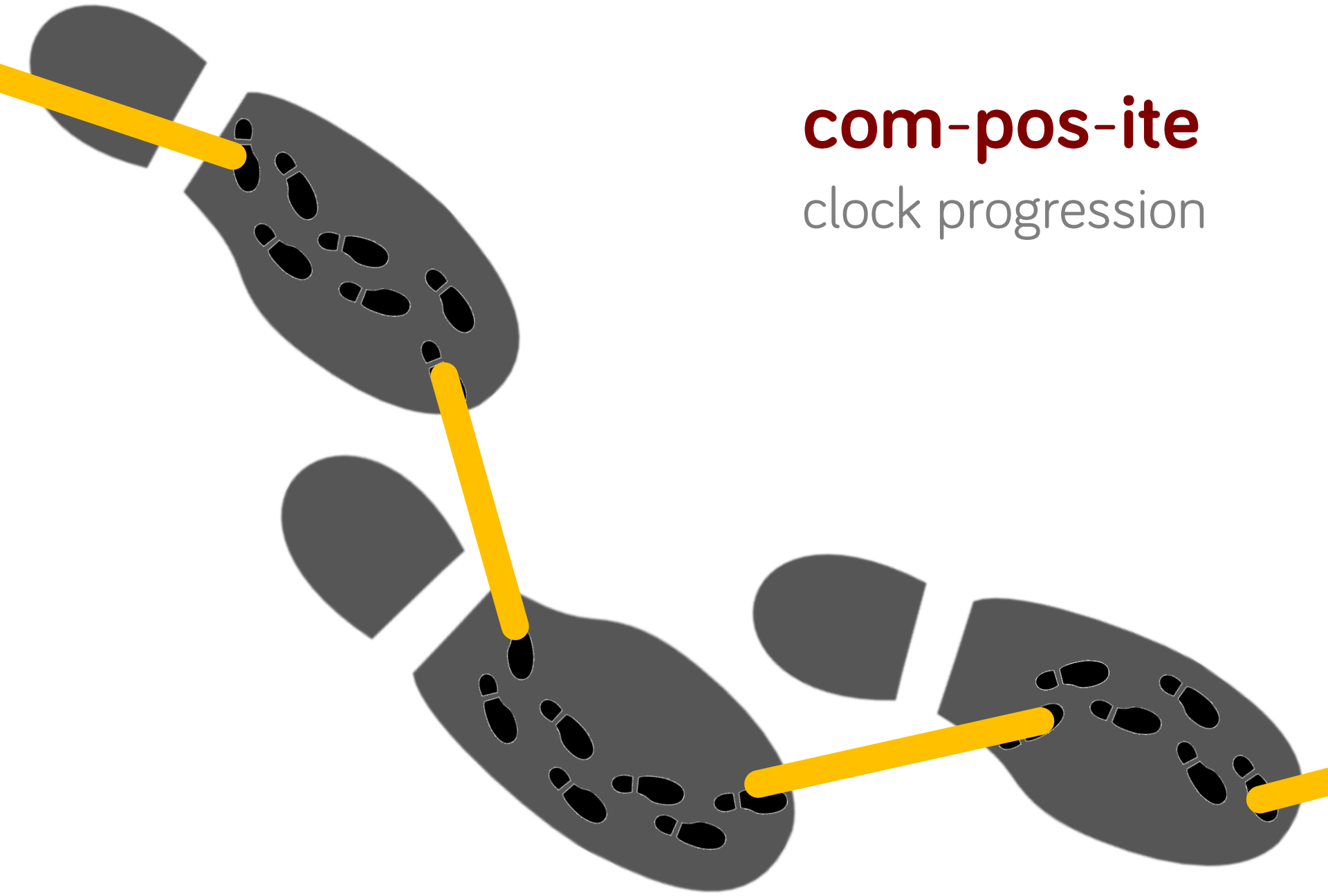
$|4\rangle_c$

$$+ \begin{array}{cc} 100 & 101 \\ 011 & \end{array} \quad + \begin{array}{cc} 100 & 101 \\ 011 & \end{array} \quad + \begin{array}{cc} 100 & 101 \\ 011 & \end{array} \quad 101$$



com-pos-ite

clock progression



3 Constructing clocks

- advance the clock

2-locally

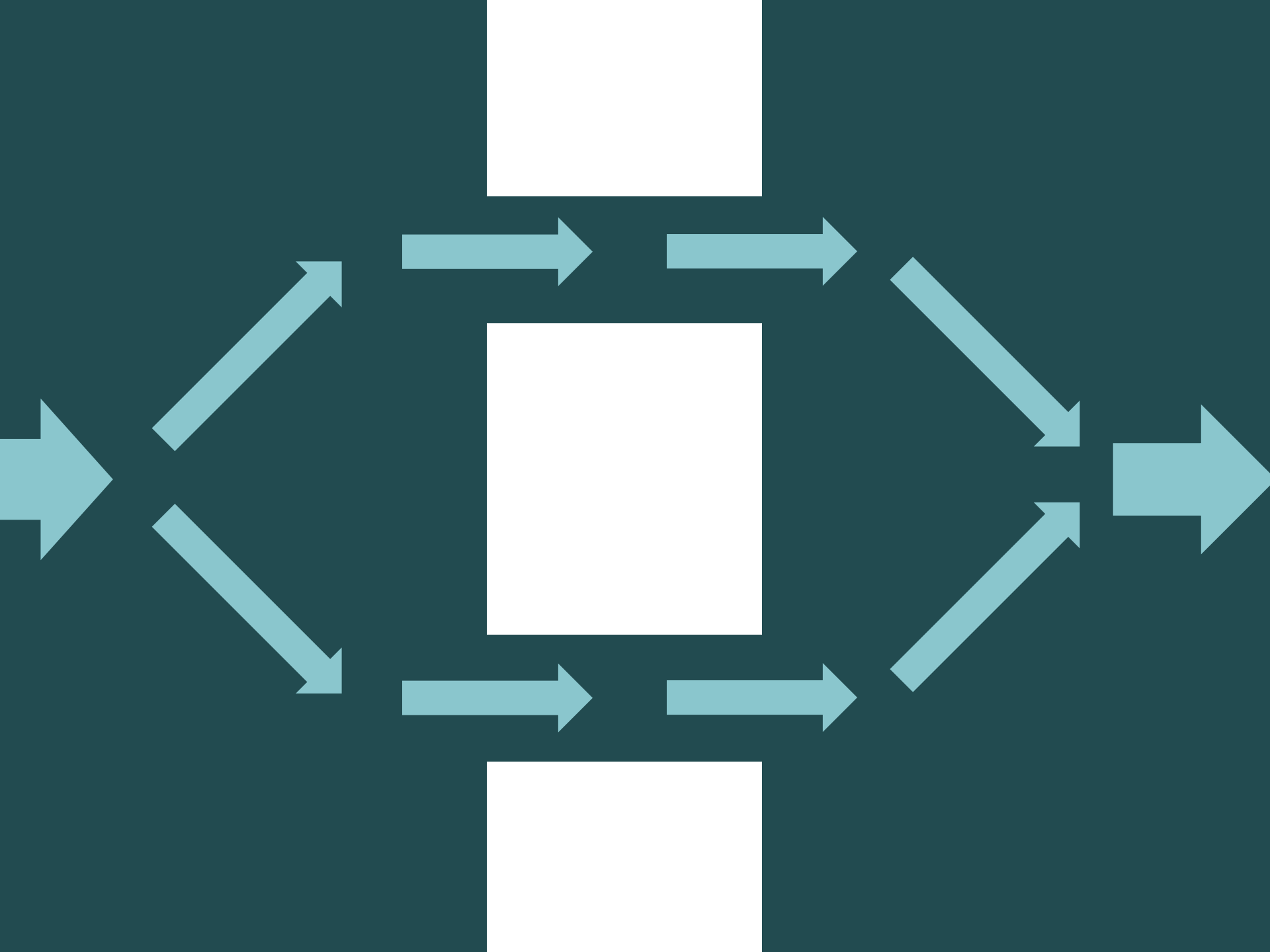
- find out I'm late

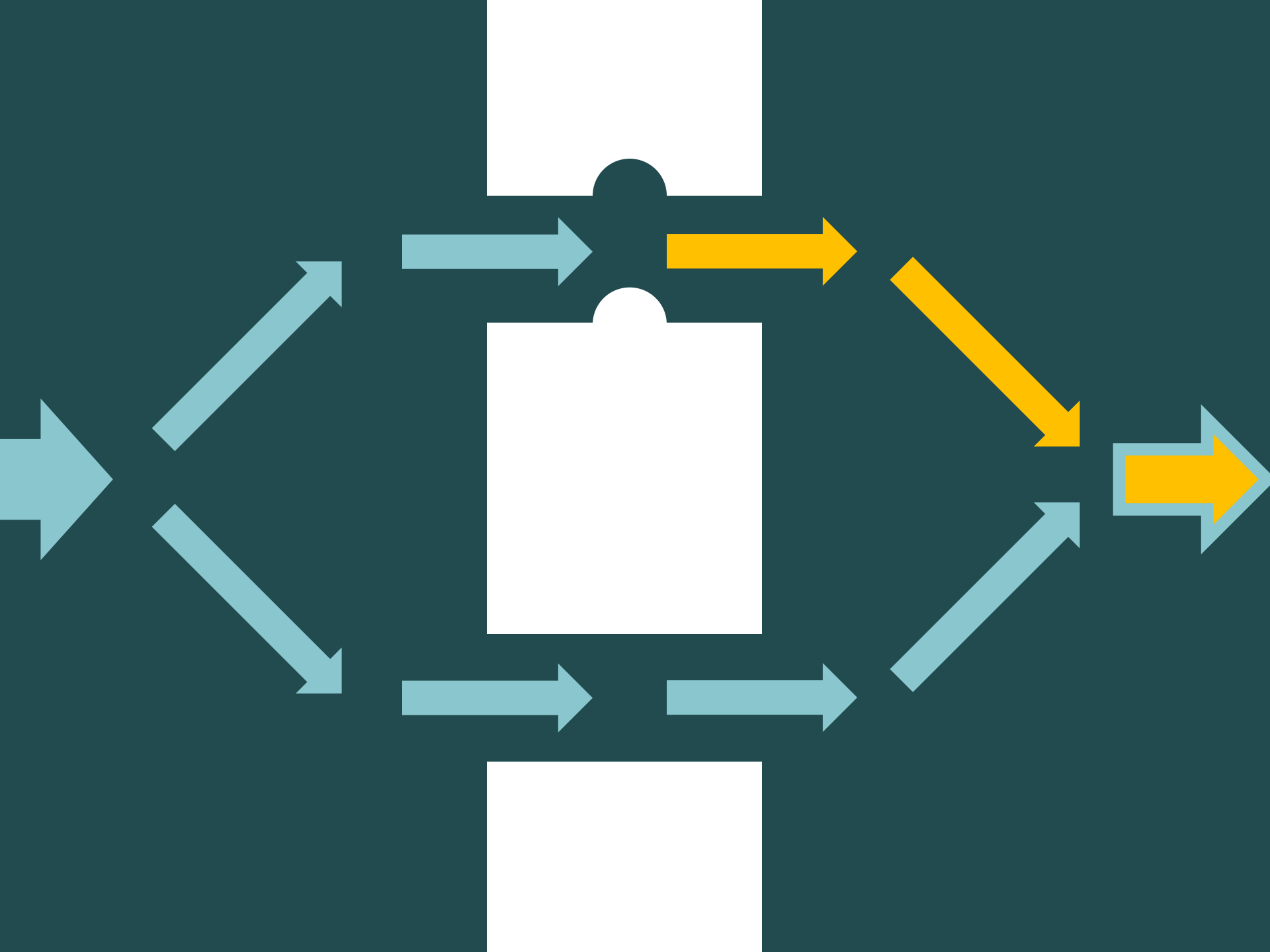
1-locally

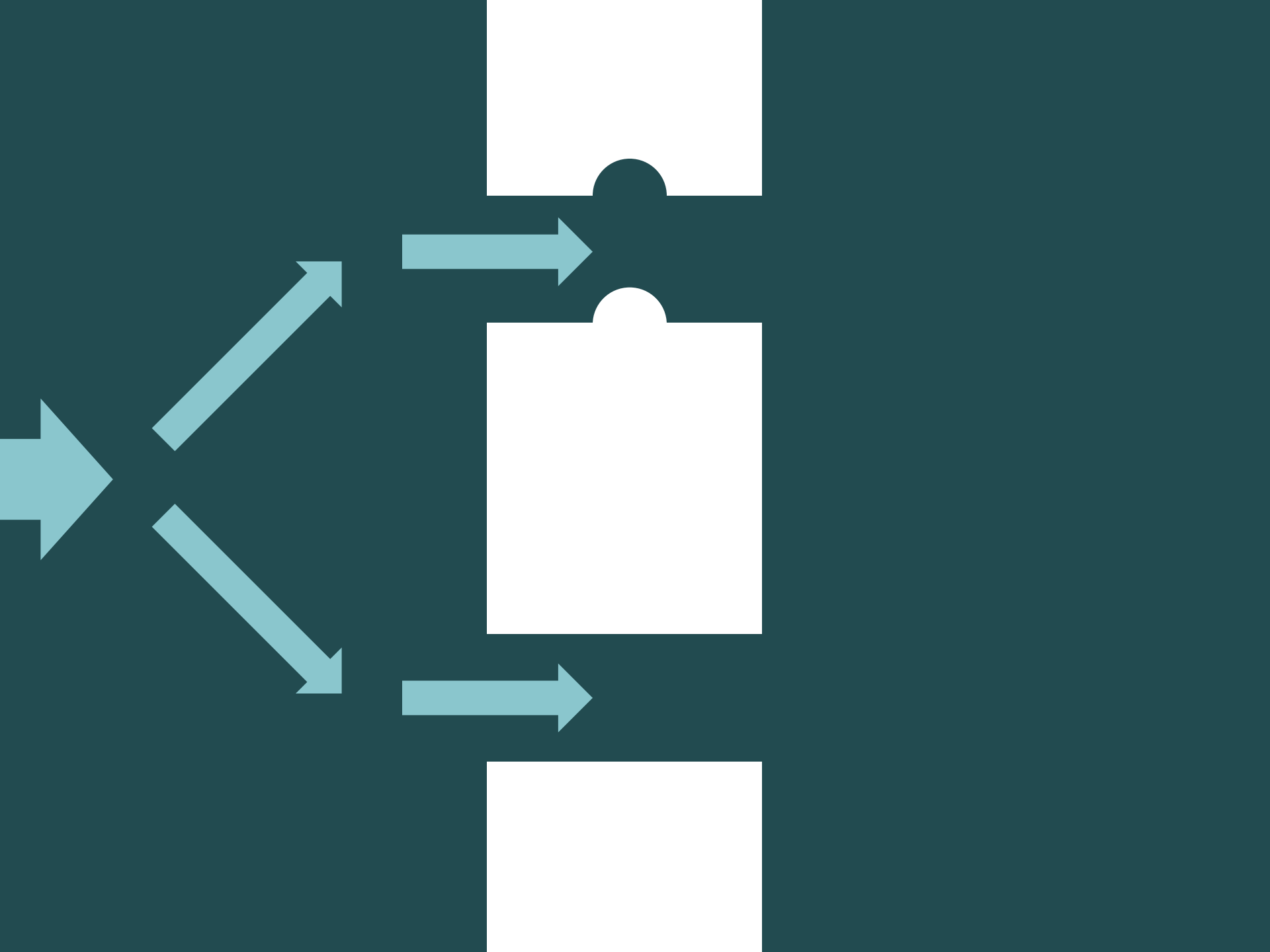
- apply a CNOT

3-locally?

$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	110	000	000	000	000	000	10000
$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	101	000	000	000	000	000	11000
$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	110	000	000	000	000	11000
$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	101	000	000	000	000	11000
$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	110	000	000	000	11100
$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	101	000	000	000	11100
$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	110	000	000	11100
$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	$\begin{smallmatrix} 100 \\ 011 \end{smallmatrix}$	101	000	11110



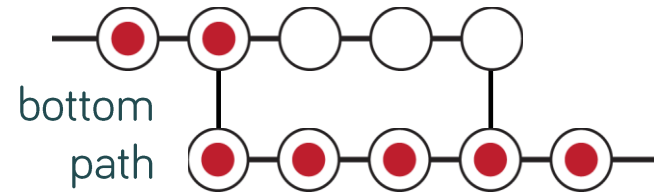
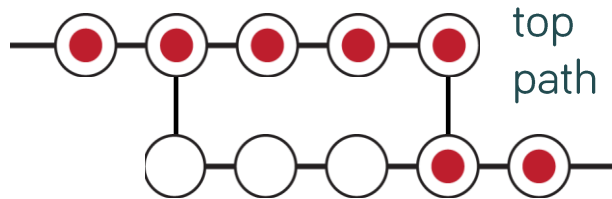
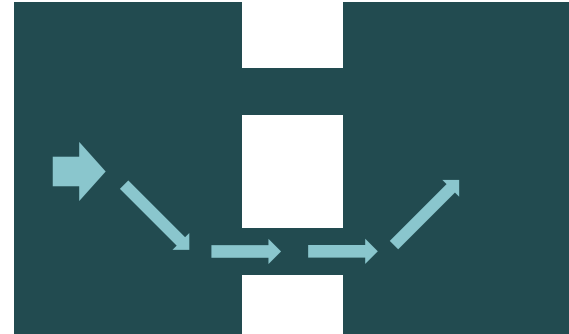
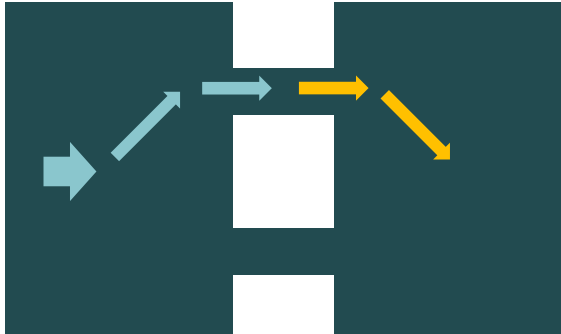




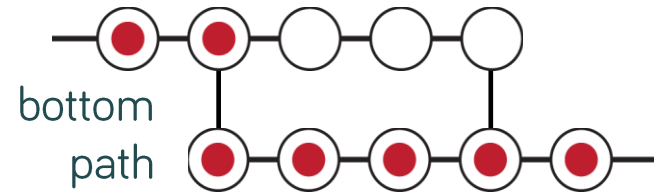
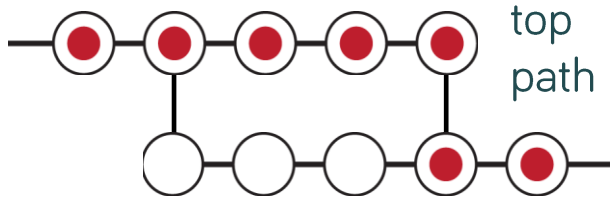
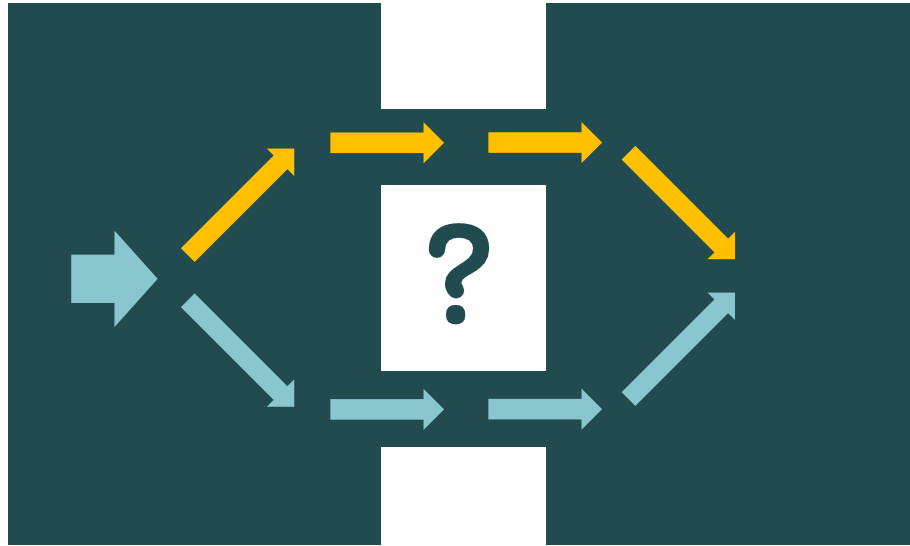




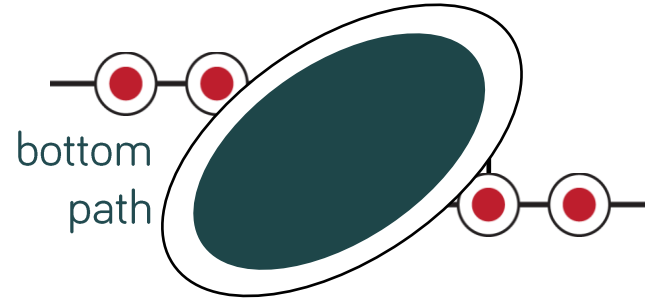
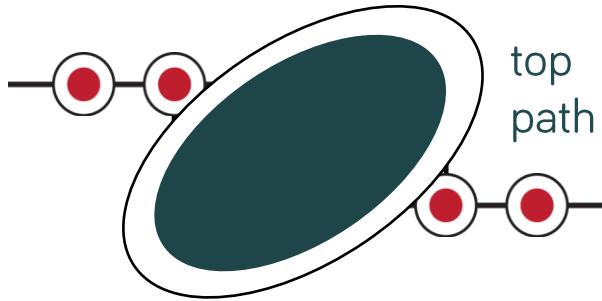
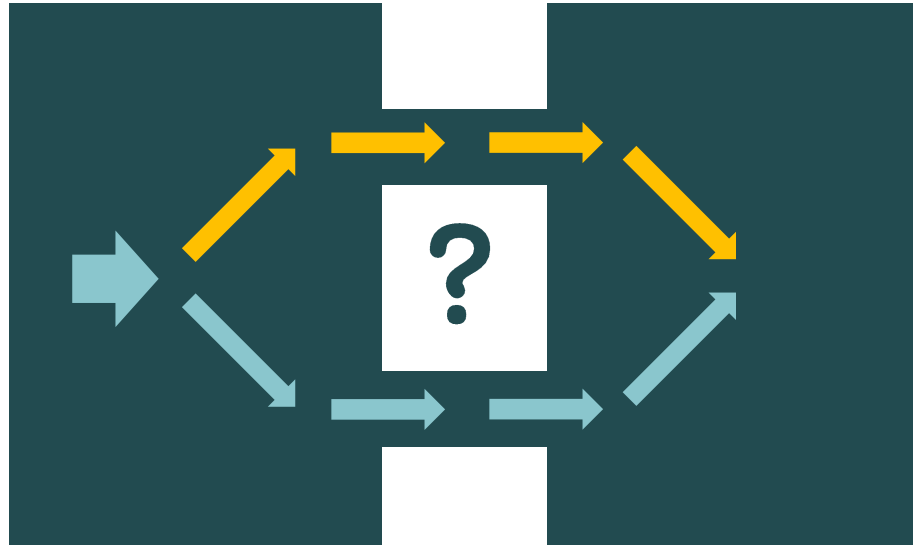
4 A botched double-slit experiment



4 Can we make it work?

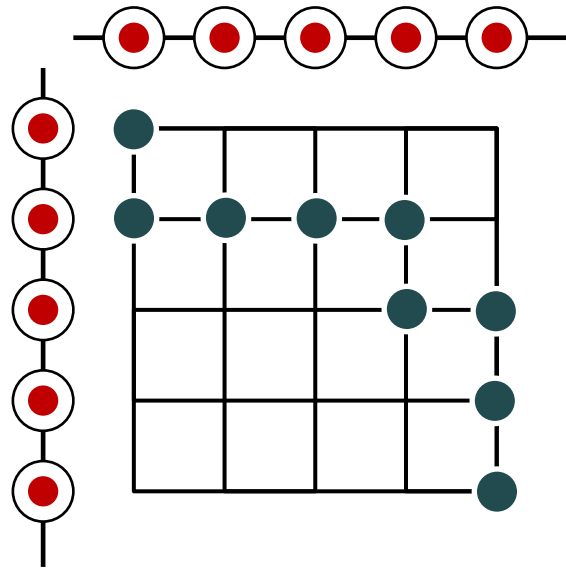


4 Can we make it work?



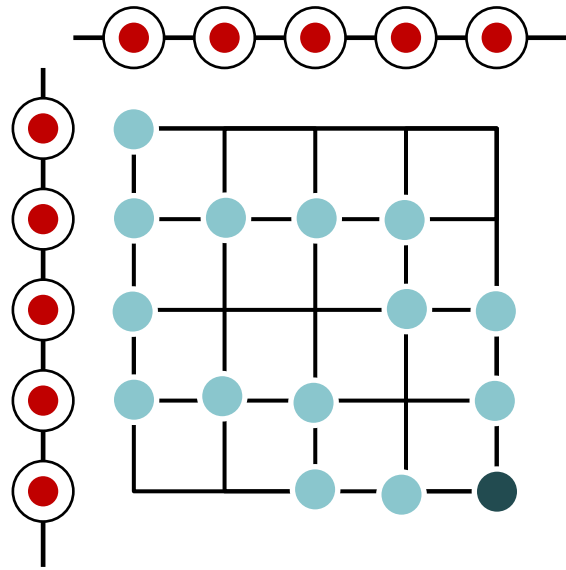
4 2D time

- two clocks



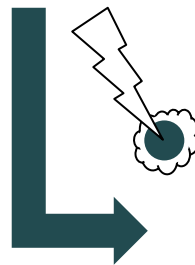
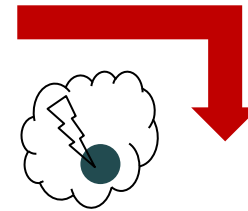
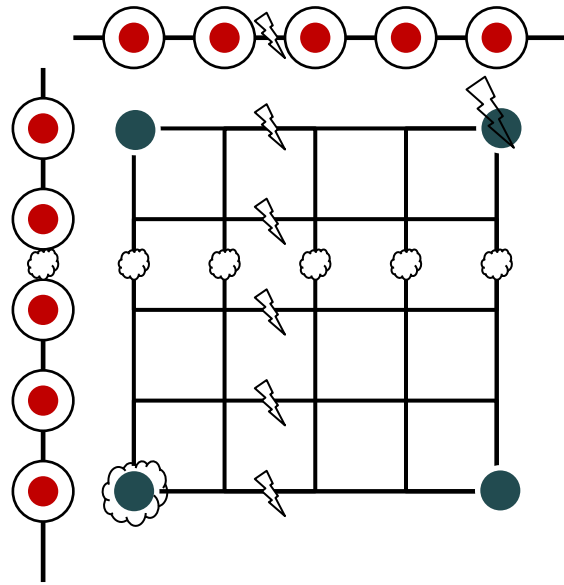
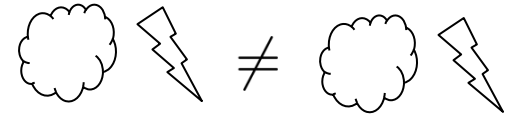
4 2D time

- two clocks



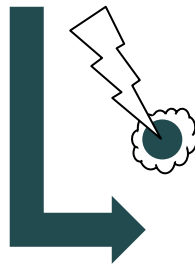
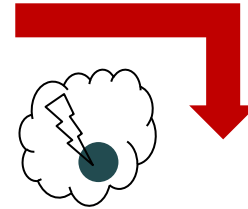
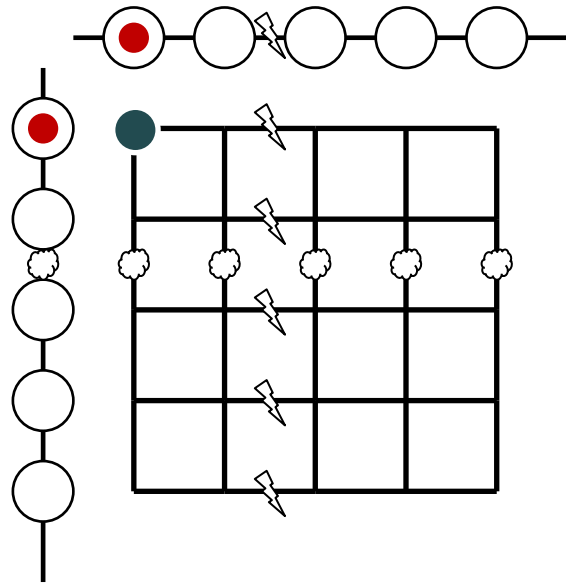
4 2D time

- non-commuting (data) operations



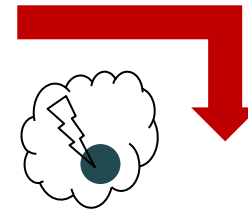
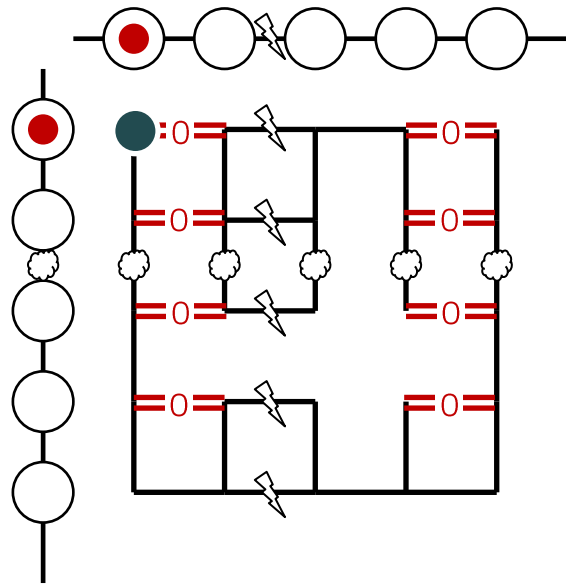
4 2D time + data

- remove/control transitions

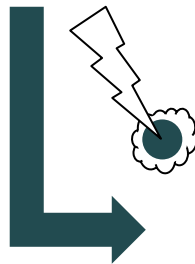


4 2D time + data

- remove/control transitions

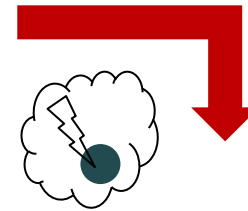
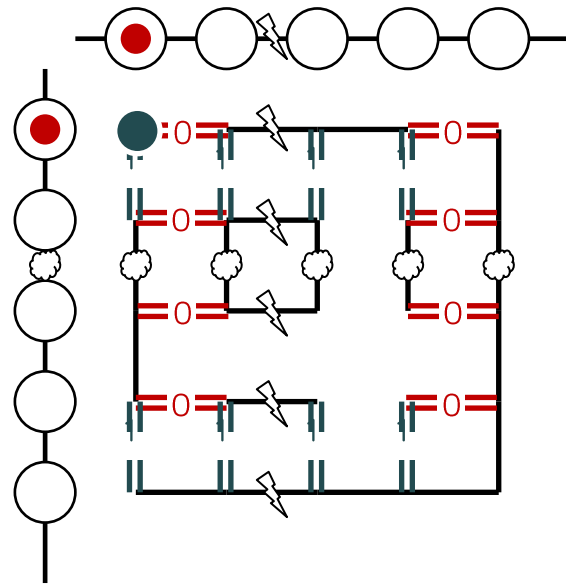


control qubit: **0**



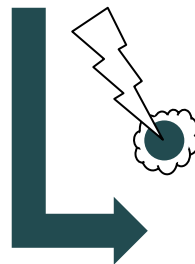
4 2D time + data

- remove/control transitions



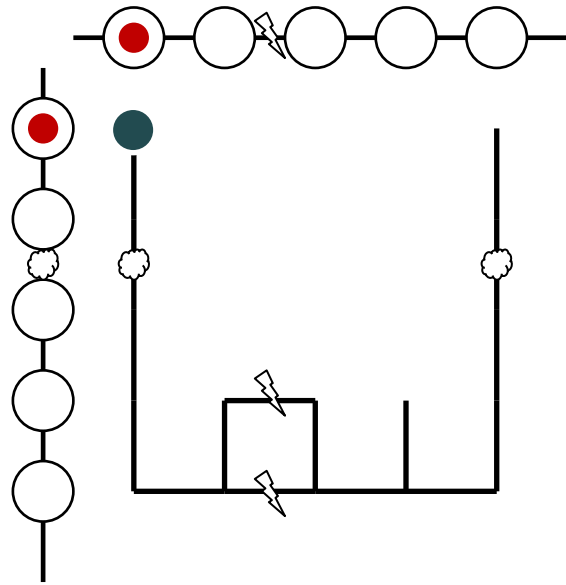
control qubit: **0**

control qubit: **1**

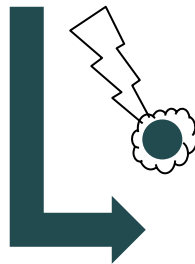


4 2D time + data

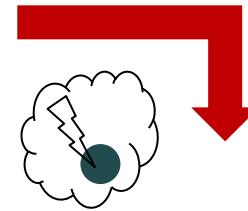
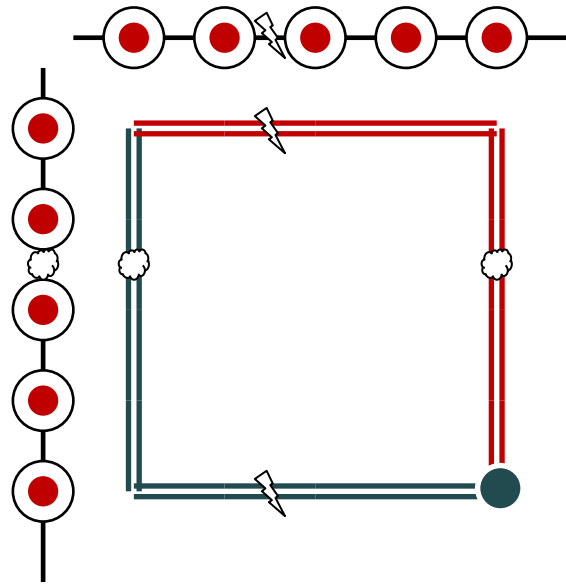
- remove/control transitions



control
qubit:



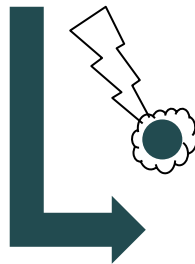
4 Check a controlled gate

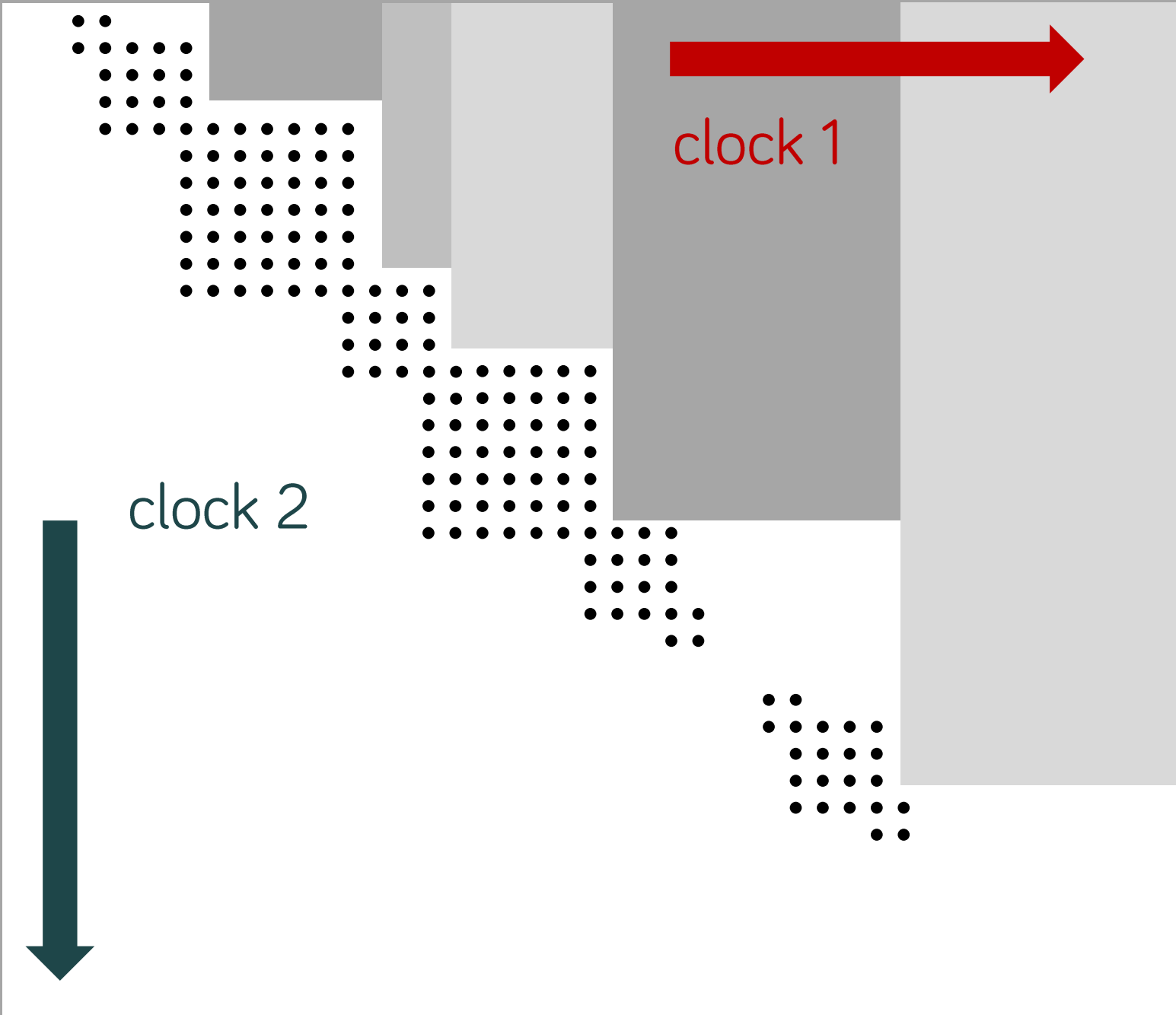


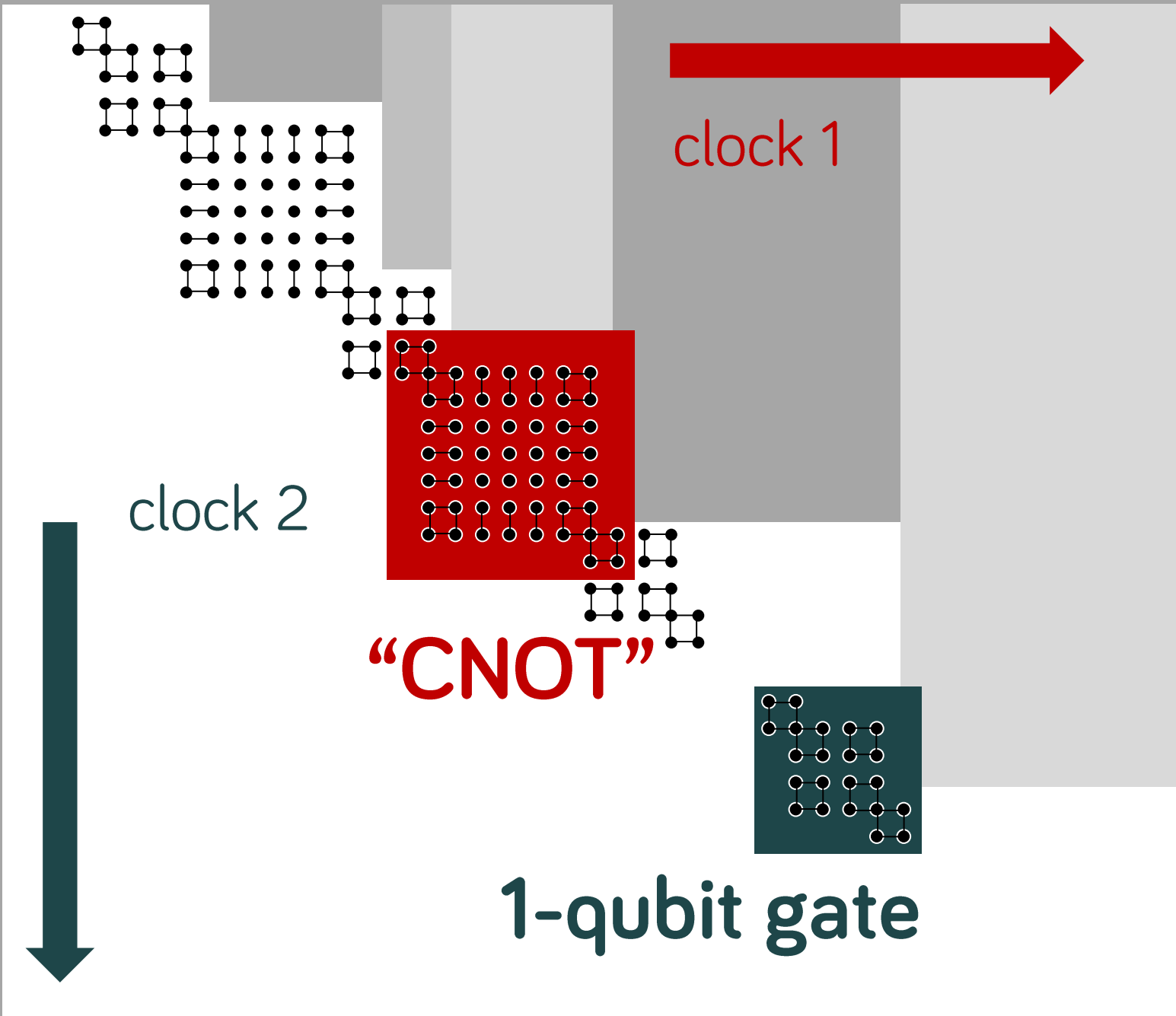
control qubit: **0**

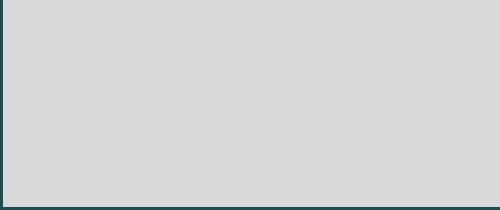
control qubit:

1

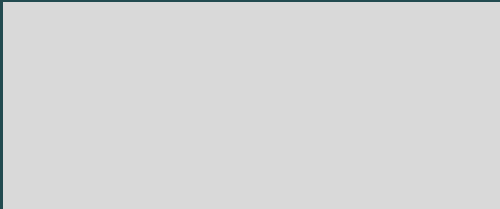
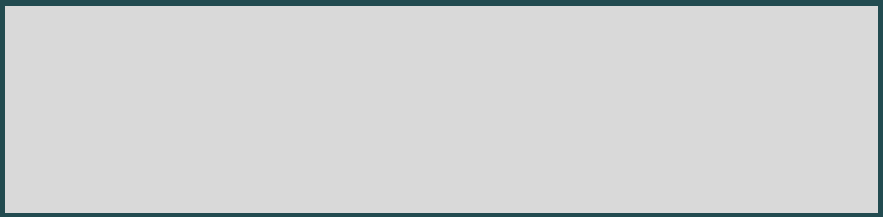
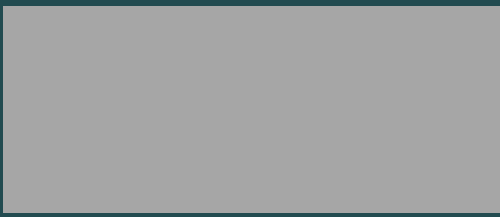
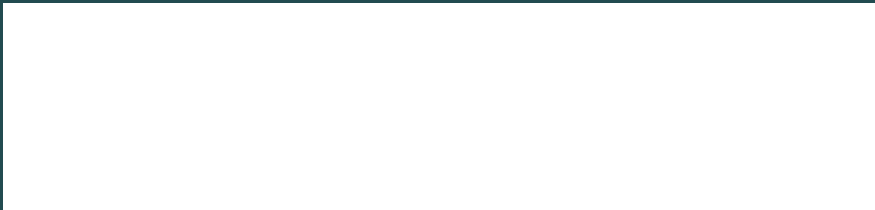








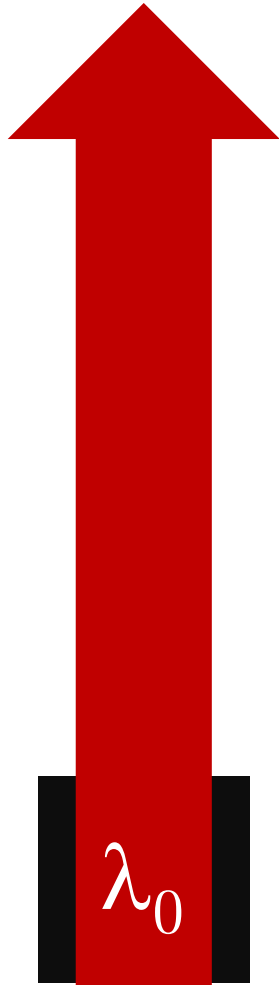
soundness



Is the smallest eigenvalue large?

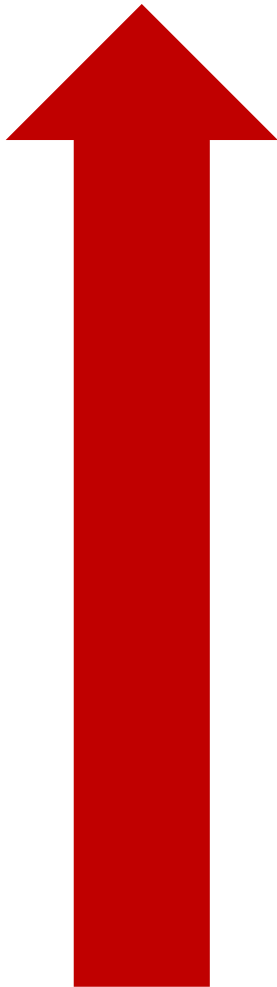
$$H_A + H_B + H_C + H_D + H_E$$





$$H_A + H_B + H_C + H_D + H_E + H_F$$

no solution?
all states have
a high energy



quantum 3-SAT
is QMA_1 -complete

[Gosset, N. '13]

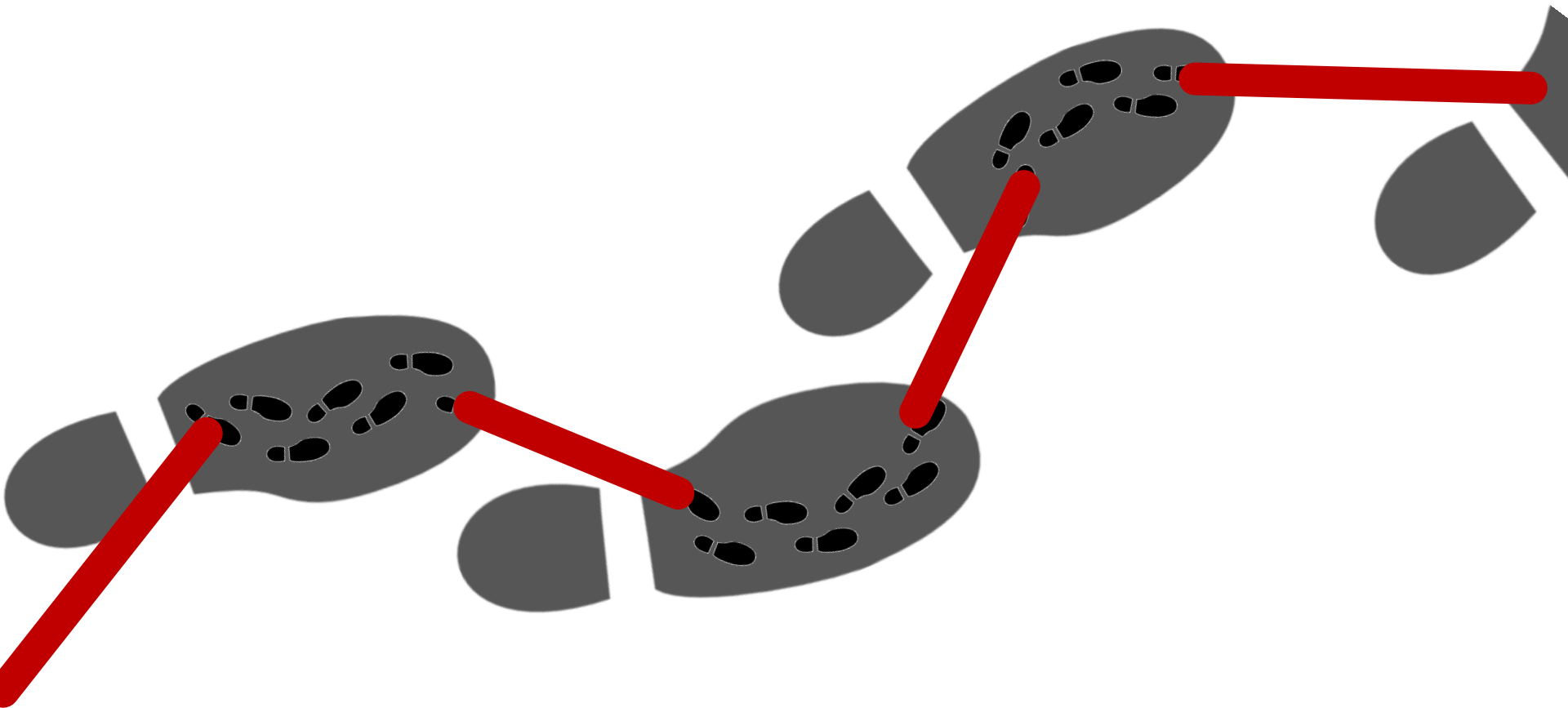
quantum
3-sat

&

projectors
no frustration
computation

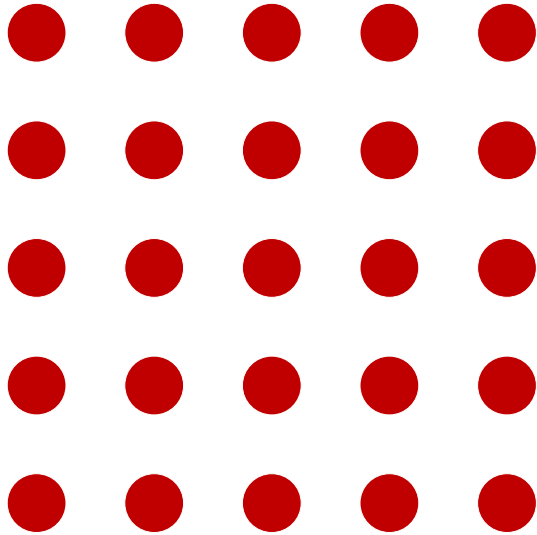


a clock

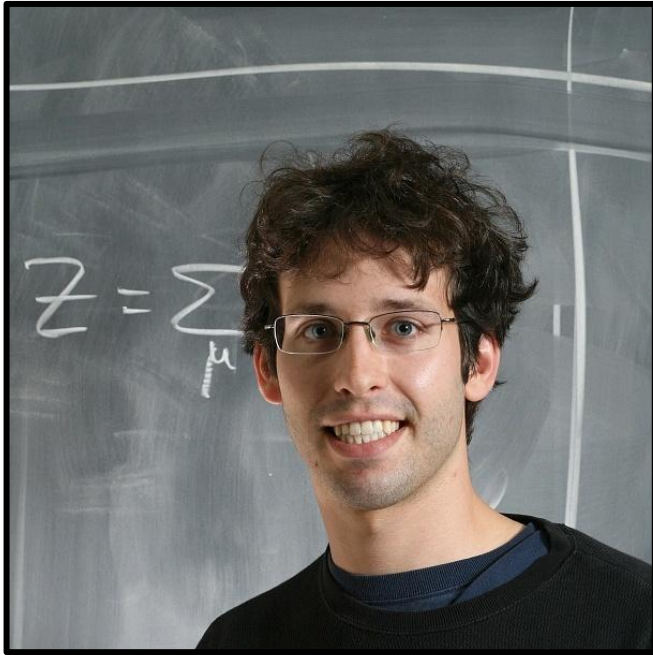


a clock

for quantum
computation



3-local
composite
runs in 2D



Quantum 3-SAT

is QMA_1 -complete

FOCS '13

David Gosset 

Daniel Nagaj  universität
wien