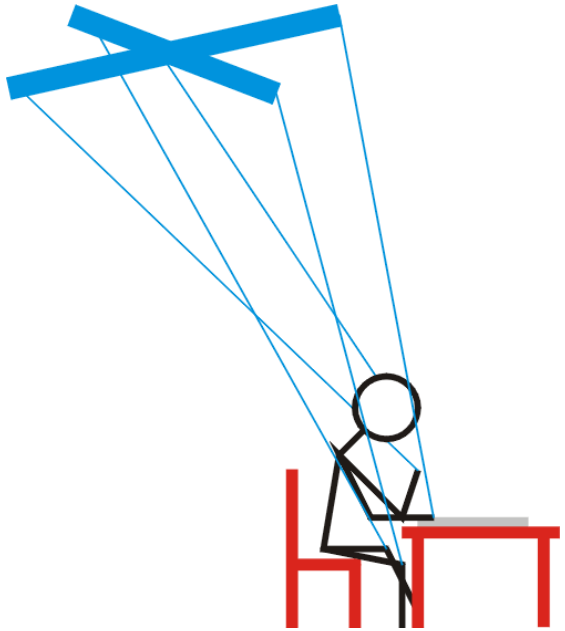


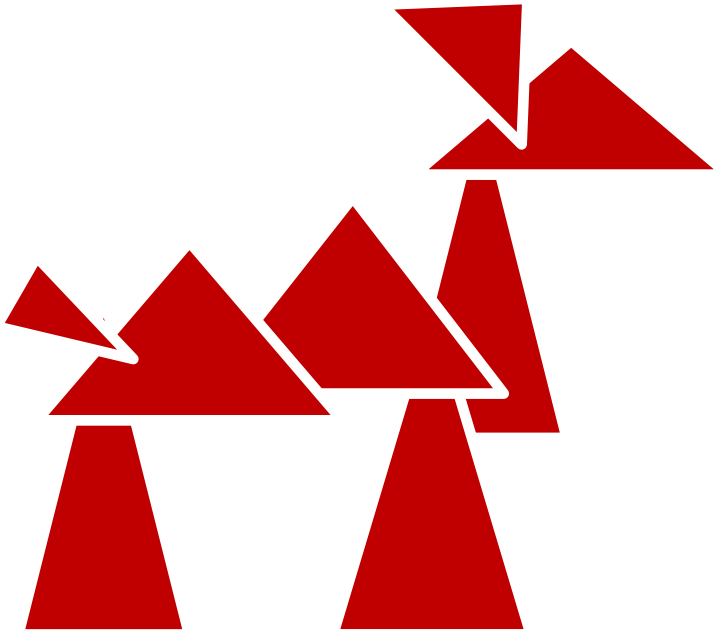
An introduction to

Local Hamiltonians & Quantum Complexity

D. Gosset, S. Hallgren, S. Narayanaswami,
M. Bardoscia, A. Scardicchio,
S. Mozes, P. Wocjan, O. Regev, P. Love, P. Shor, S. Jordan,
S. Lloyd, A. Landahl, S. Irani, D. Gottesman, S. Bravyi,
E. Farhi, J. Goldstone, D. Aharonov, T. Morimae,
R. Movassagh, F. Brandao, J. Eisert, J. Whitfield, ...

Daniel Nagaj

universität wien



Recent news about

Quantum Satisfiability

D. Gosset, S. Hallgren, S. Narayanaswami,
M. Bardoscia, A. Scardicchio,
S. Mozes, P. Wocjan, O. Regev, P. Love, P. Shor, S. Jordan,
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R. Movassagh, F. Brandao, J. Eisert, J. Whitfield, ...

Daniel Nagaj

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Quantum SAT

[wisegeek.com]

YESTERDAY QMA

TODAY QMA_1

0 why Hamiltonians

1 a little complexity

2 QMA, ground states and local Hamiltonians

3 what can we do in 1D

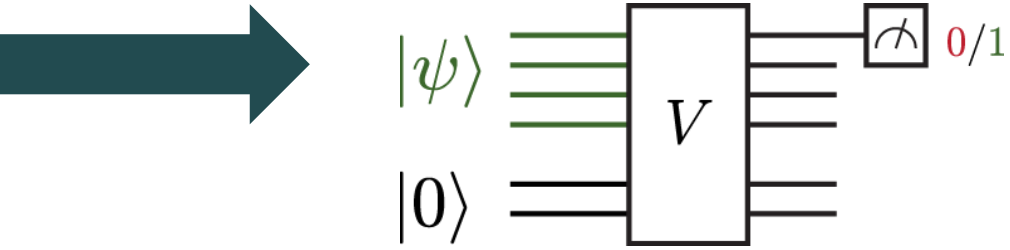
4 no frustration & perfect verifiers

5 quantum 3-SAT

6 random quantum SAT

7 quantum PCP

0 Review of QMA and LH



$$H_{clock} + H_{init} + H_{prop} + H_{out}$$

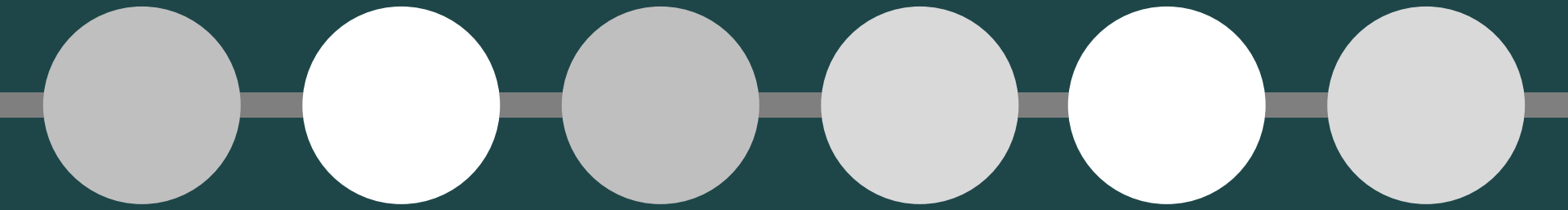
NO no witness accepted by V more likely than ε

any state has a high energy

YES there is a proof accepted by V with probability 1-ε

the history state of V acting on the proof has a low energy

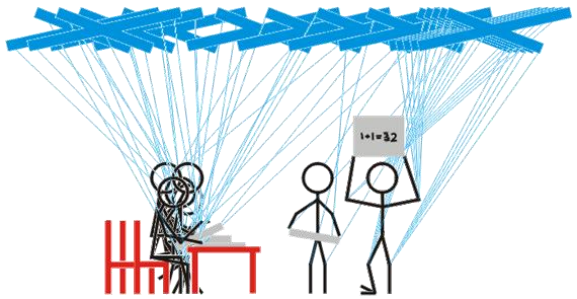
l-H-a-m-i-l-t-o-n-i-a-n-s-i-n-1-D-L-o-c-a-l-H-a-m-i-l-t-o-n-i-a-n-s-i-n-1-D-L-o-c-a-l-H-a-m



3 History states in 1D

- the history state: a ground state

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\psi_t\rangle$$

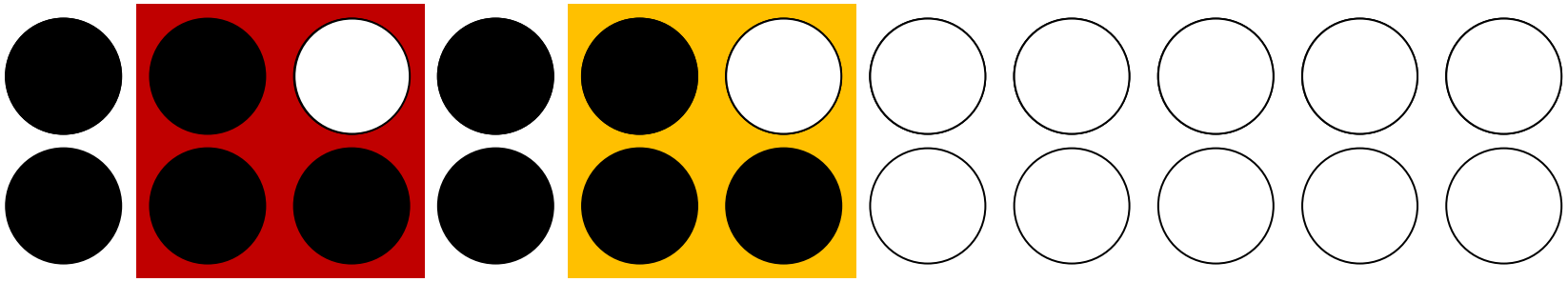


- the goal: a propagation Hamiltonian with 2-locally checkable transitions

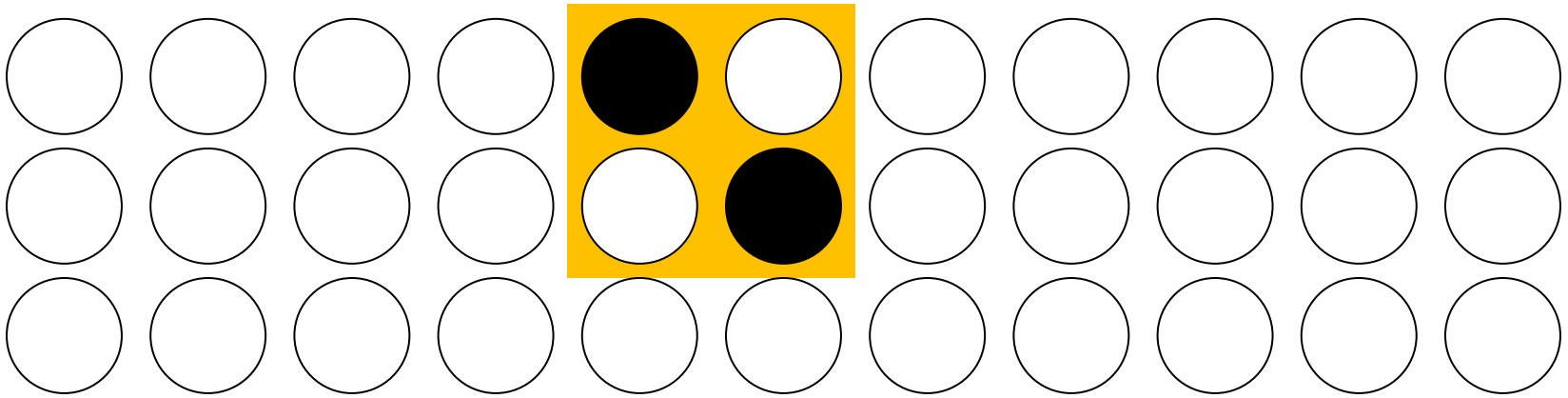
$$\frac{1}{2}\left(|ab\rangle - |cd\rangle\right)\left(\langle ab| - \langle cd|\right)$$

$$\cdots |ab\rangle \cdots$$
$$+ \cdots |cd\rangle \cdots$$

3 Moving data in 1D

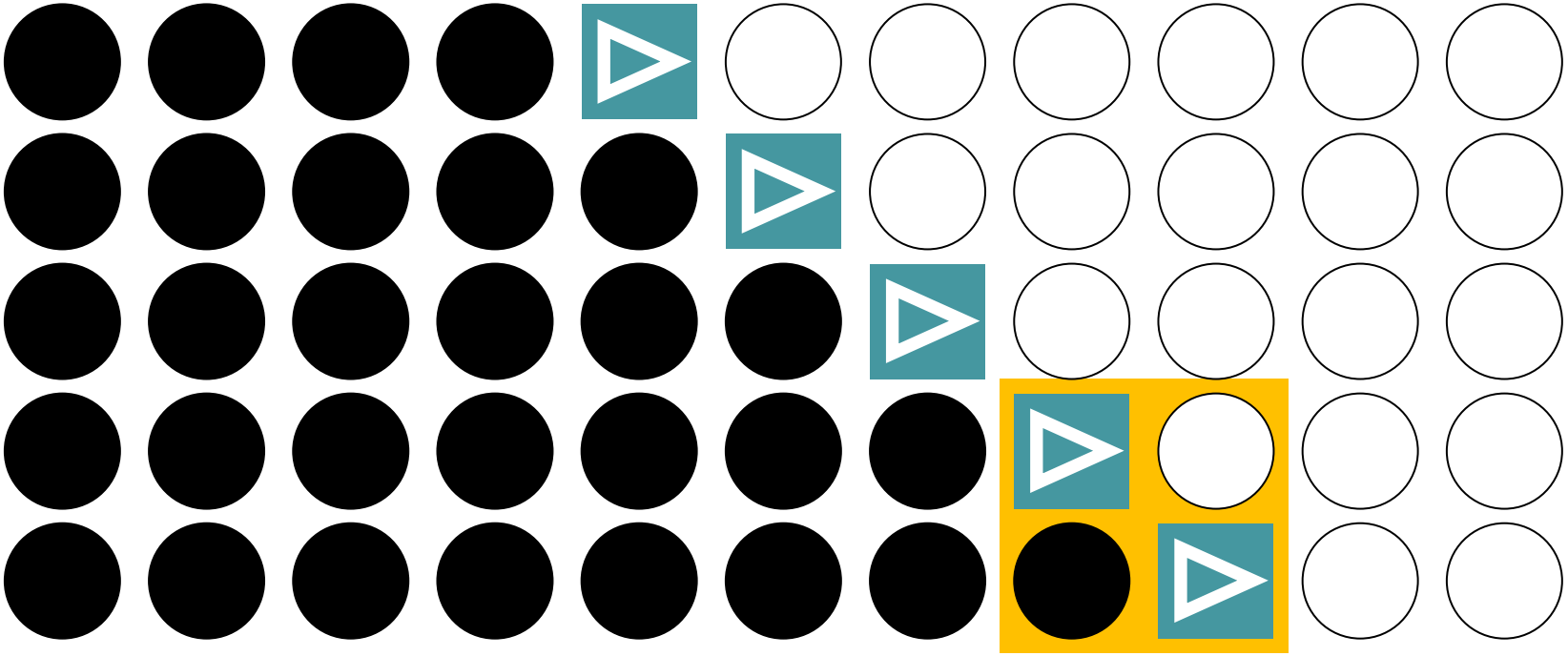


a domain wall can't move uniquely

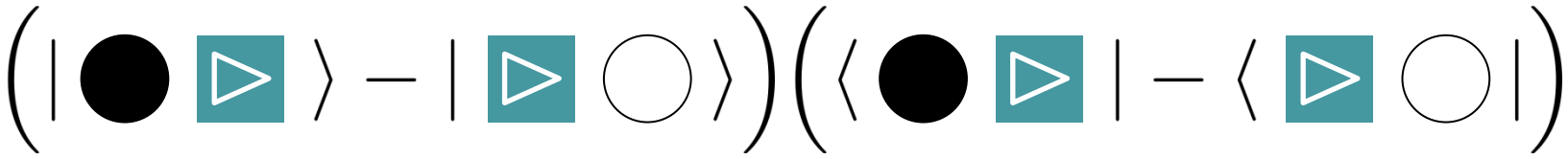


a pulse is bad, because of dead states

using more states (qudits) helps



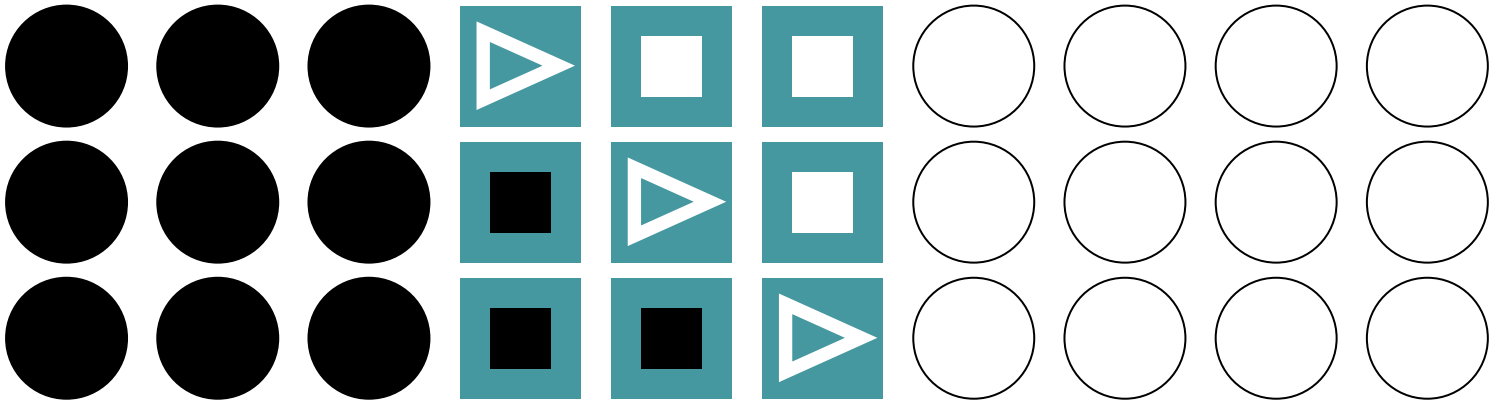
a surfer on a domain wall



[Aharonov, Gottesman, Irani, Kempe]

the **power** of quantum systems on a line

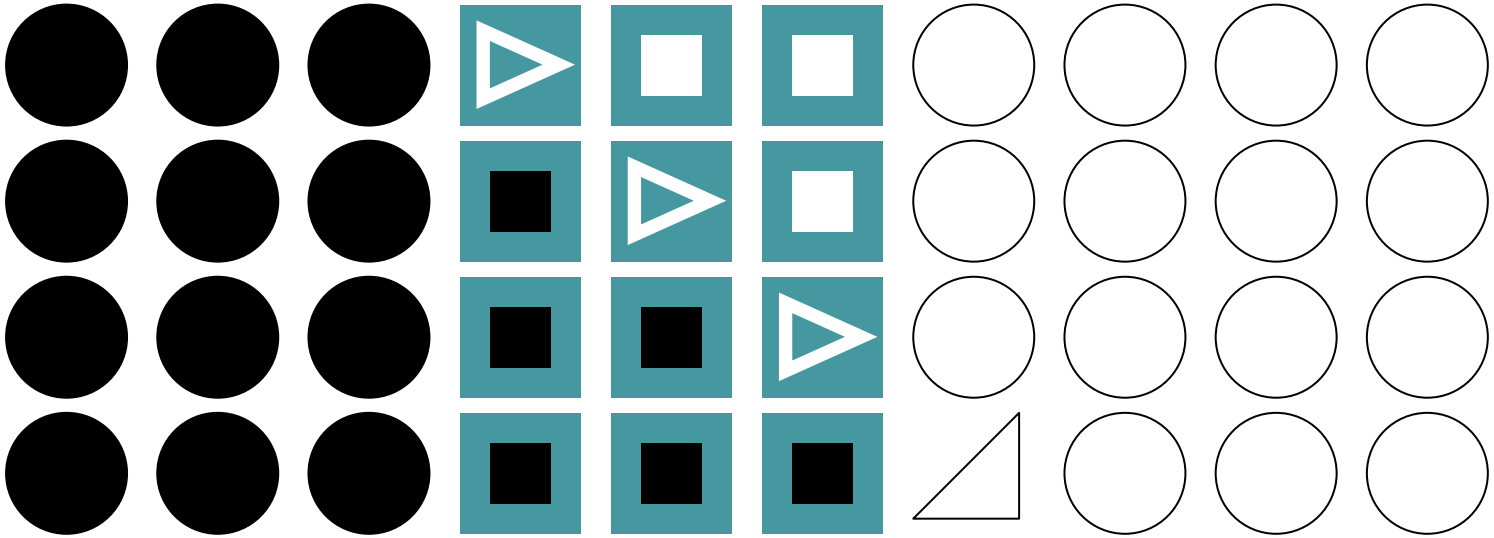
[Aharonov, Gottesman, Irani, Kempe]



A DOMAIN wall structure

the **power** of quantum systems on a line

[Aharonov, Gottesman, Irani, Kempe]

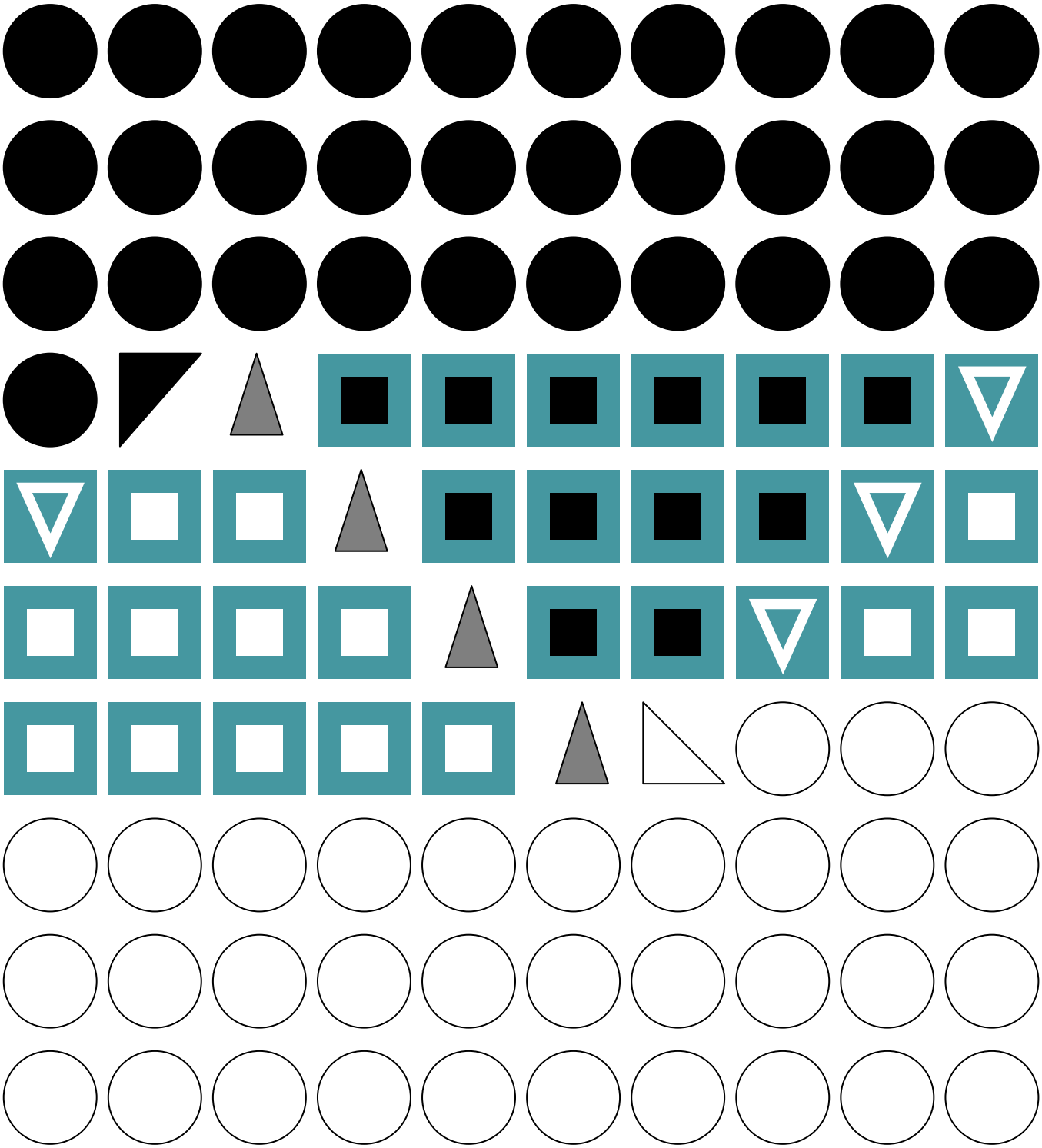


$$\cdots |ab\rangle \cdots$$
$$+ \cdots |cd\rangle \cdots$$

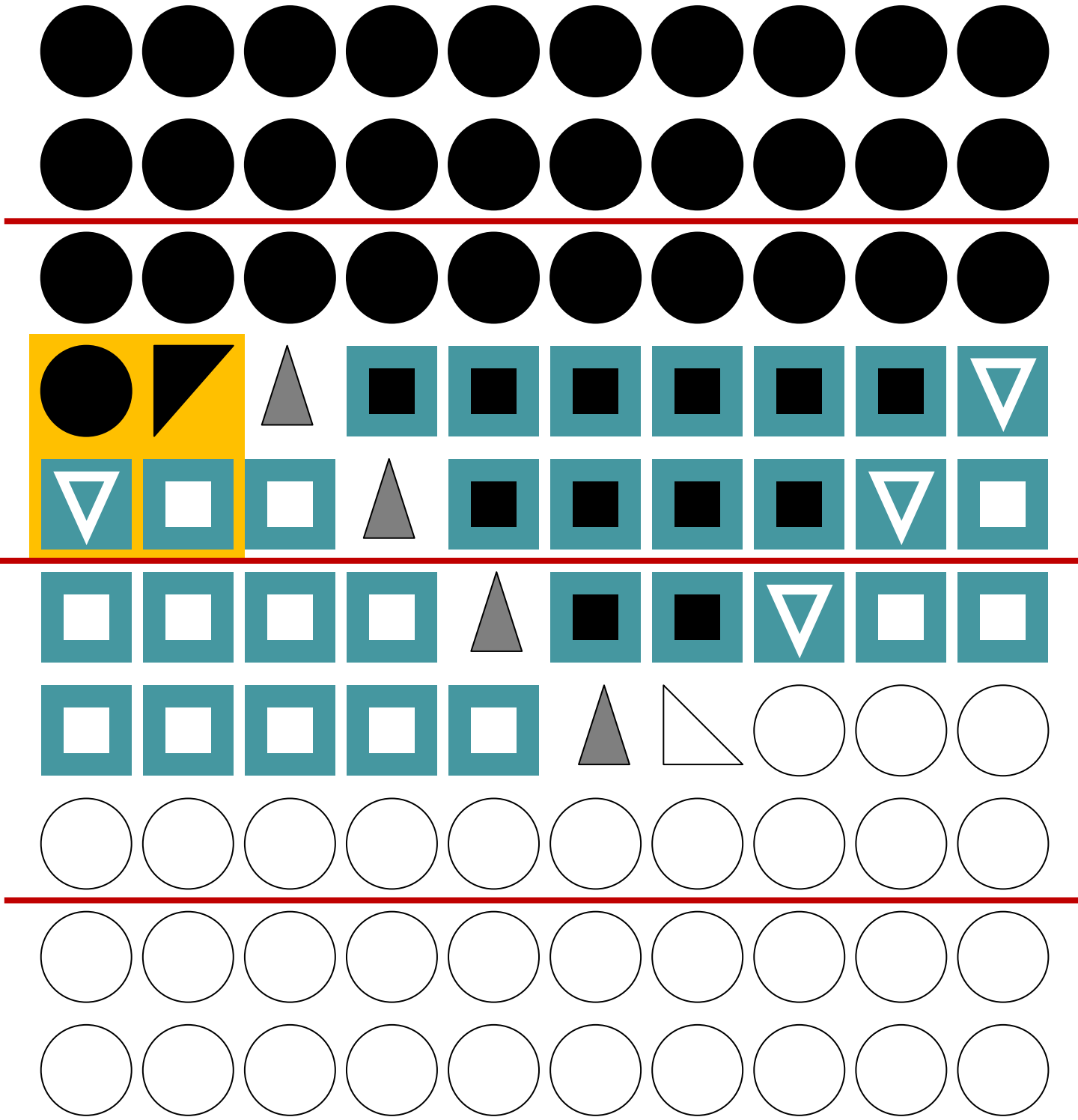
$$\tfrac{1}{2}\big(|ab\rangle - |cd\rangle\big)\big(\langle ab| - \langle cd|\big)$$

moving the data

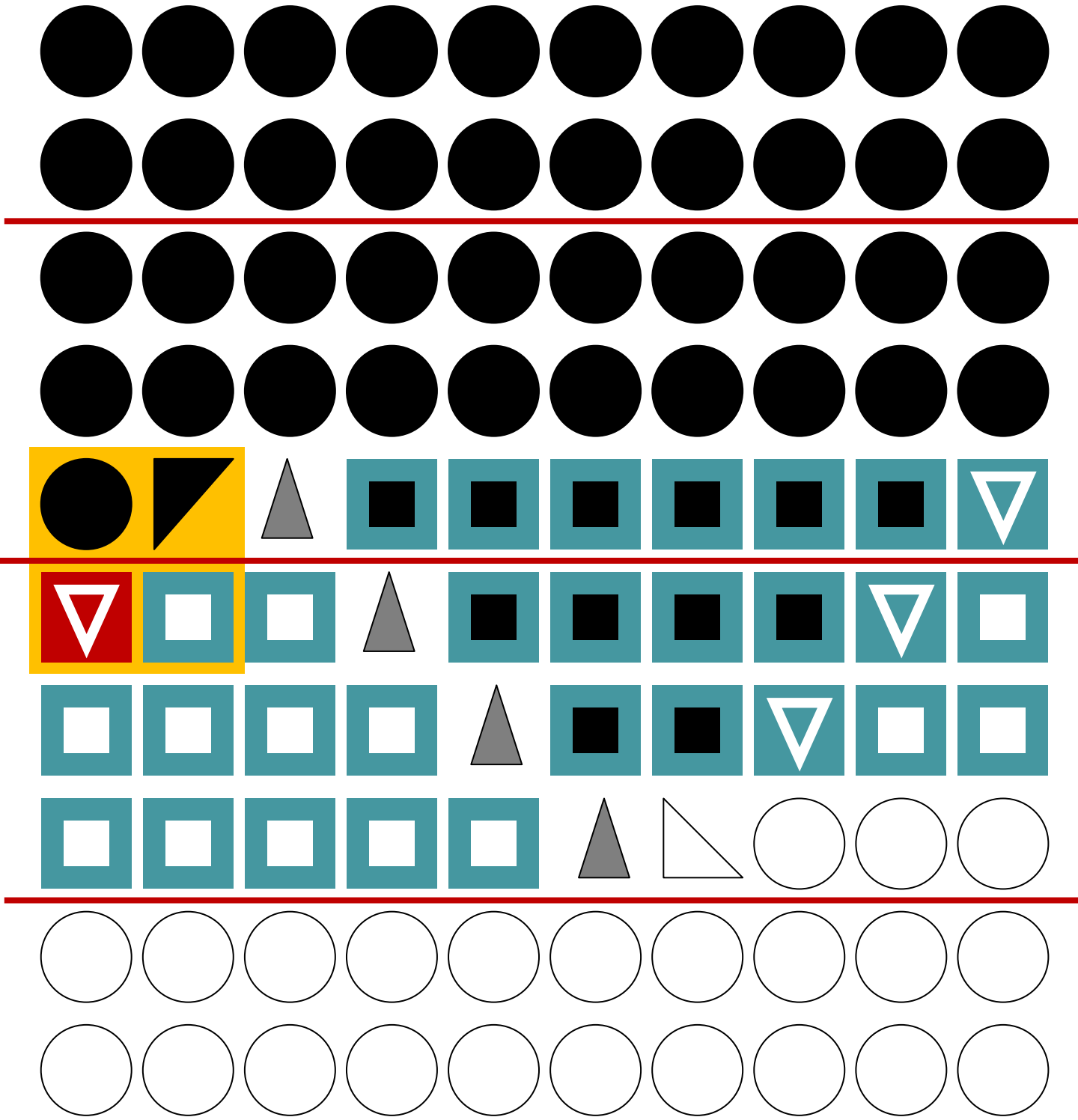
[Aharonov, Gottesman, Irani, Kempe]



[Aharonov, Gottesman, Irani, Kempe]



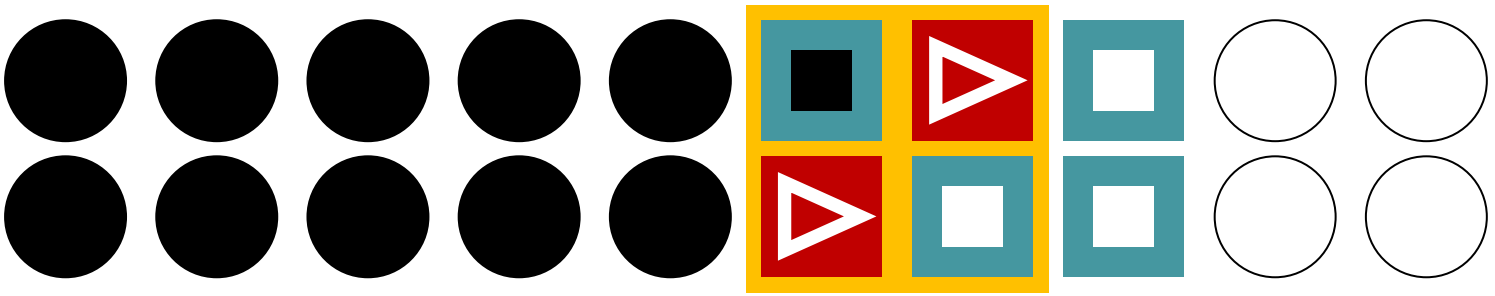
[Aharonov, Gottesman, Irani, Kempe]



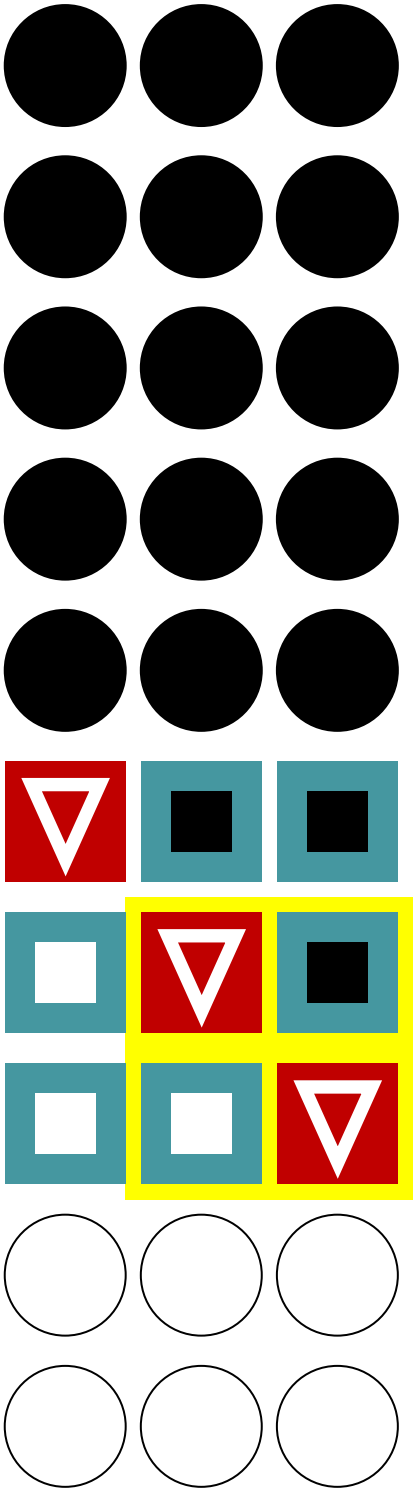
applying gates

[Aharonov, Gottesman, Irani, Kempe]

U_{ab}

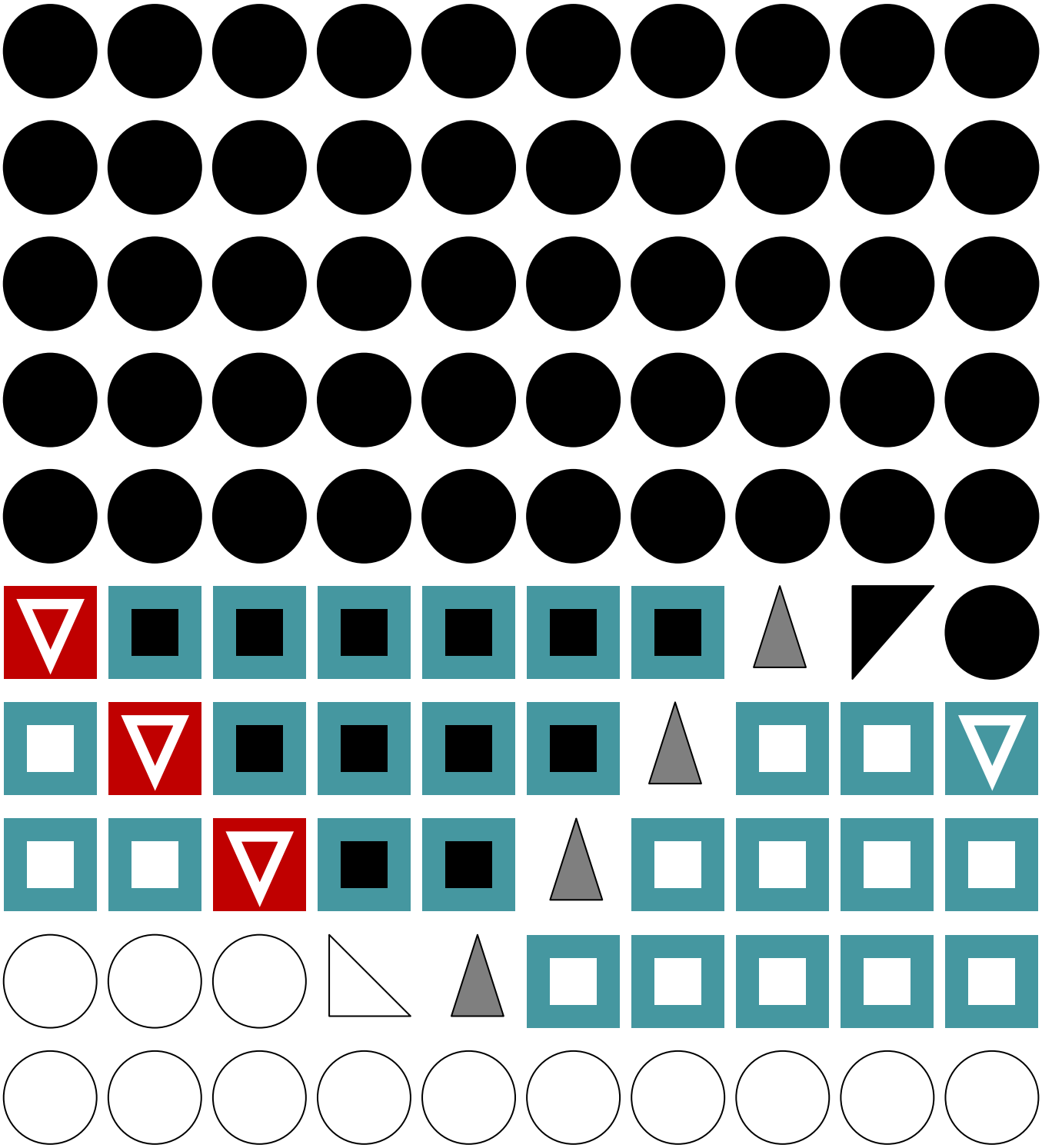


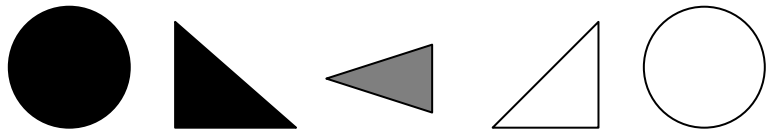
[Aharonov, Gottesman, Irani, Kempe]



U_{bc}

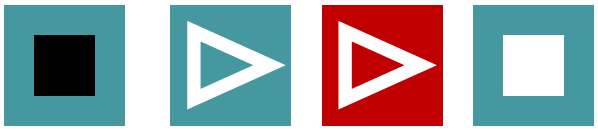
[Aharonov, Gottesman, Irani, Kempe]



5 

\+

=13

4×2 

3 Local Hamiltonian in 1D with qudits

- a unique state progression

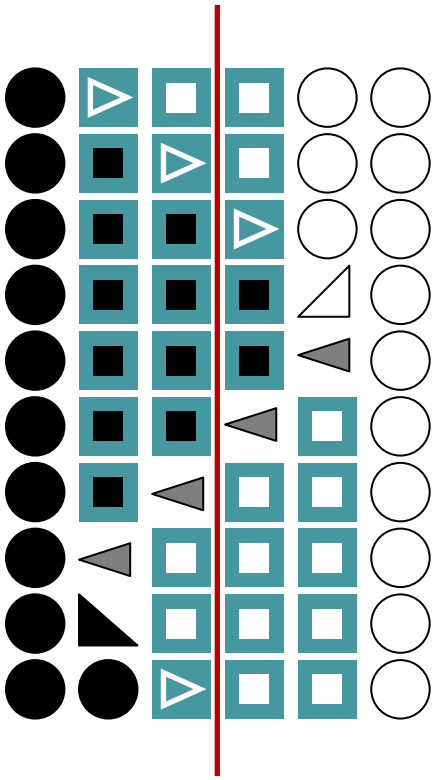
  every legal state goes
  to exactly 2 states

- an entangled ground state

  even running a classical
  computation [Schuch]

- clairvoyance

  allowed but illegal states
  evolve to forbidden ones

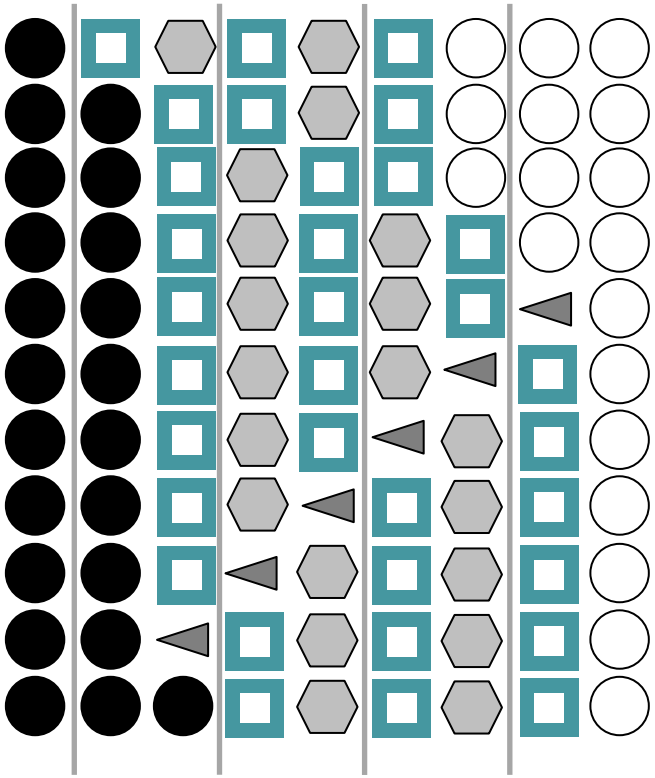


QMA-complete

d=13

[AGIK '06]

3 LH in 1D with smaller qudits



- non-unique transitions

  bad ones caught immediately

$$\langle \psi_{hist} | H^{(8)}_{prop} | \psi_{hist} \rangle = 0$$

- parity helps to lower d

- clairvoyance

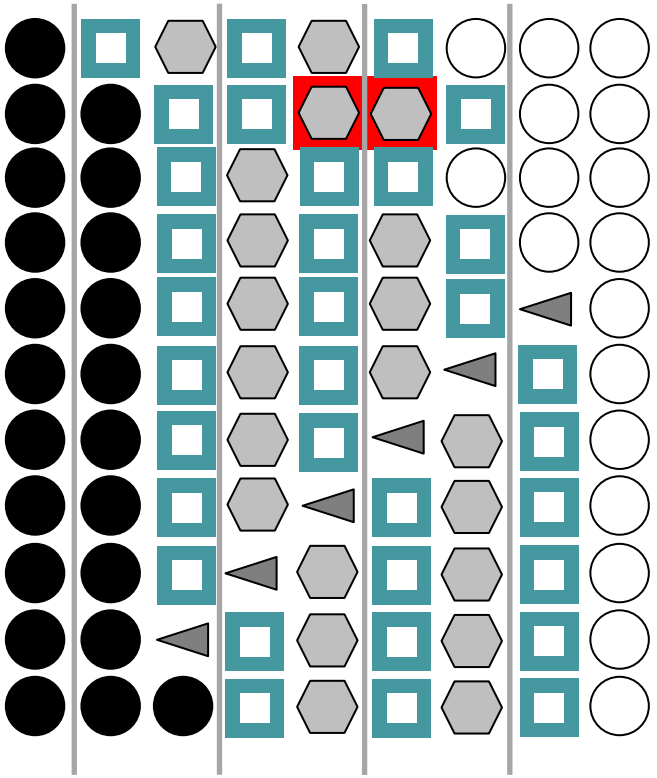
  allowed but illegal states evolve to forbidden ones

QMA-complete

d=8

[Hallgren+ '13]

3 LH in 1D with smaller qudits



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[Hallgren+ '13]

3 LH in 1D with smaller qudits

- non-unique transitions

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$$\langle \psi_{hist} | H^{(8)}_{prop} | \psi_{hist} \rangle = 0$$

- parity helps to lower d

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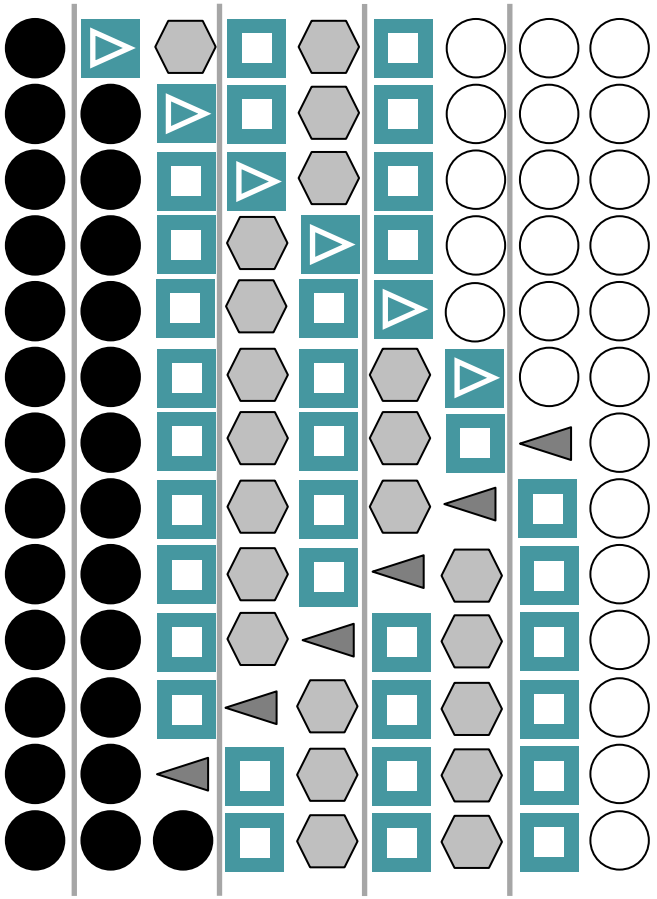
  allowed but illegal states evolve to forbidden ones

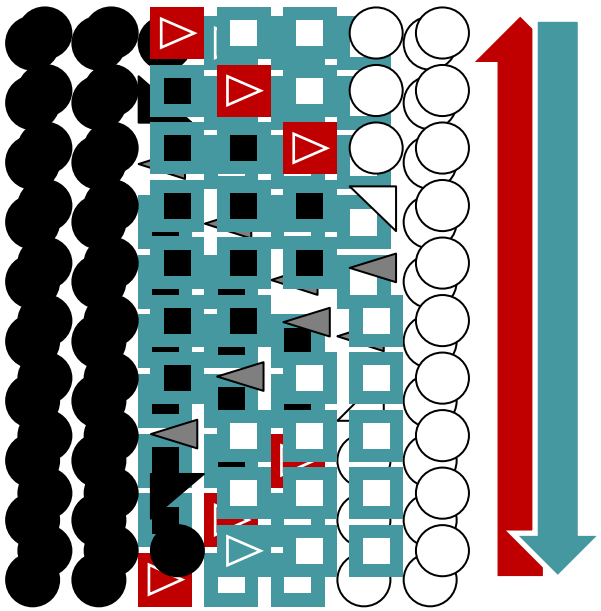


QMA-complete

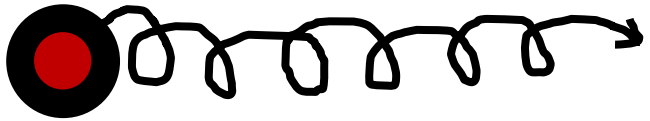
d=8

[Hallgren+ '13]

3 Translationally invariant 2-local Hamiltonian and tiling



layer 1



runs a counting machine that counts and writes out N, the system size

layer 2

runs a universal TM on input N

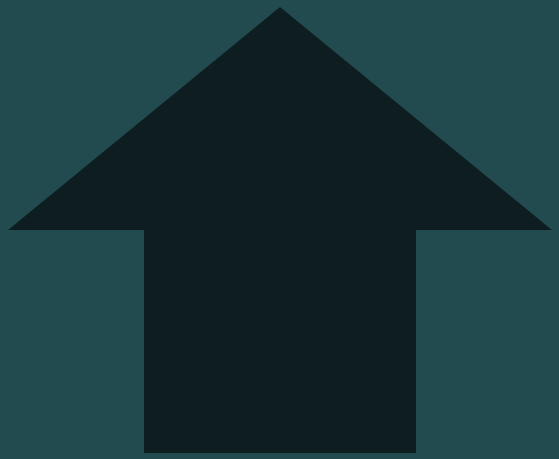
- big qudits, many kinds of “tiles” (rules)
- a system of size N corresponds to a log(N) QMA verifier circuit

QMA_{EXP}-complete

[Gottesman, Irani ’09]

QMA

[Jordan, Kobayashi, Le Gall, Nishimura, N.]



QMA_1

[Jordan, Kobayashi, Le Gall, Nishimura, N.]

[Pereszlényi]

4 Probabilistic checks

Did dinosaurs exist?

Sometimes reject a genuine proof?

Accept a fake?

4 Perfect completeness

Did dinosaurs exist?

YES?

Accept some proof without any doubt.

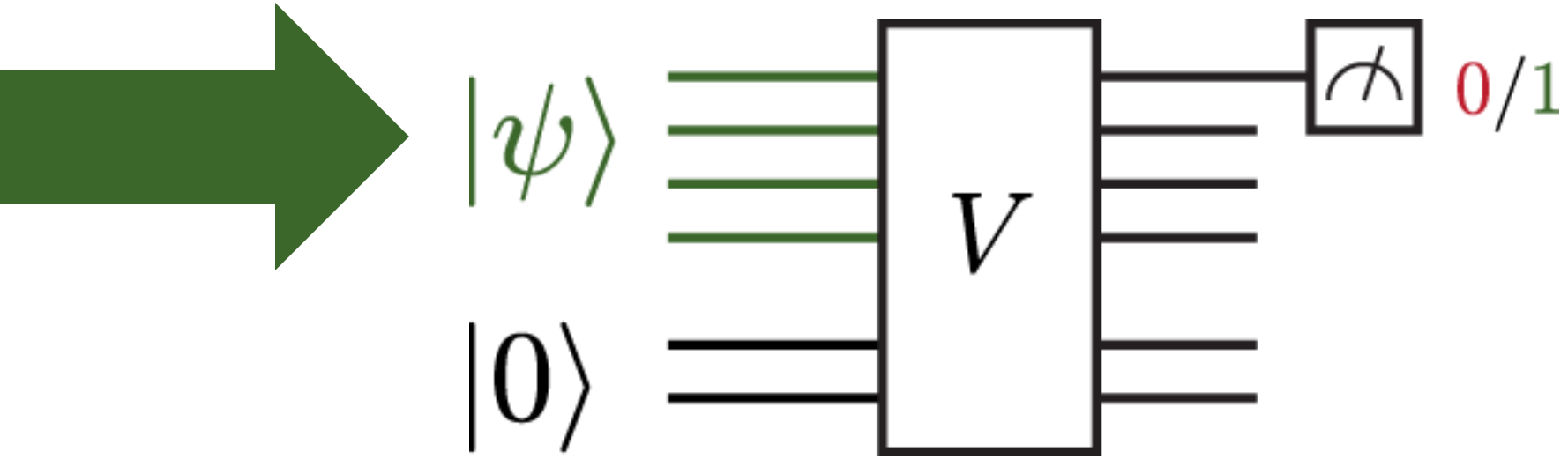
Never reject a genuine proof?

NO?
Still don't get fooled easily.

YES?
Accept some proof without any doubt.

perfect completeness

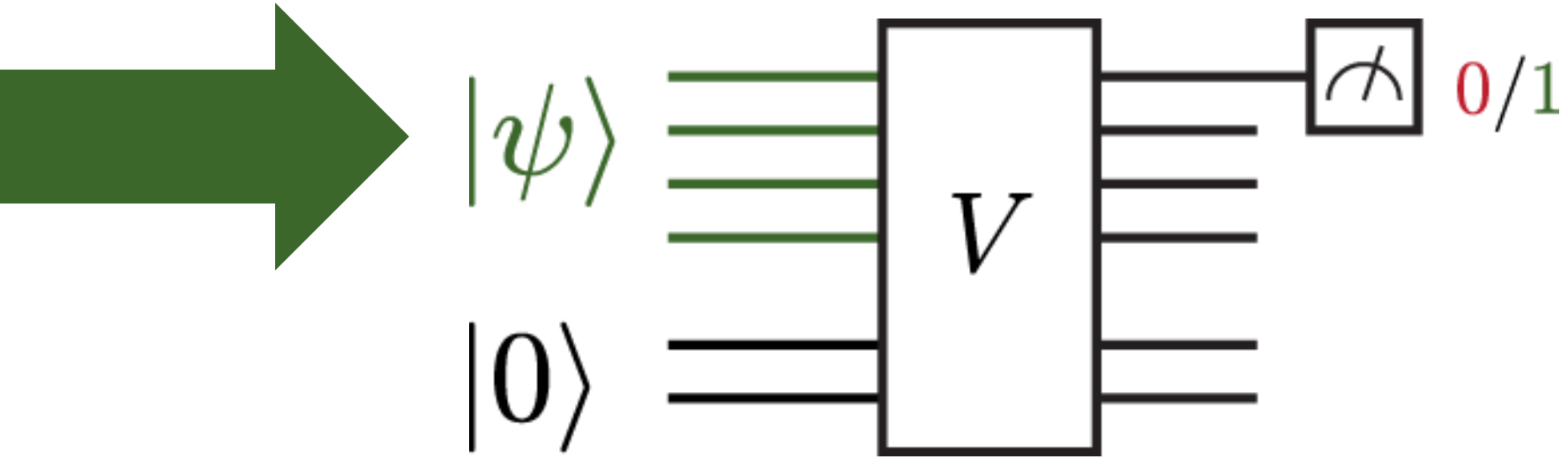
4 The QMA protocol



YES? Accept a good proof with p > a.

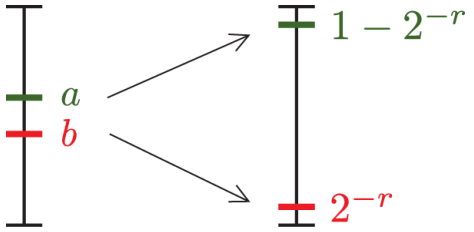
NO? Probability of accepting p < b.

4 The QMA protocol: amplification

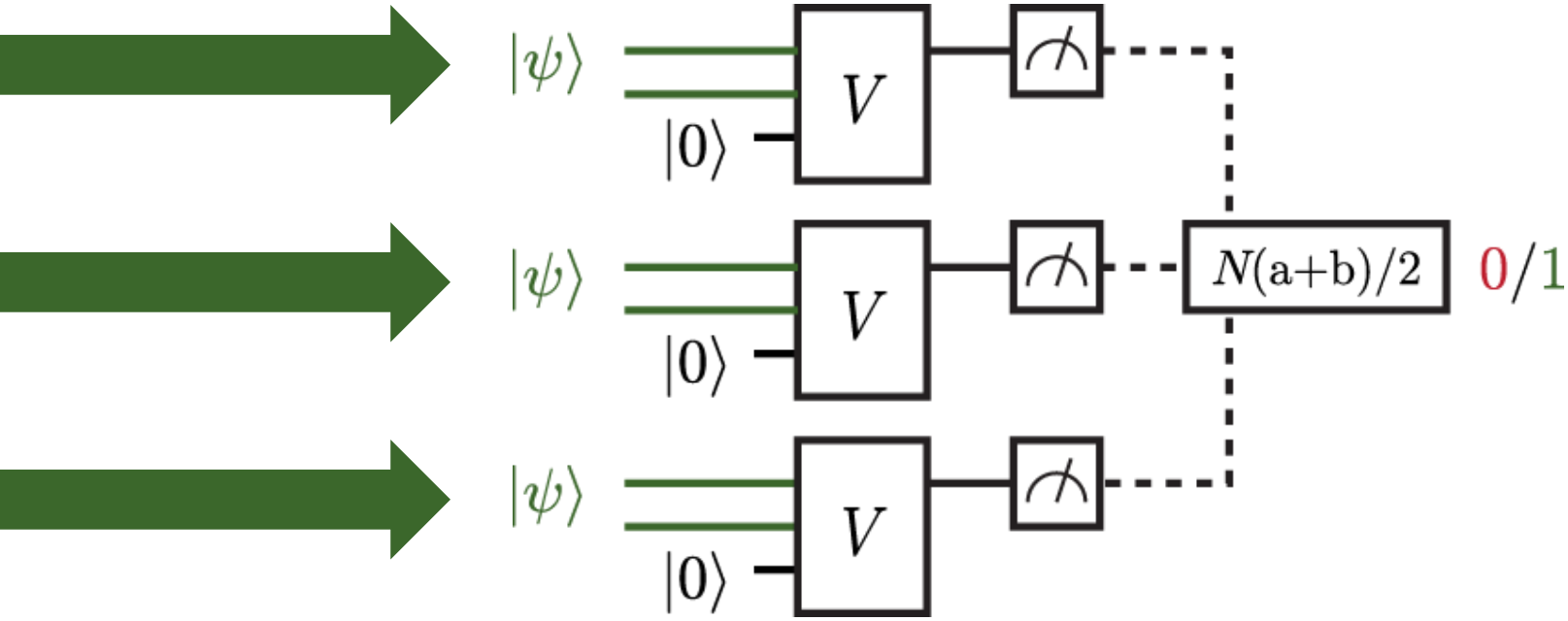


YES? Accept a good proof with p > a.

NO? Probability of accepting p < b.

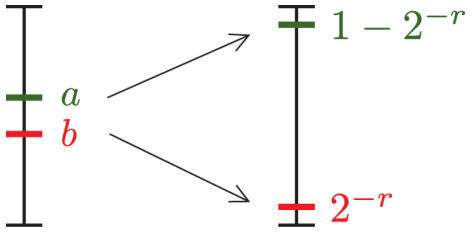


4 The QMA protocol: amplification [Kitaev]



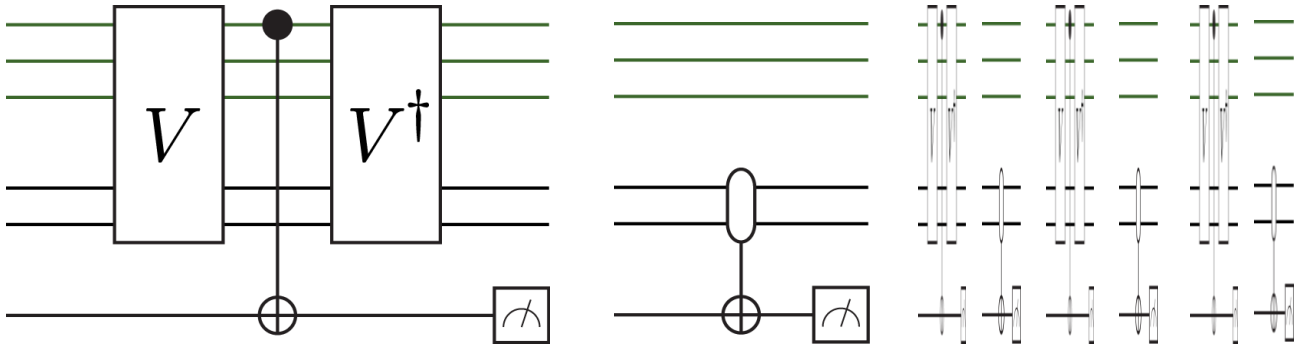
YES? Accept a good proof with p > a.

NO? Probability of accepting p < b.



4 The QMA protocol: amplification [Mariott-Watrous]

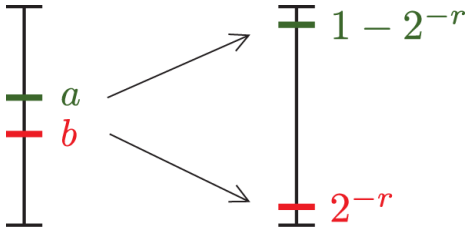
alternating projections P, Q



1, 0, 0, 0, 1, 1, 0, 1 …

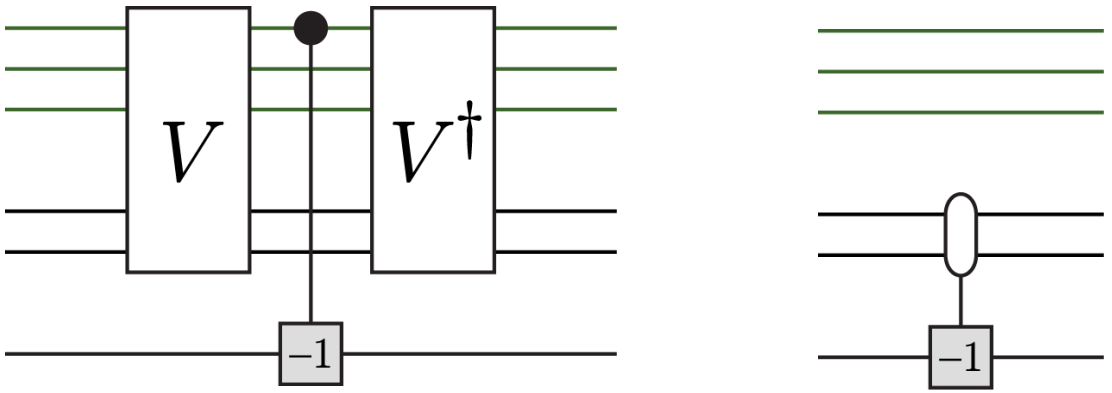
Life in a 2D subspace. [Jordan]

How many 00's and 11's?

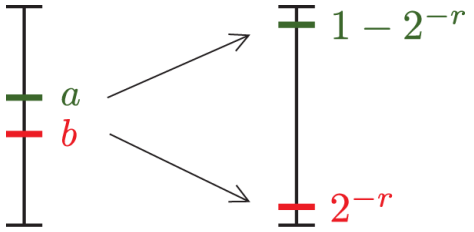


4 The QMA protocol: fast amplification [N.-Wocjan-Zhang]

alternating reflections R, S



Together: a rotation.
Phase estimation of RS.



4 The QMA protocol: fast amplification [N.-Wocjan-Zhang]

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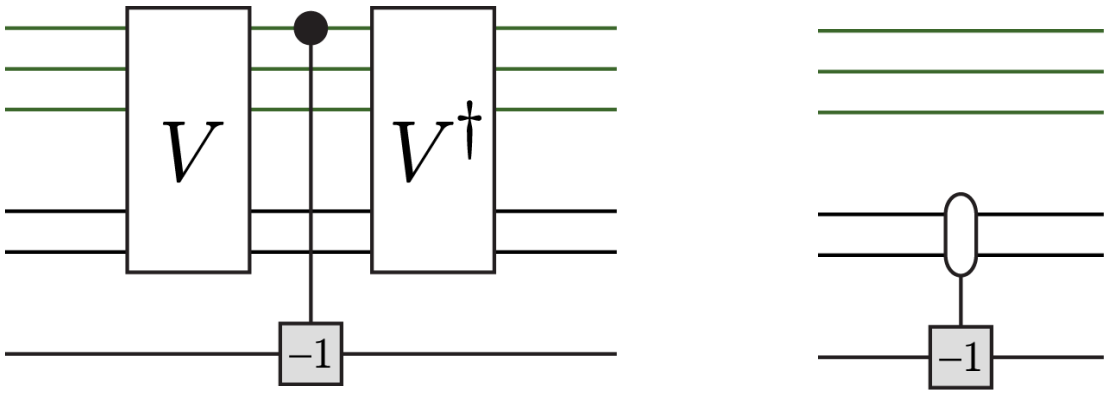
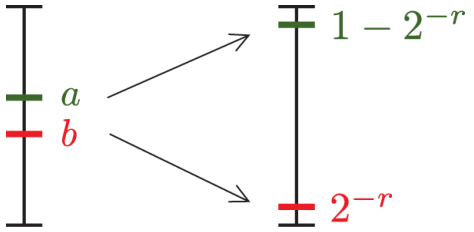


Together: a rotation.

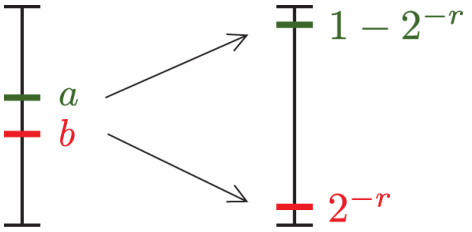
Perfect phase estimation of RS?



amplification

YES? Accept with p almost 1.

NO? Get fooled with small p.

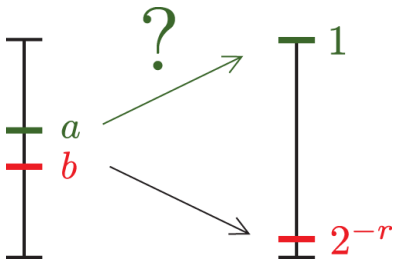


4 Perfect amplification for MA & QMA?

perfect amplification

YES? Accept a good proof.

NO? Get fooled with small p.



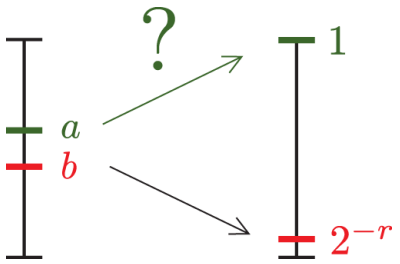
4 Perfect amplification for MA.

perfect classical amplification

$$\mathrm{MA} = \mathrm{MA}_1$$ [Zachos & Fürer]

YES? Accept a good proof.

NO? Get fooled with small p.



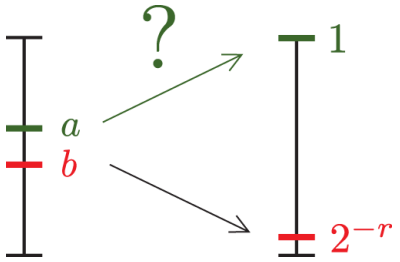
4 Perfect amplification for QMA?

perfect quantum amplification

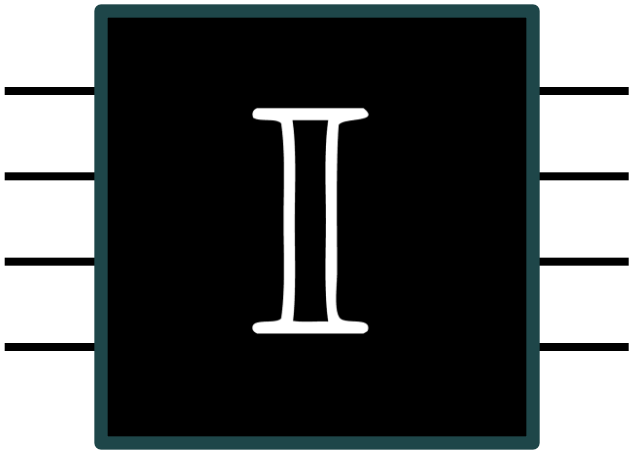
$$\text{QMA} \overset{?}{=} \text{QMA}_1$$

YES? Accept a good proof.

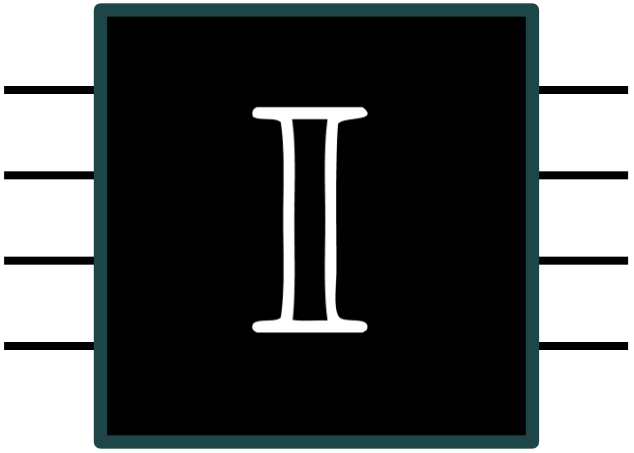
NO? Get fooled with small p.



An oracle separation of QMA & QMA_1



An oracle separation of QMA & QMA_1



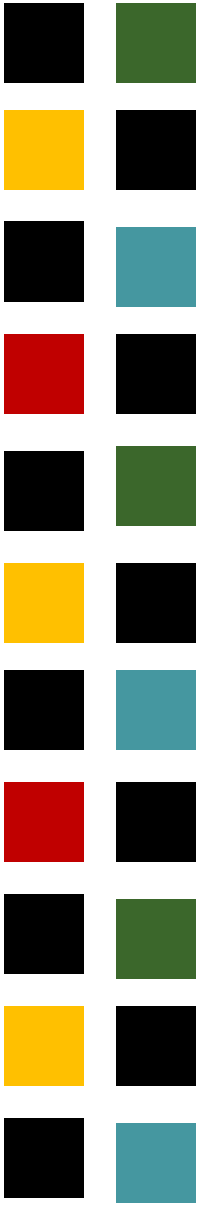
a continuous range of angles

Accept something without a doubt?

=

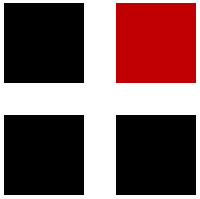
Accept everything…
[Aaronson '08]

Quantum Computing



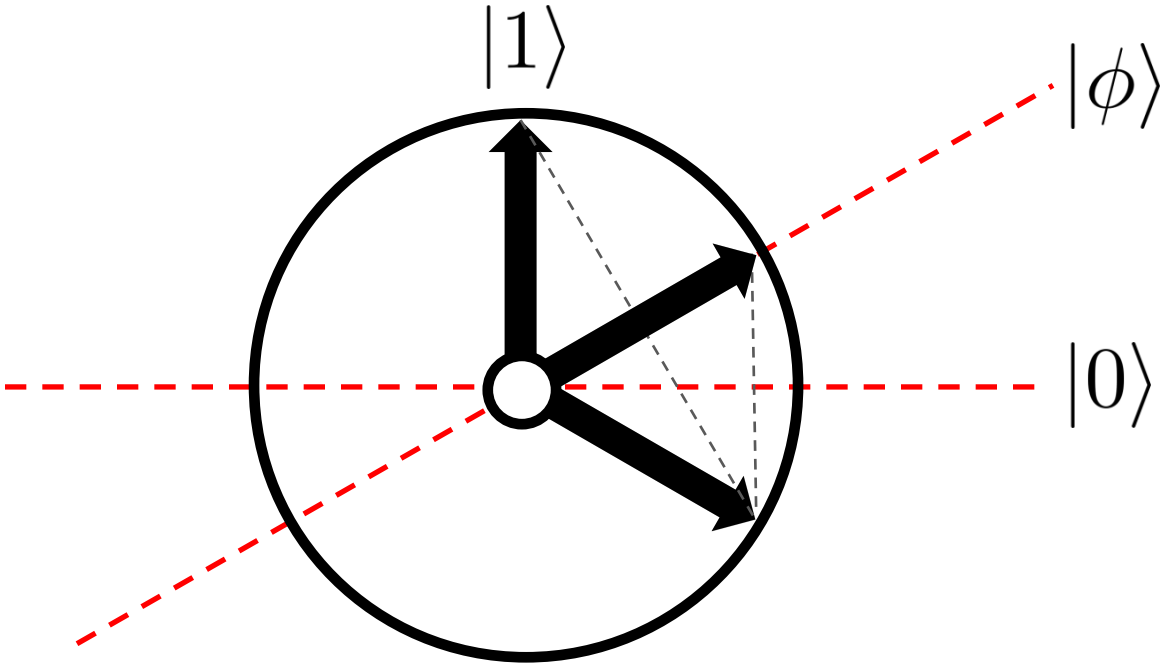
Exact

Quantum Computing



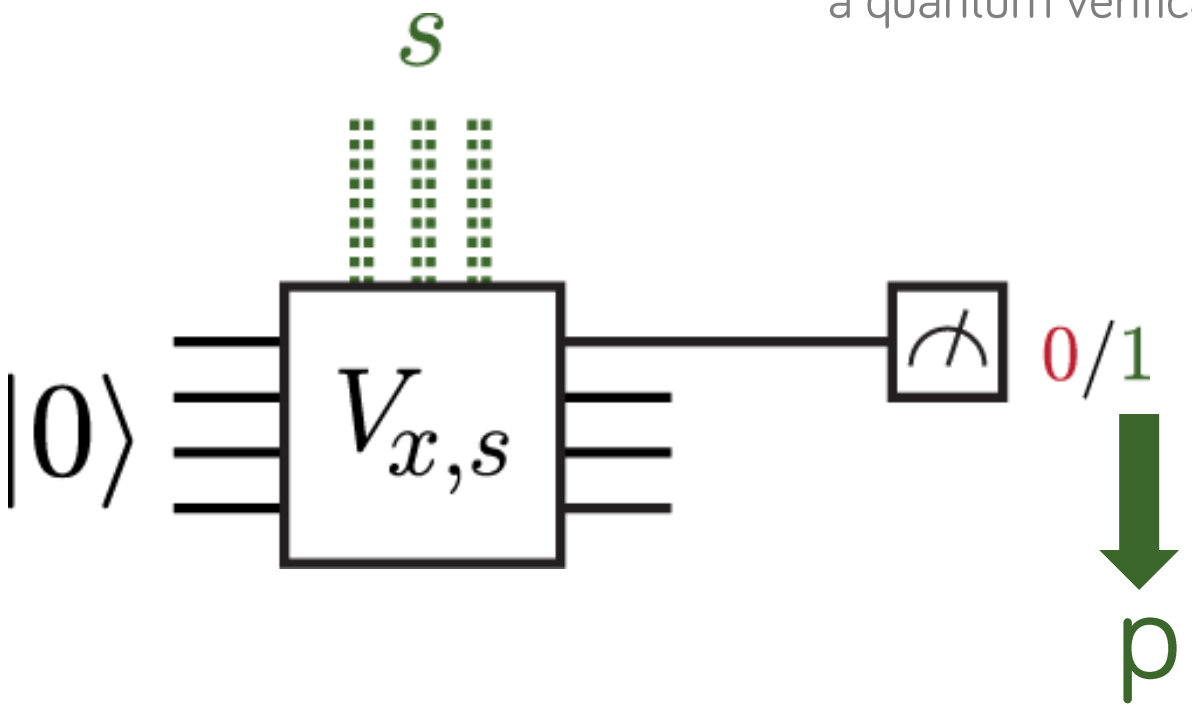
Exact Grover's search [Grover]

$$|\phi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$



4 QCMA (MQA)

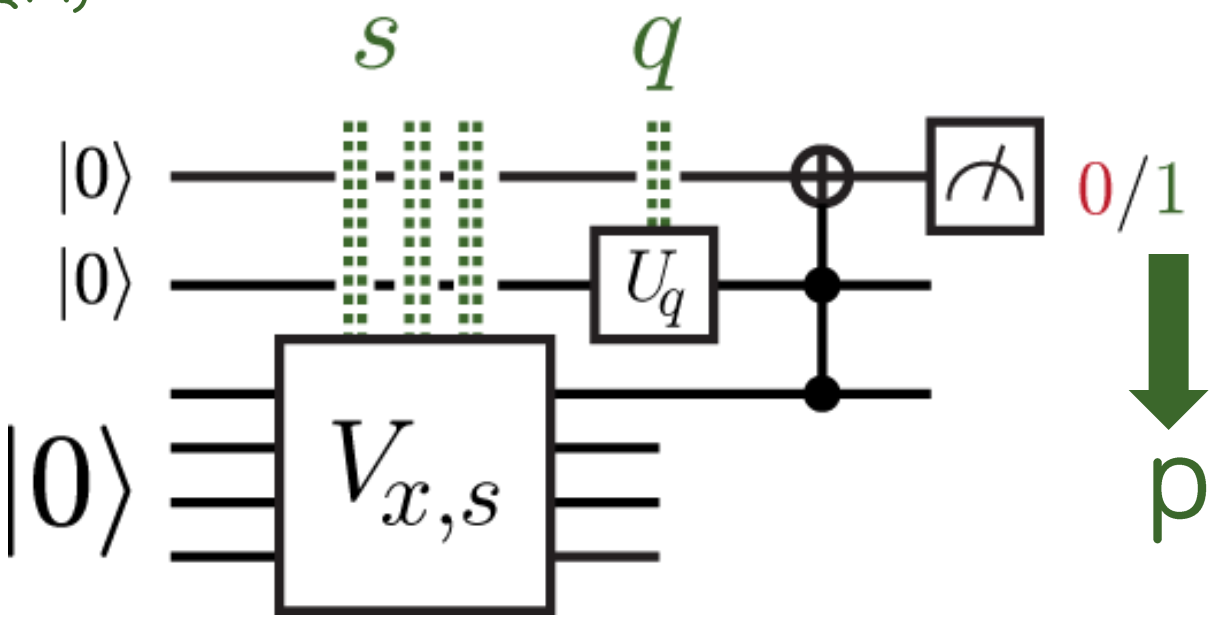
A classical witness,
a quantum verification.



Knowing how to prepare the witness...
we can reflect about it.

$$\sqrt{1-p}|\cdots 0\rangle + \sqrt{p}|\cdots 1\rangle$$

4 QCMA (MQA)

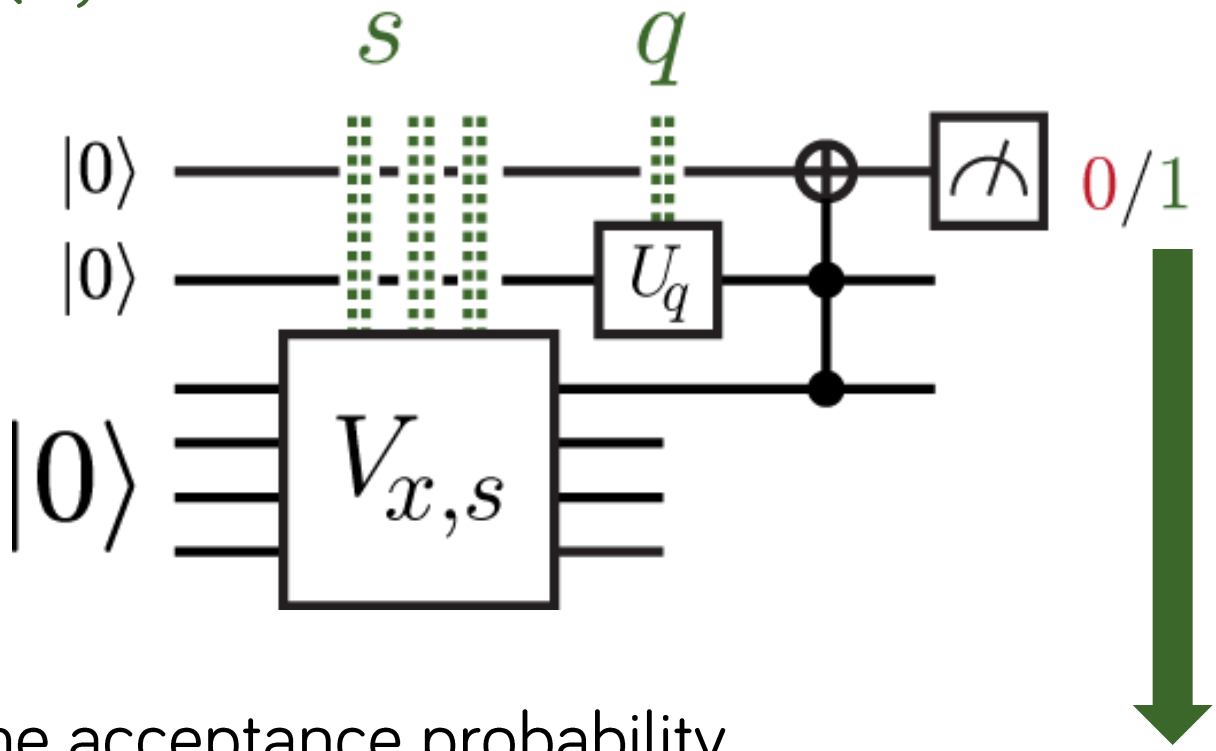


Knowing the acceptance probability...
add a rotated ancilla.

$$\sqrt{1-q}|0\rangle + \sqrt{q}|1\rangle$$

$$\sqrt{1-p}|\cdots 0\rangle + \sqrt{p}|\cdots 1\rangle$$

4 QCMA (MQA)



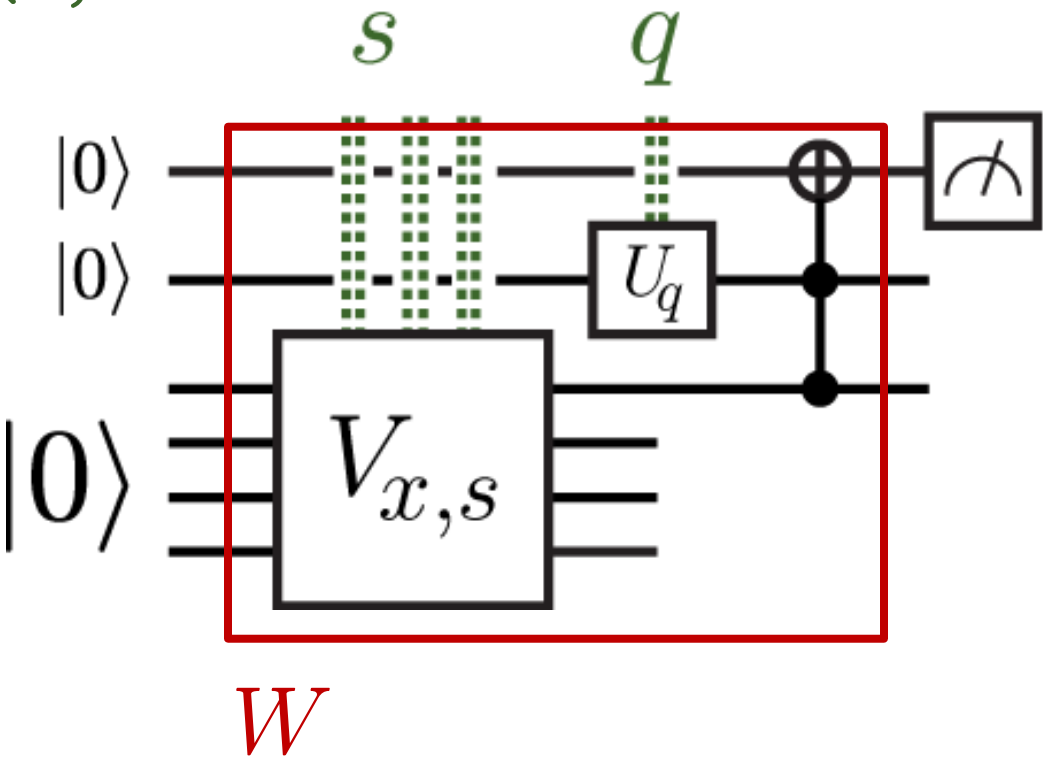
Knowing the acceptance probability...
add a rotated ancilla, get ¼.

p'=pq=¼

$$\sqrt{1-q}|0\rangle + \sqrt{q}|1\rangle$$

$$\sqrt{1-p}|\cdots 0\rangle + \sqrt{p}|\cdots 1\rangle$$

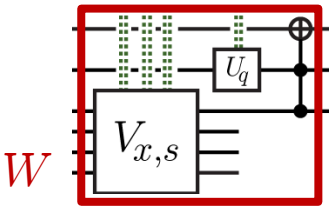
4 QCMA (MQA)

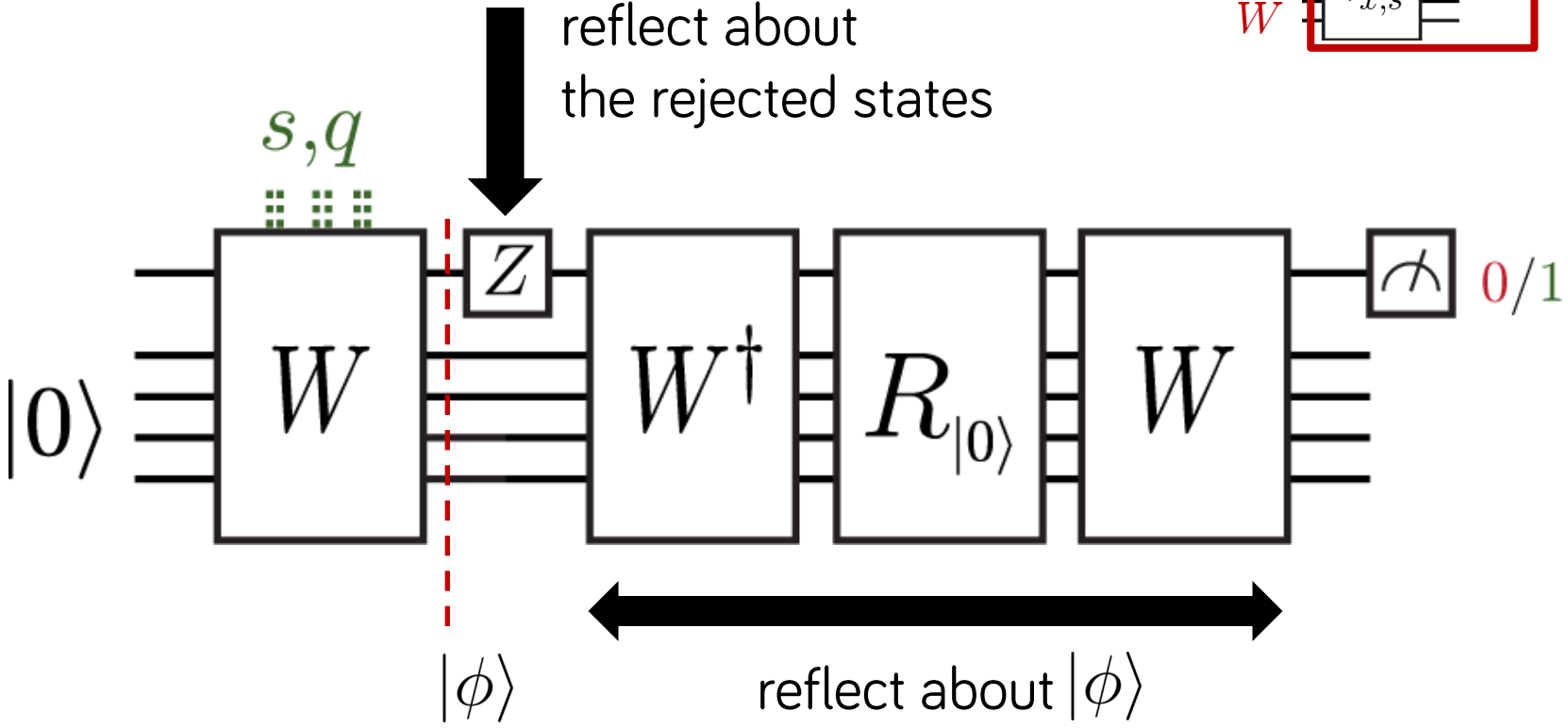


p'=pq=¼

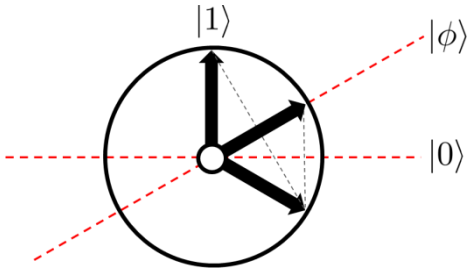
Gates with rational-number elements are universal.
Both p and q are rational.
It's doable.

4 QCMA (MQA) with perfect completeness

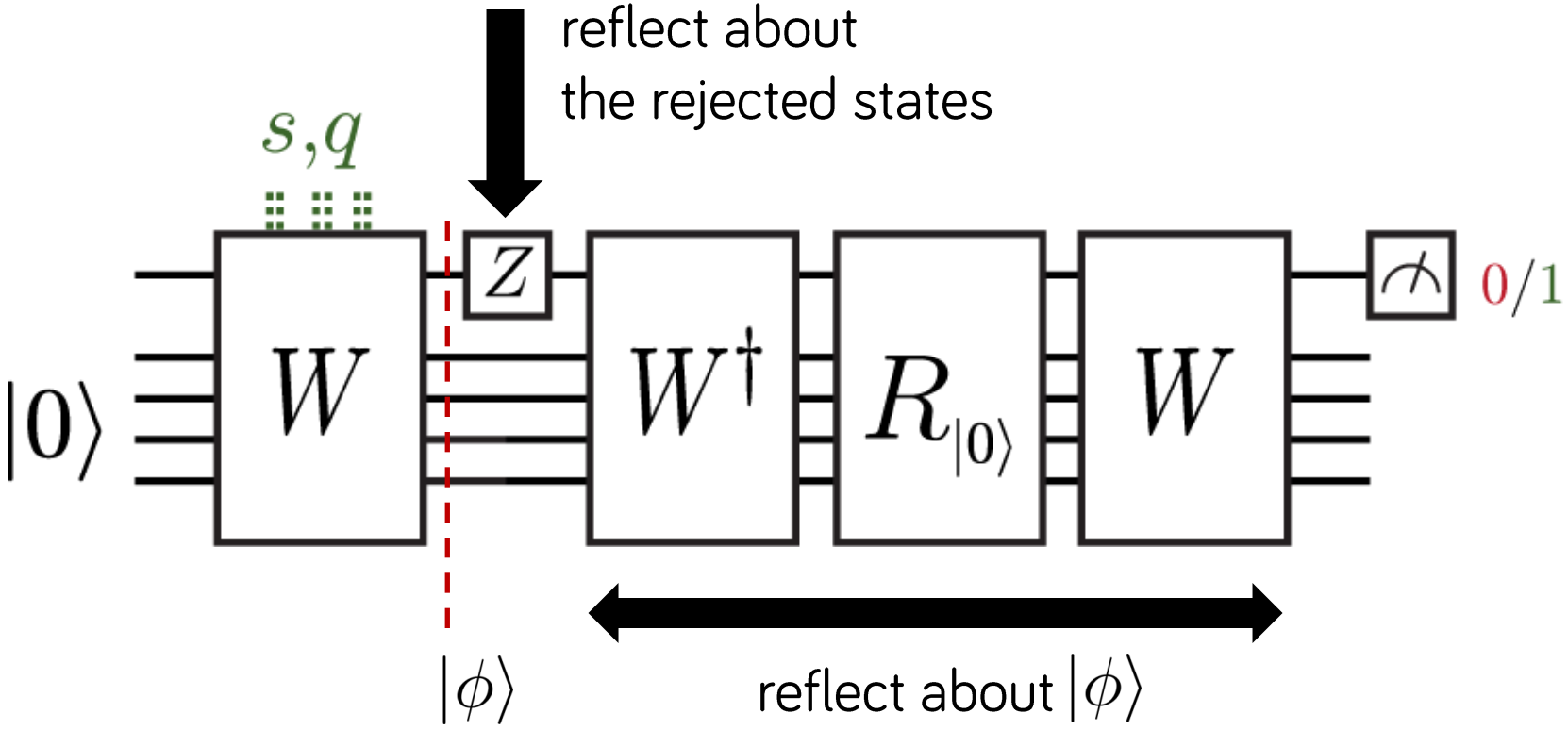




Perfectly accepts solid proofs.

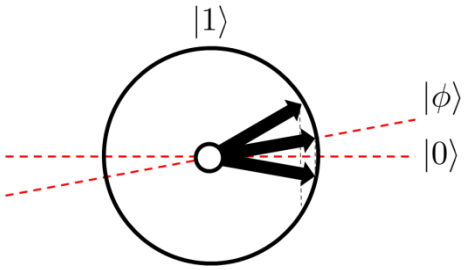


4 QCMA (MQA) with perfect completeness

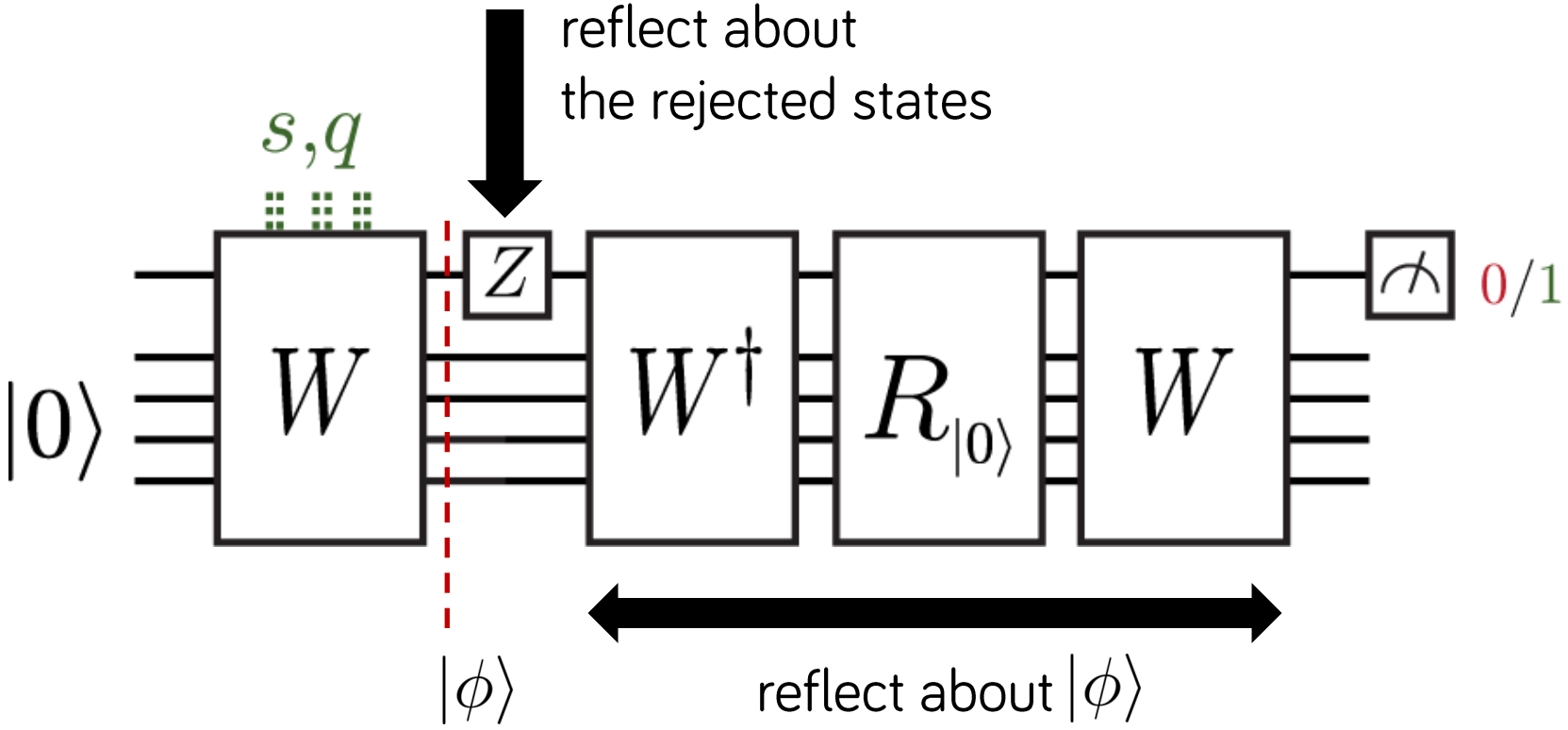


Perfectly accepts solid proofs.

The soundness doesn't break.



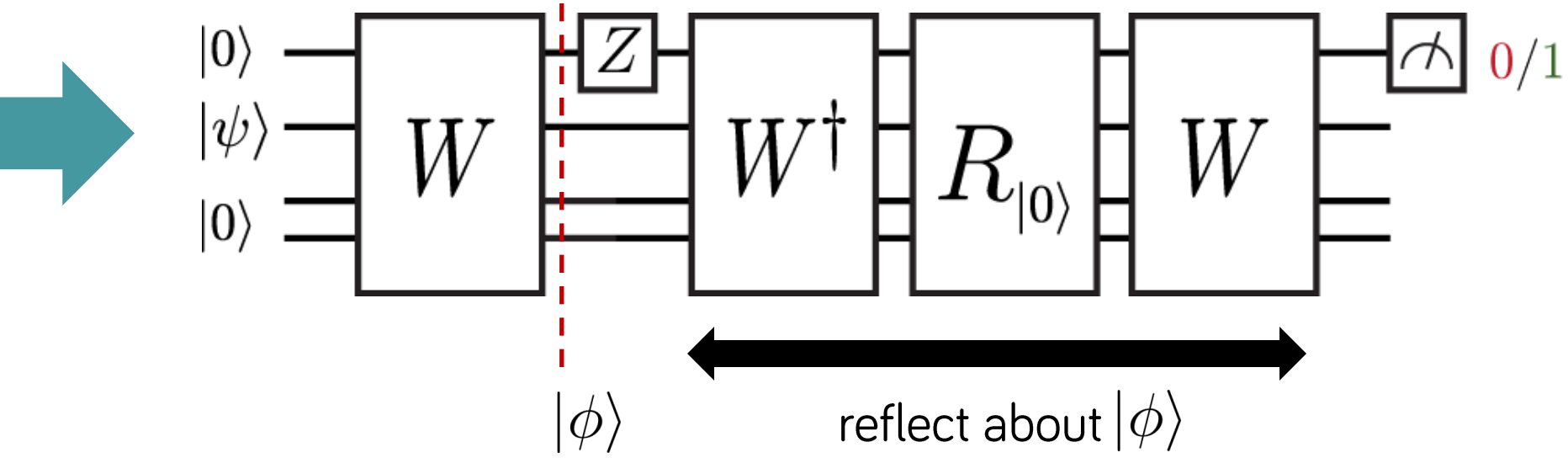
4 QCMA (MQA) with perfect completeness



$$\mathrm{QCMA}_1 = \mathrm{QCMA}$$

4 Towards perfect completeness in QMA...

Let's try the same with a quantum witness.



How to

correct p to something nice?

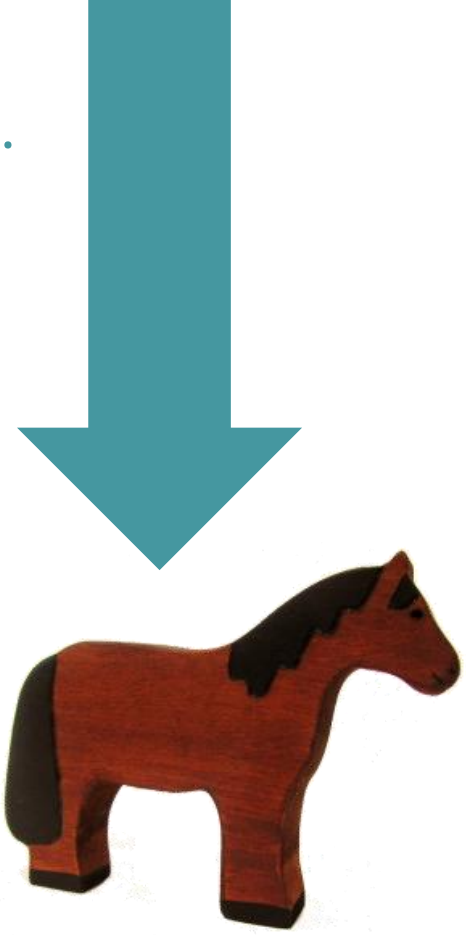
reflect about the unknown witness?

4 Towards perfect completeness in QMA...

Send us the witness.

Send us its acceptance probability p?

a correction q?



How to correct p to something nice?

reflect about the unknown witness?

4 Towards perfect completeness in QMA...

Send us the witness.

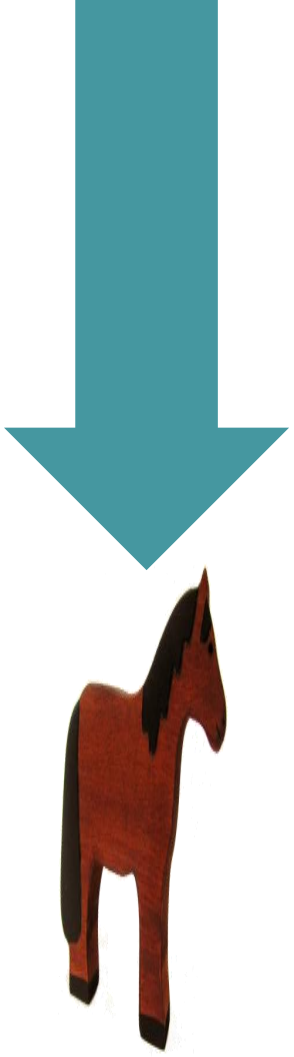
Send us its acceptance probability p?

a correction q?

a trustworthy encoding of q?

$$\sqrt{1-q}|0\rangle + \sqrt{q}|1\rangle$$

We need a few EPR pairs to be safe.



How to

correct p to something nice?

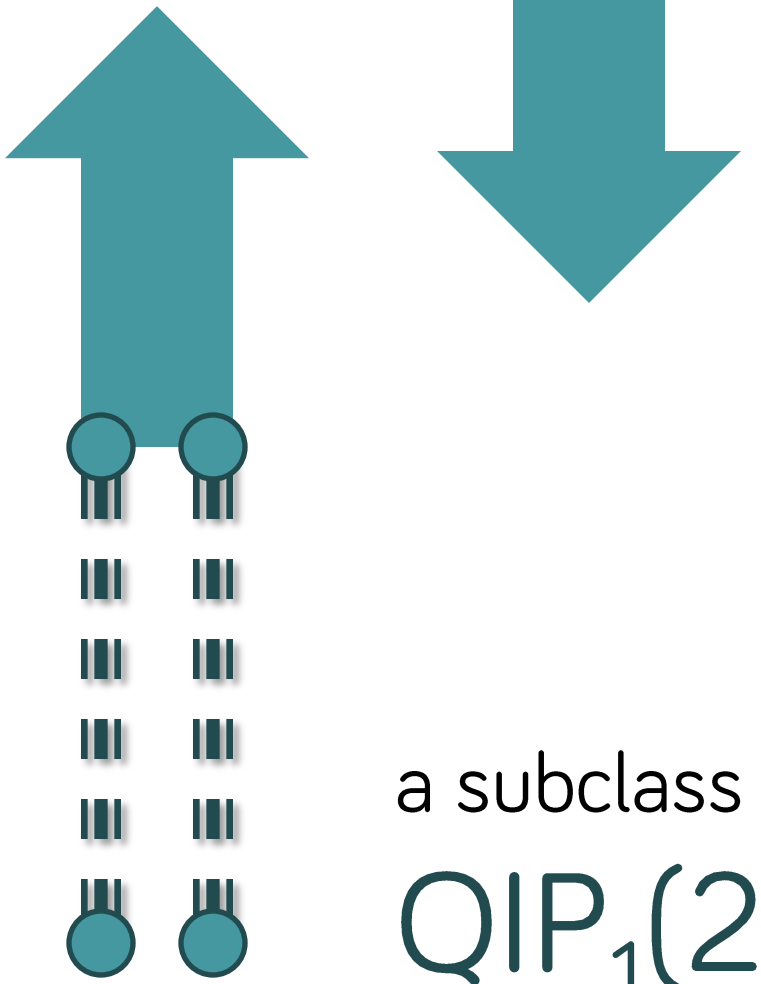
reflect about the unknown witness?

4 A slightly interactive proof

Hey, Merlin, could you carve something from this material?

IP(2)

QMA_1 const. EPR



a subclass of

$QIP_1(2)$

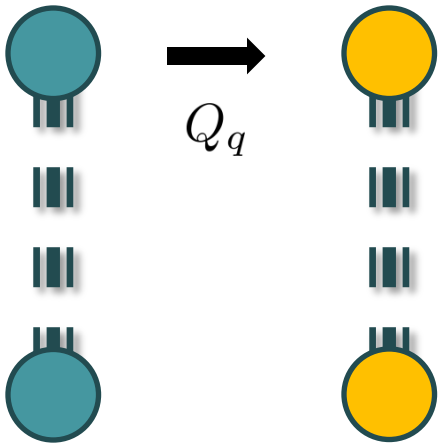
4 Correcting p to something "nice".

A "correcting" state $\sqrt{1-q}|0\rangle + \sqrt{q}|1\rangle$ with $pq = \frac{1}{2}$

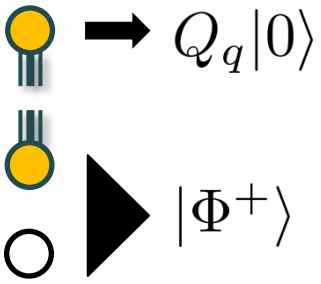
Prepared by Merlin using

$$Q_q = \begin{bmatrix} \sqrt{1-q} & \sqrt{q} \\ \sqrt{q} & -\sqrt{1-q} \end{bmatrix}$$

on a half of an EPR pair

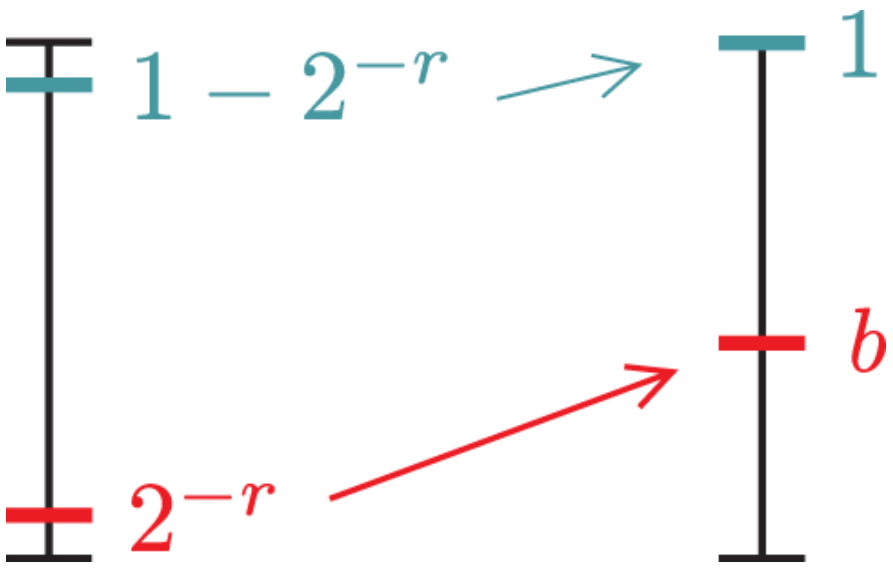


$$\sqrt{1-q}|\Phi^-\rangle + \sqrt{q}|\Psi^+\rangle$$



A Choi-Jamiołkowski state… it allows probabilistic (heralded) simulation of Q_q.

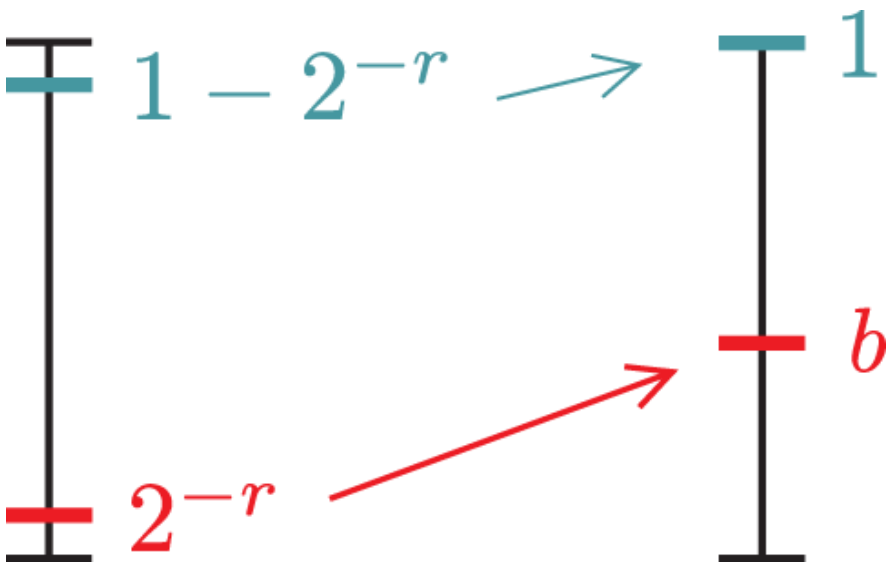
4 The second result



Probabilistically apply an unknown correction with M's help.

$$\mathrm{QMA} \subseteq \mathrm{QMA}_1^{\text{const. EPR}}$$

4 The second result



Probabilistically apply an unknown correction with M's help.

$$\mathrm{QMA} \subseteq \mathrm{QMA}_1$$

I wish...

my proof

by misunderstanding

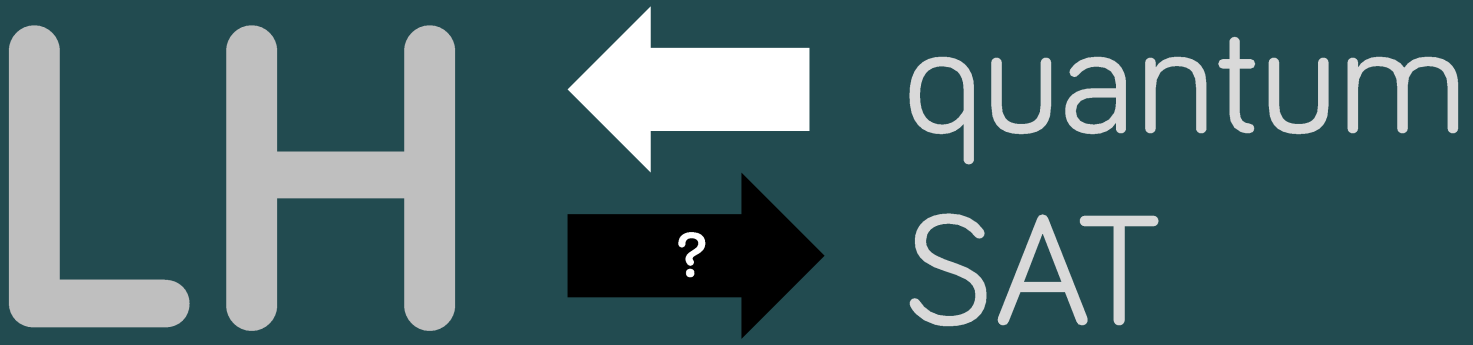
does

not work

QIP 2013

rump session

$\mathrm{QMA} \overset{?}{\subseteq} \mathrm{QMA}_1$

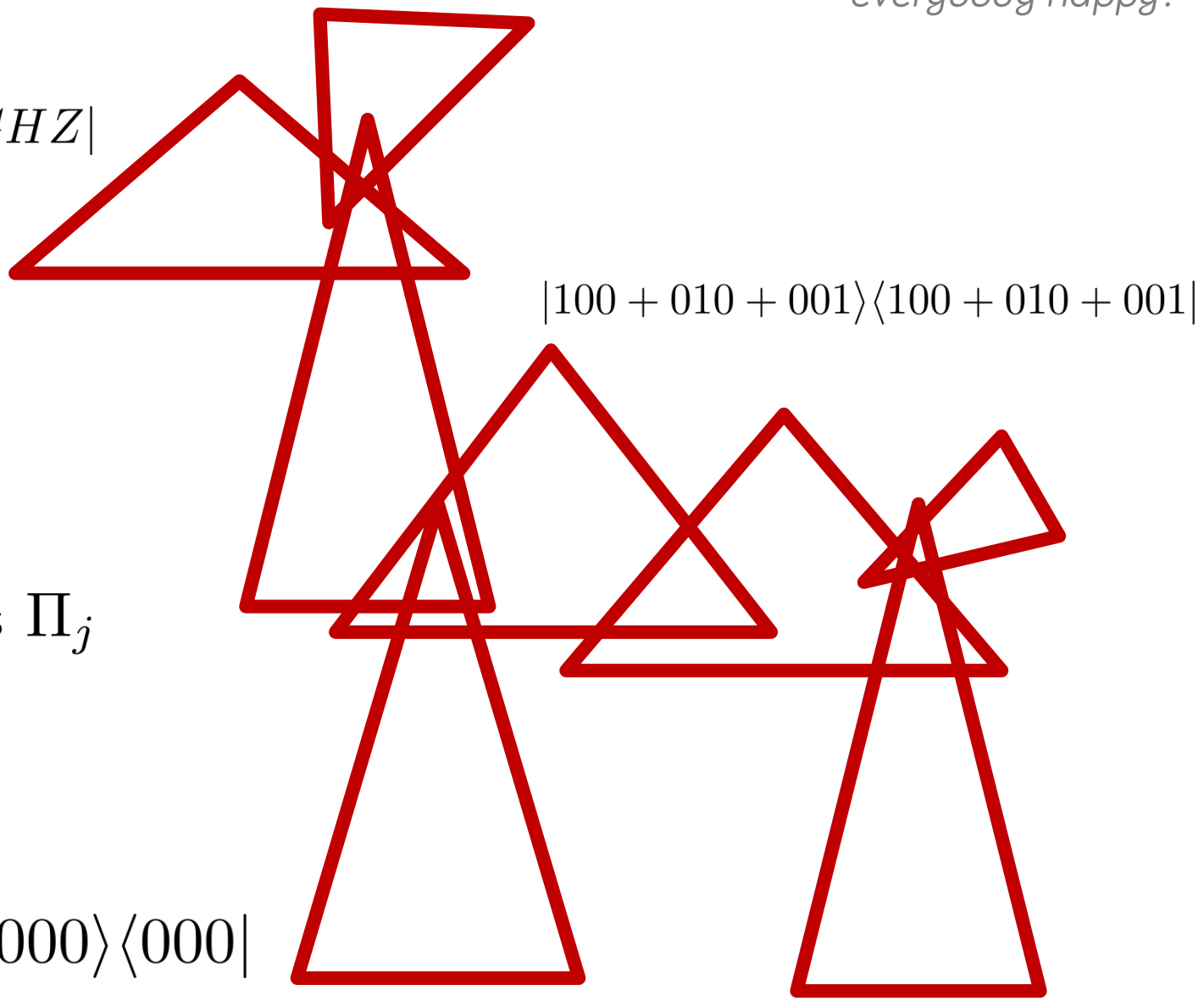


frustrated

frustration
free

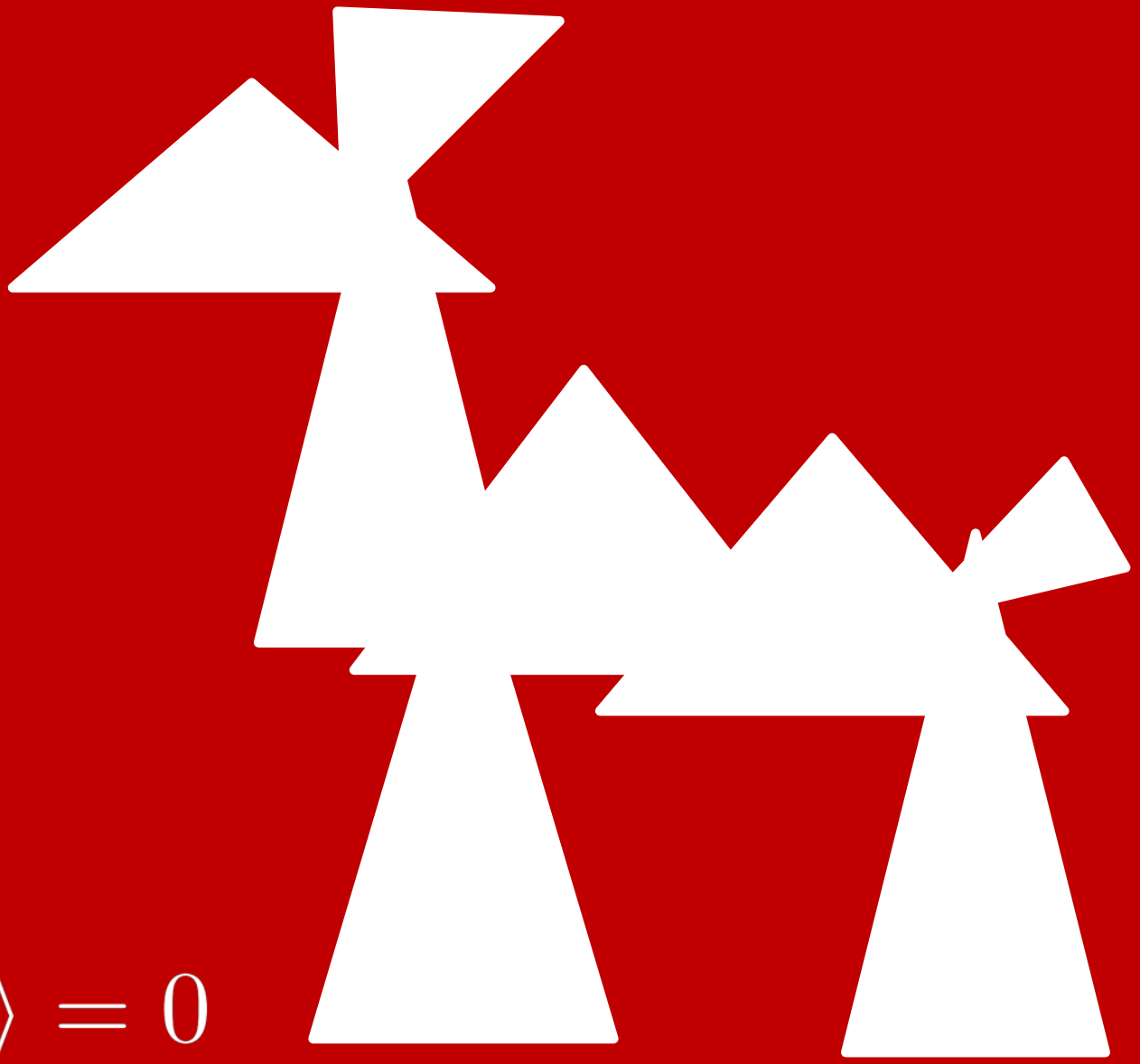
5 Quantum *k*-SAT

Can we make everybody happy?



- k-local projectors Π_j

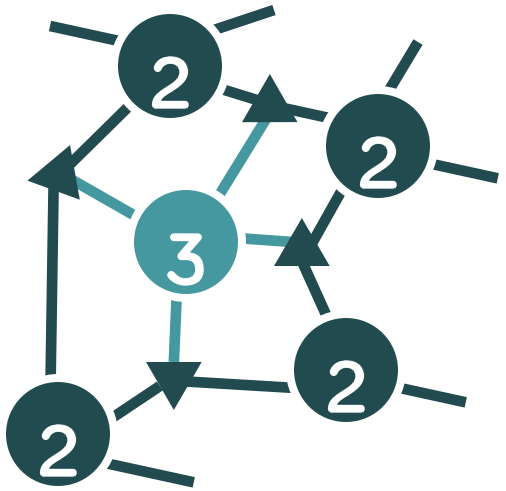
Quantum *k*-SAT



$$\Pi^j |\psi\rangle = 0$$

5 QMA_1-complete problems

[N. '08]



[Moses, N. '07]

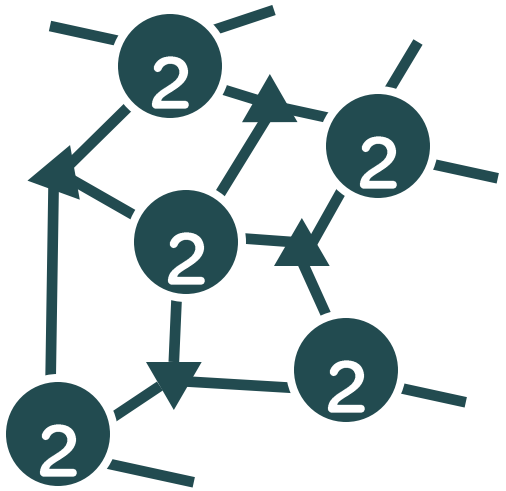
Π^j

unfrustrated

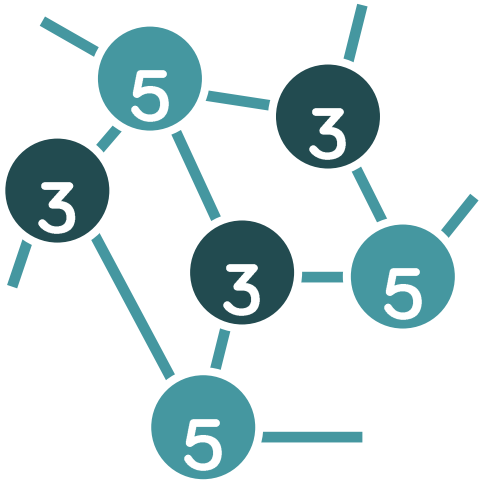
qSAT

5 QMA_1-complete problems

[N. '08]



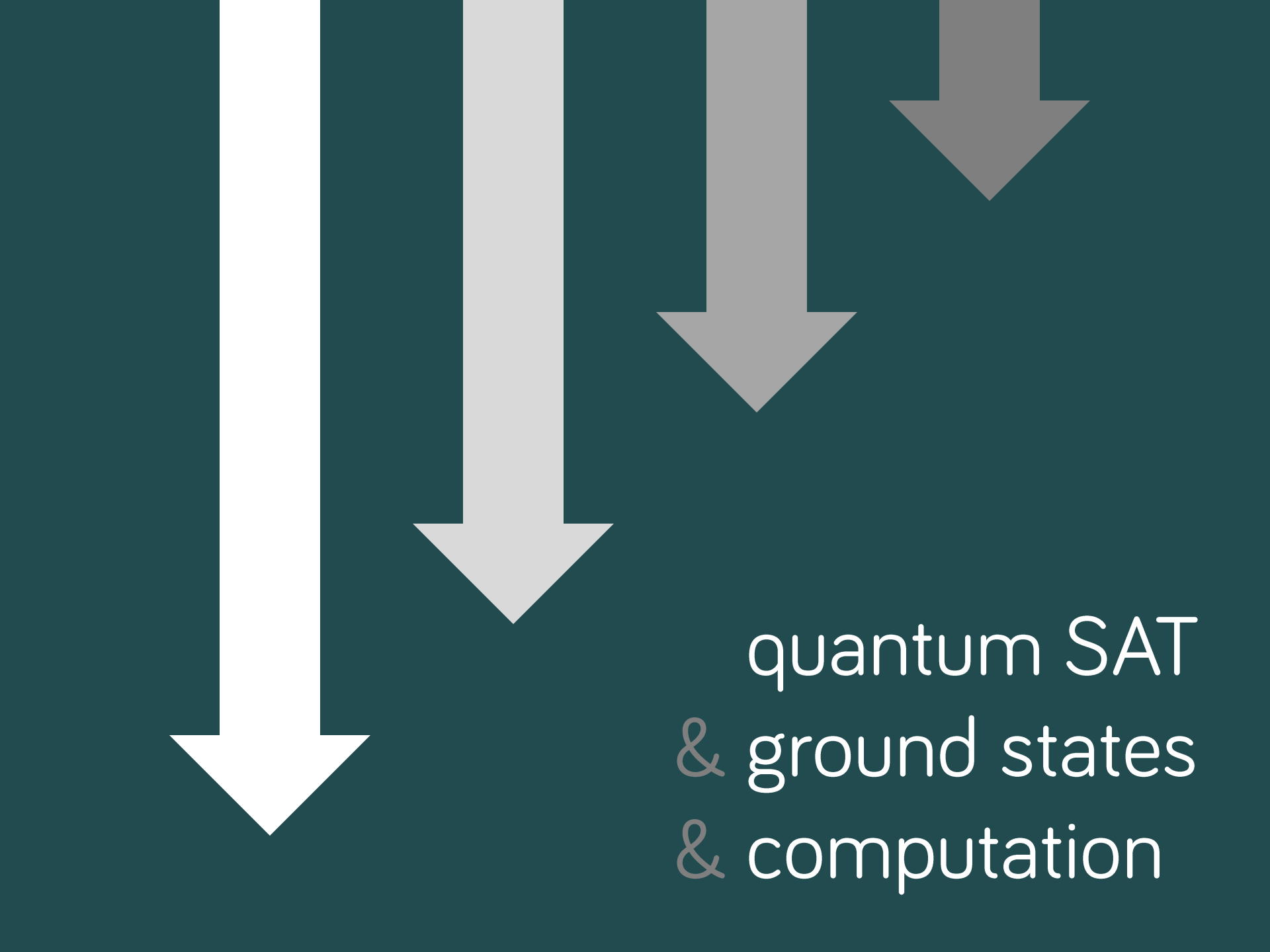
[Gosset, N. '13]



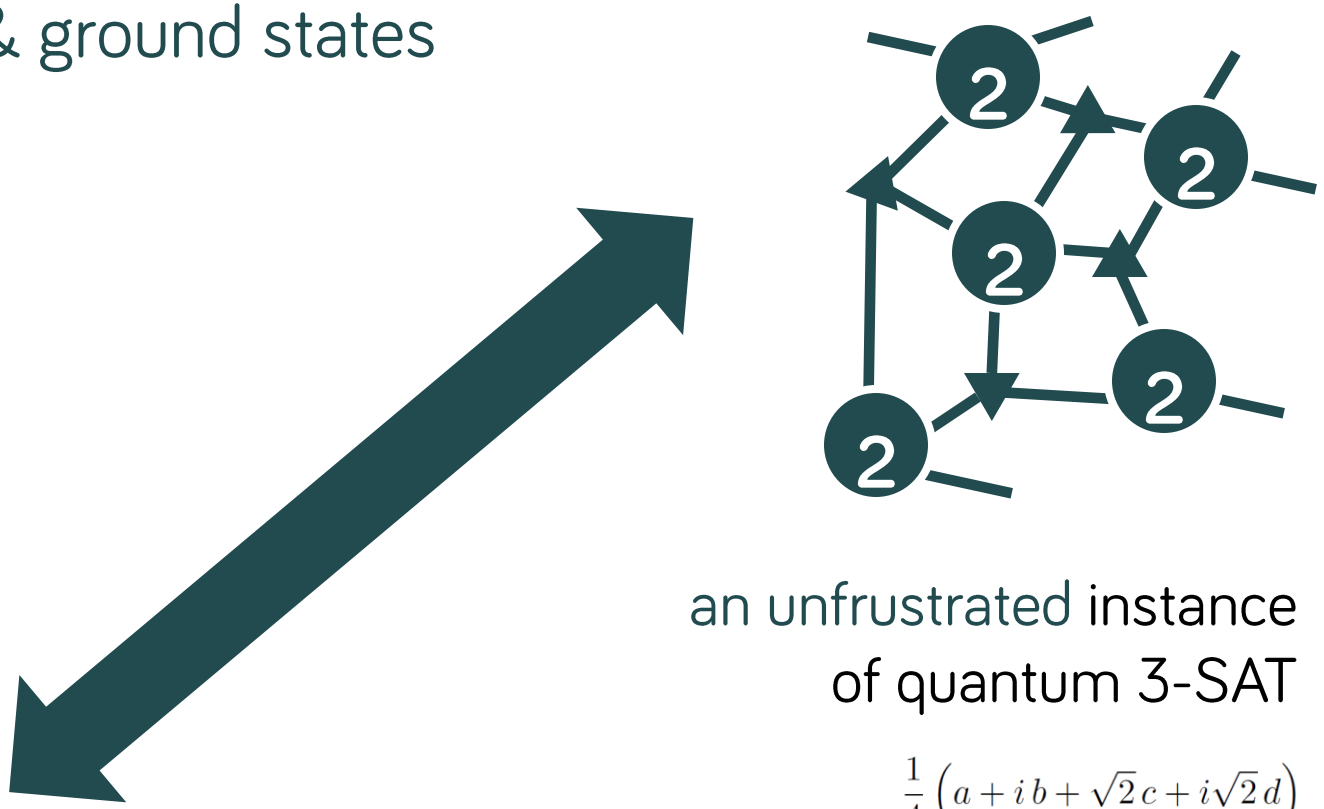
[Eldar, Regev '08]

unfrustrated

qSAT


quantum SAT
& ground states
& computation

5 Circuits & ground states



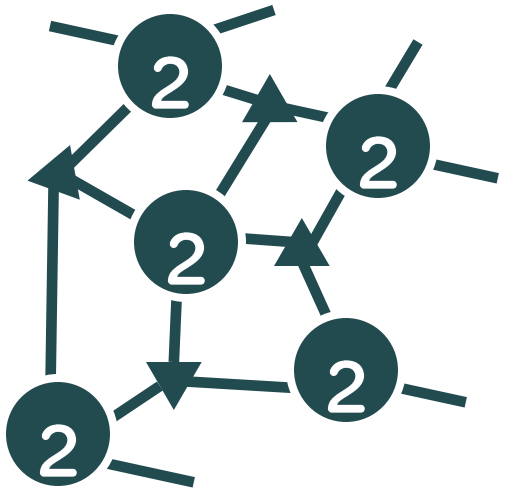
an unfrustrated instance
of quantum 3-SAT

$$\frac{1}{4}\left(a + i\,b + \sqrt{2}\,c + i\sqrt{2}\,d\right)$$

CNOT, H & T

a verifier circuit with a perfectly accepted witness

5 Circuits & ground states



a frustrated instance
of quantum 3-SAT

CNOT, H & T

a verifier circuit which doesn't like to accept anything

5 Uniquely run the clock & apply a 2-qubit gate, 3-locally?

$$|\varphi_{t-2}\rangle \otimes |t-2\rangle$$

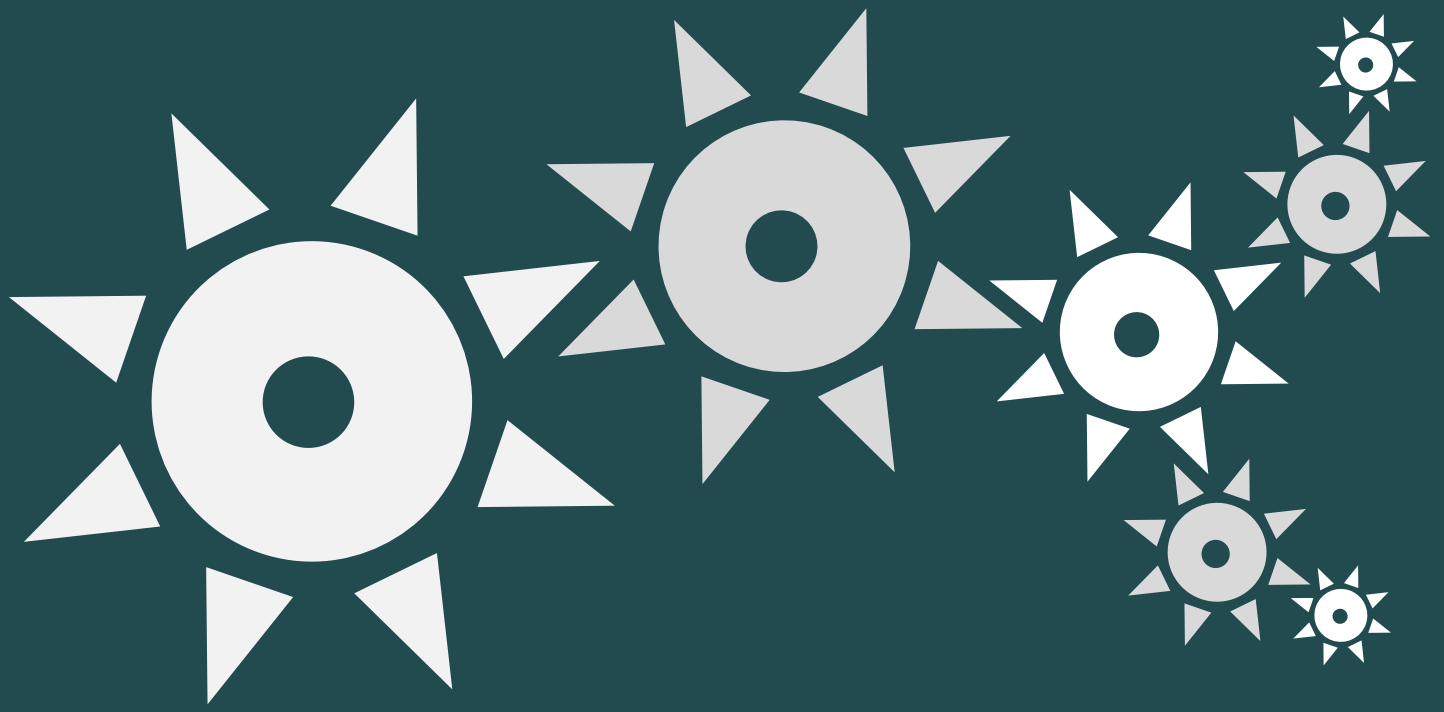
$$|\varphi_{t-1}\rangle \otimes |t-1\rangle$$

$$|\varphi_{t}\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

$$|\varphi_{t+2}\rangle \otimes |t+2\rangle$$

$$|\varphi_{t+3}\rangle \otimes |t+3\rangle$$



a clock workshop #2

5 Constructing clocks

- the domain wall 

$$|t\rangle = |4\rangle$$
$$= |11110\rangle$$

- clock-checking terms $|01\rangle\langle 01|$

5 Constructing clocks

- the domain wall  transitions: 3-local

$$|t\rangle = |3\rangle$$
$$= |11100\rangle$$

- clock-checking terms $|01\rangle\langle 01|$

- checking/applying the transitions $|100-110\rangle\langle 100-110|$

5 Constructing clocks

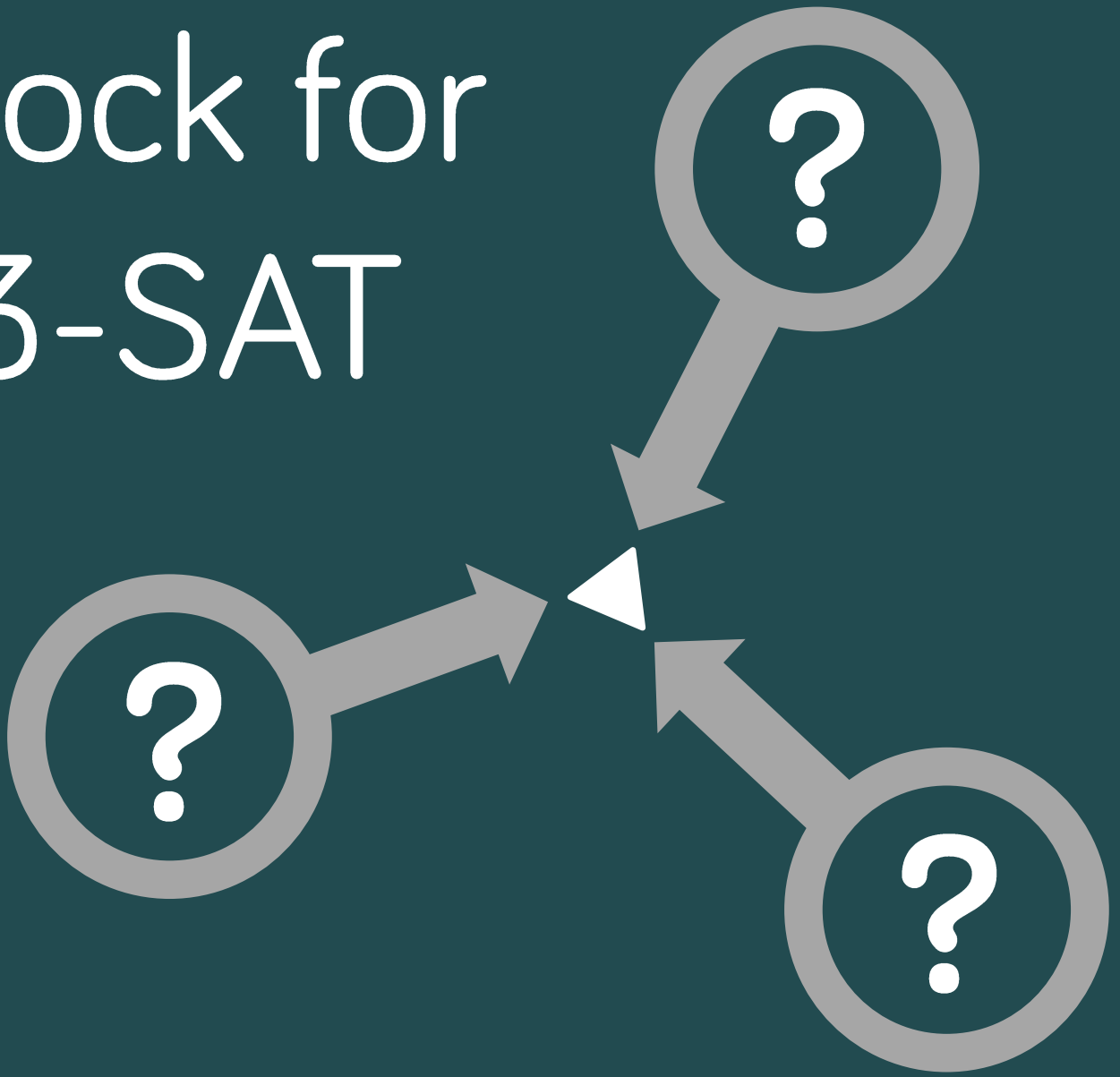
- the domain wall  transitions: 3-local
- the pulse  transitions: 2-local
  needs initialization)-:

5 Constructing clocks

- the domain wall  transitions: 3-local
- the pulse  transitions: 2-local
  needs initialization)-:

- make a good clock that ticks *2-locally?*

a clock for Q-3-SAT



5 Constructing a composite clock with 2-local progress

1111000000000
1111100000000
1111110000000
1111111000000
1111111100000
1111111110000
1111111111000
1111111111100
1111111111110

5 Constructing a composite clock with 2-local progress

111 100 000 000
111 110 000 000
111 111 000 000
111 111 100 000
111 111 110 000
111 111 111 000
111 111 111 100
111 111 111 110

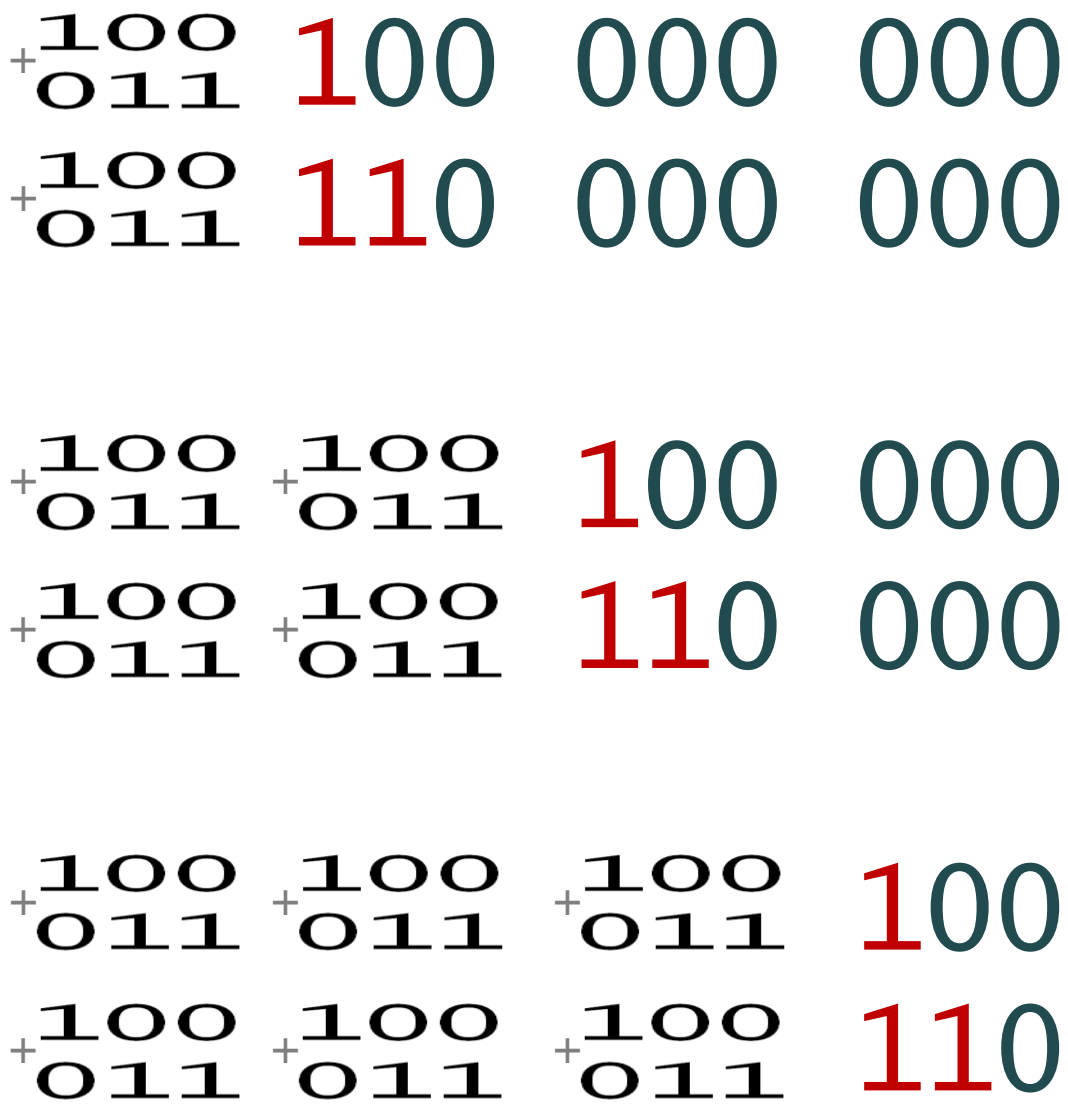
5 Constructing a composite clock with 2-local progress

111 100 000 000
111 110 000 000

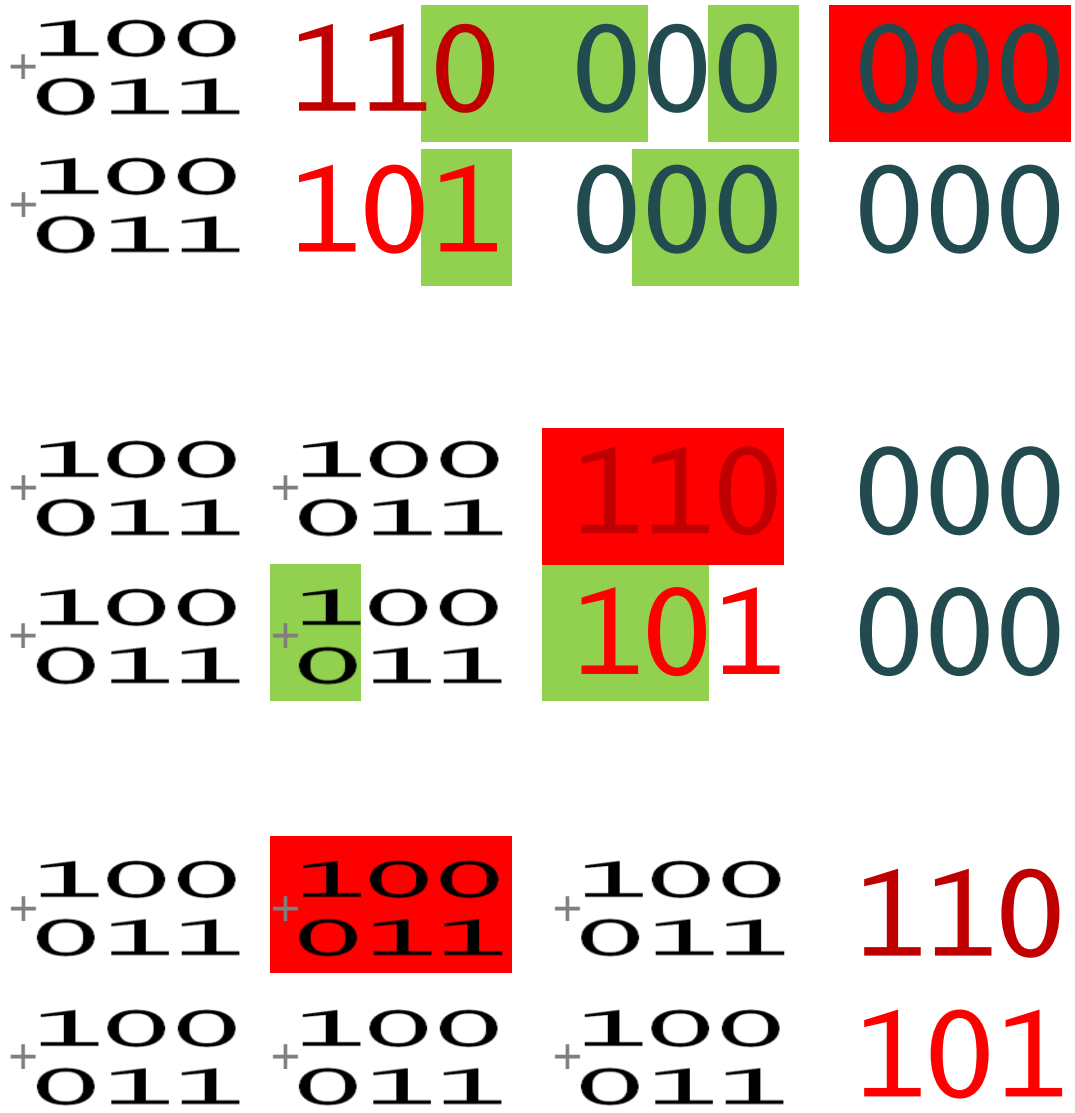
111 111 100 000
111 111 110 000

111 111 111 100
111 111 111 110

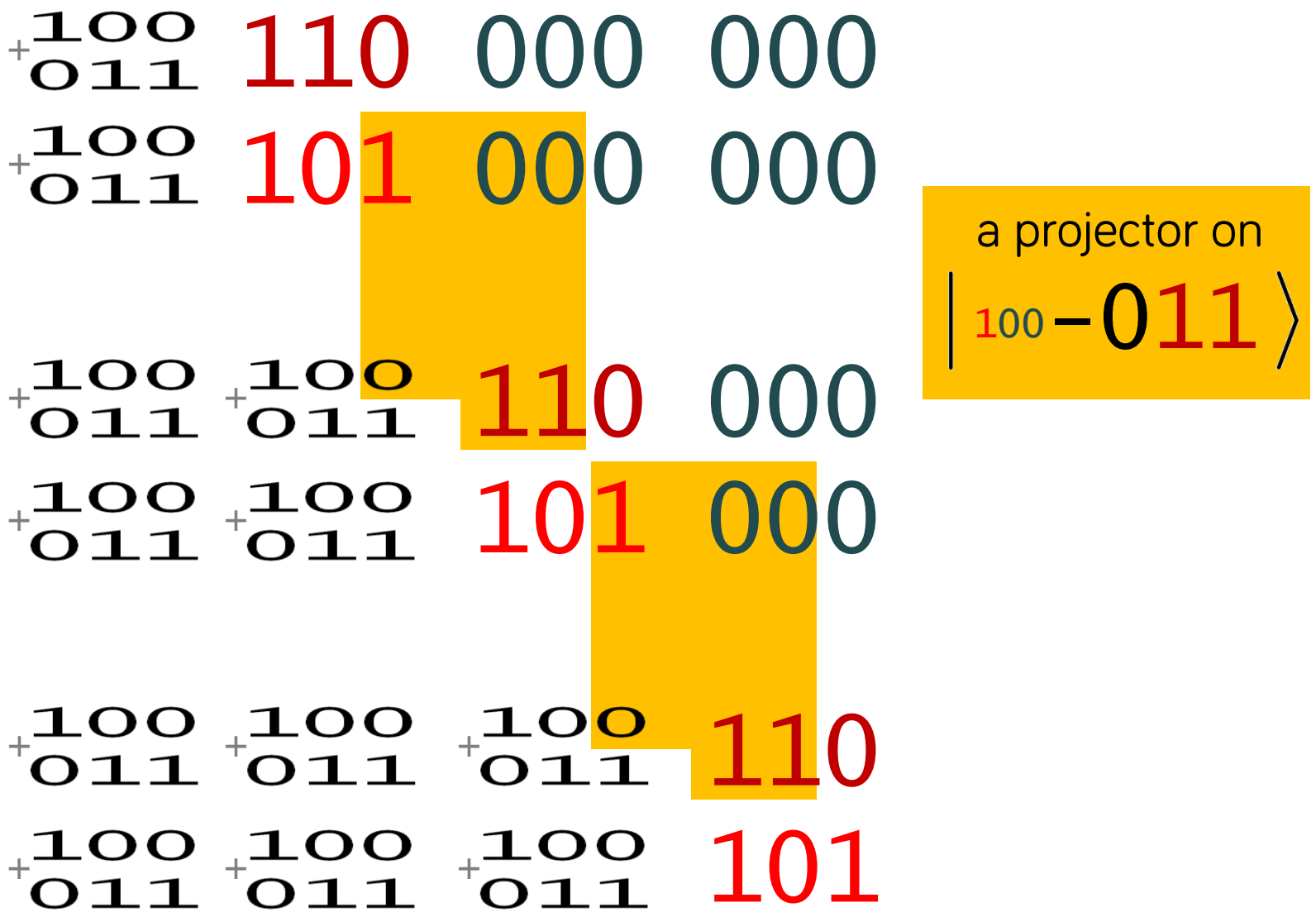
5 Constructing a composite clock with 2-local progress



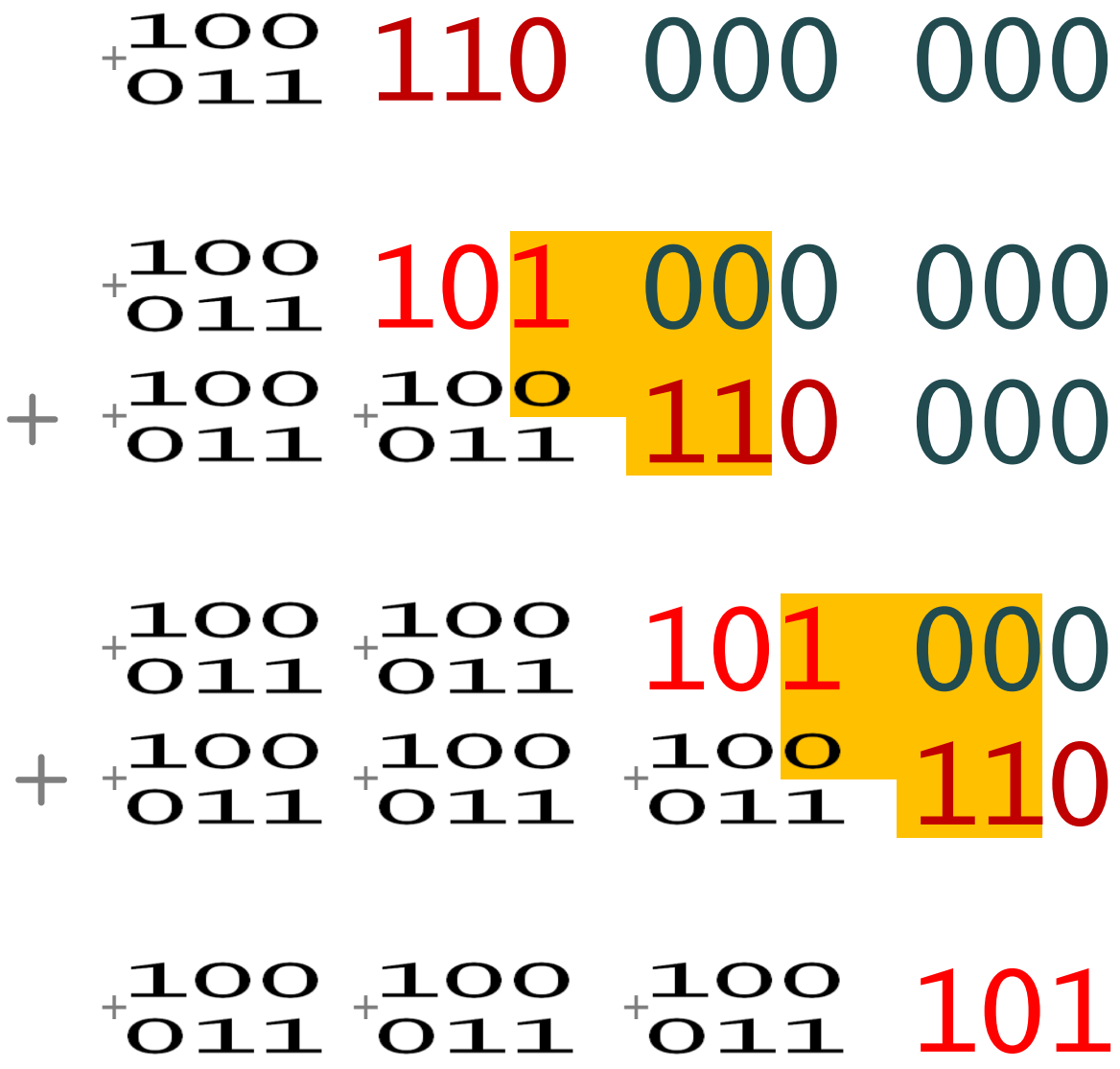
5 Look at triplets



5 Look at triplets

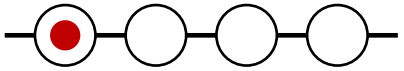


5 Look at triplets, combine some lines



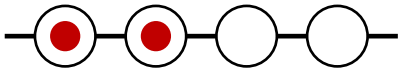
5 A composite clock made of triplets

$|1\rangle_c$ $+^{100}_{011}$ 110 000 000



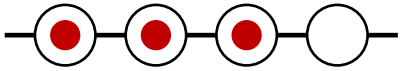
$|2\rangle_c$ $+^{100}_{011}$ 101 000 000

$+^{100}_{011}$ $+^{100}_{011}$ 110 000

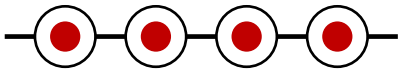


$|3\rangle_c$ $+^{100}_{011}$ $+^{100}_{011}$ 101 000

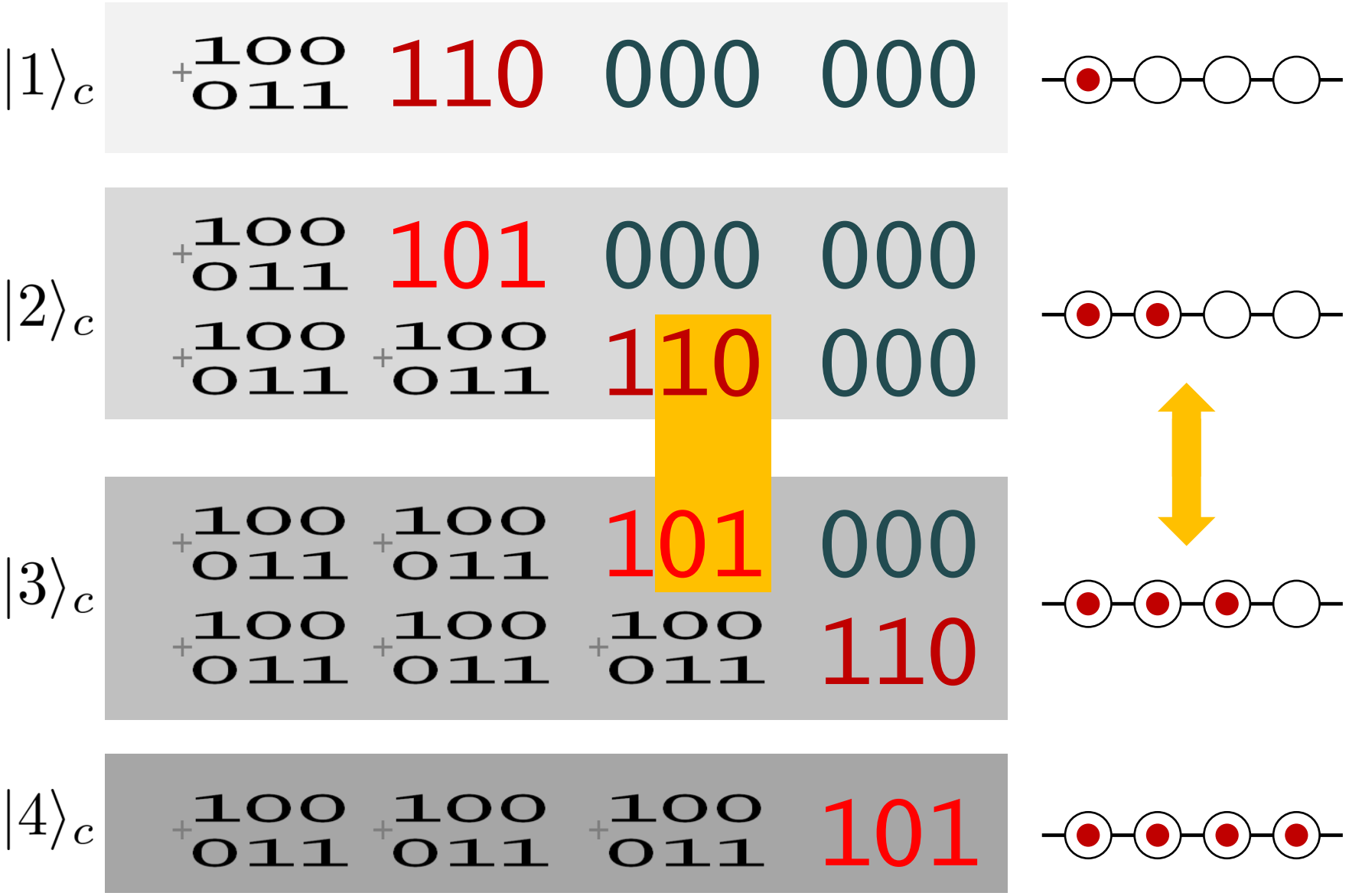
$+^{100}_{011}$ $+^{100}_{011}$ $+^{100}_{011}$ 110

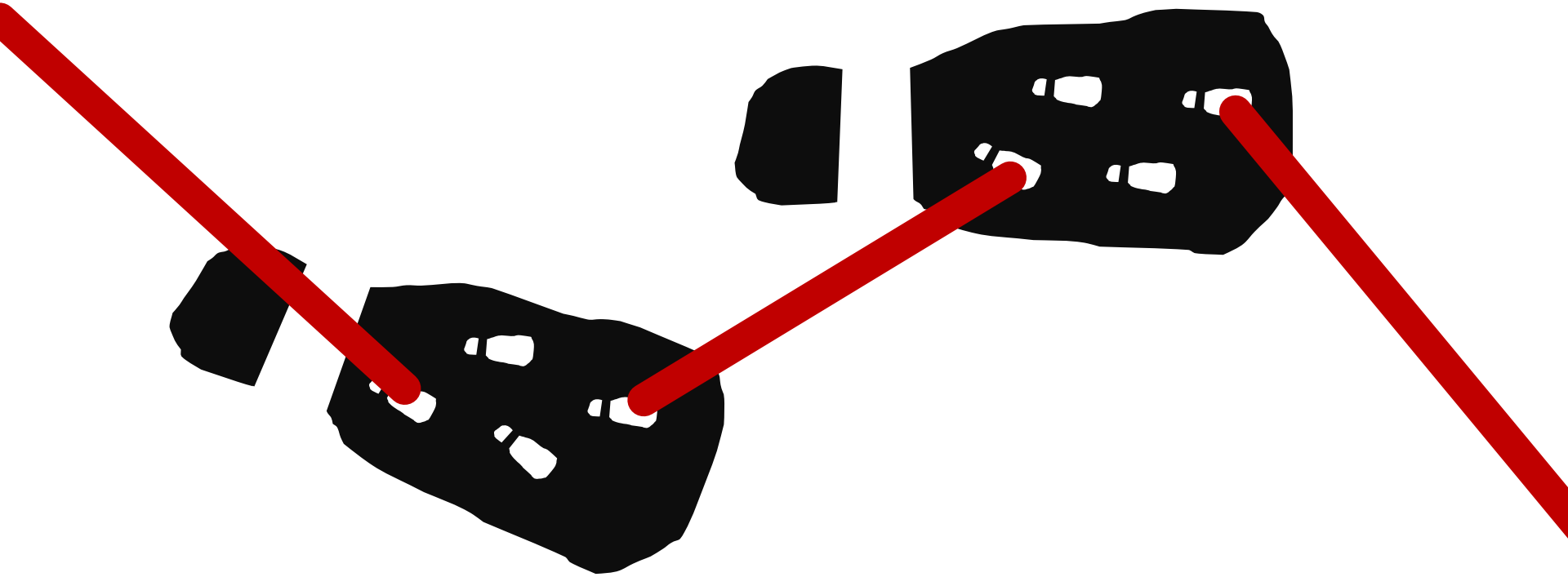


$|4\rangle_c$ $+^{100}_{011}$ $+^{100}_{011}$ $+^{100}_{011}$ 101



5 A composite clock with **2-local progress**

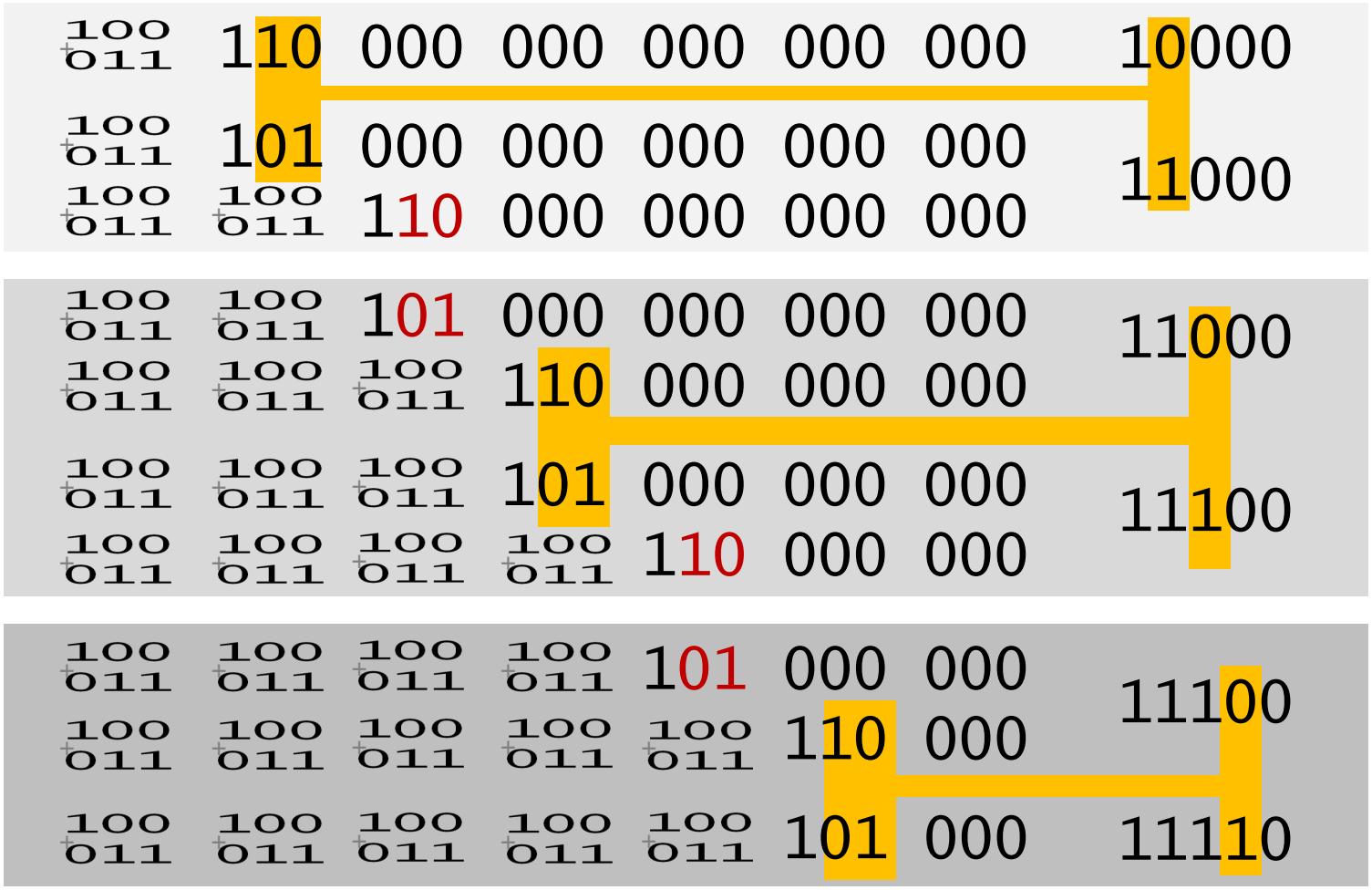




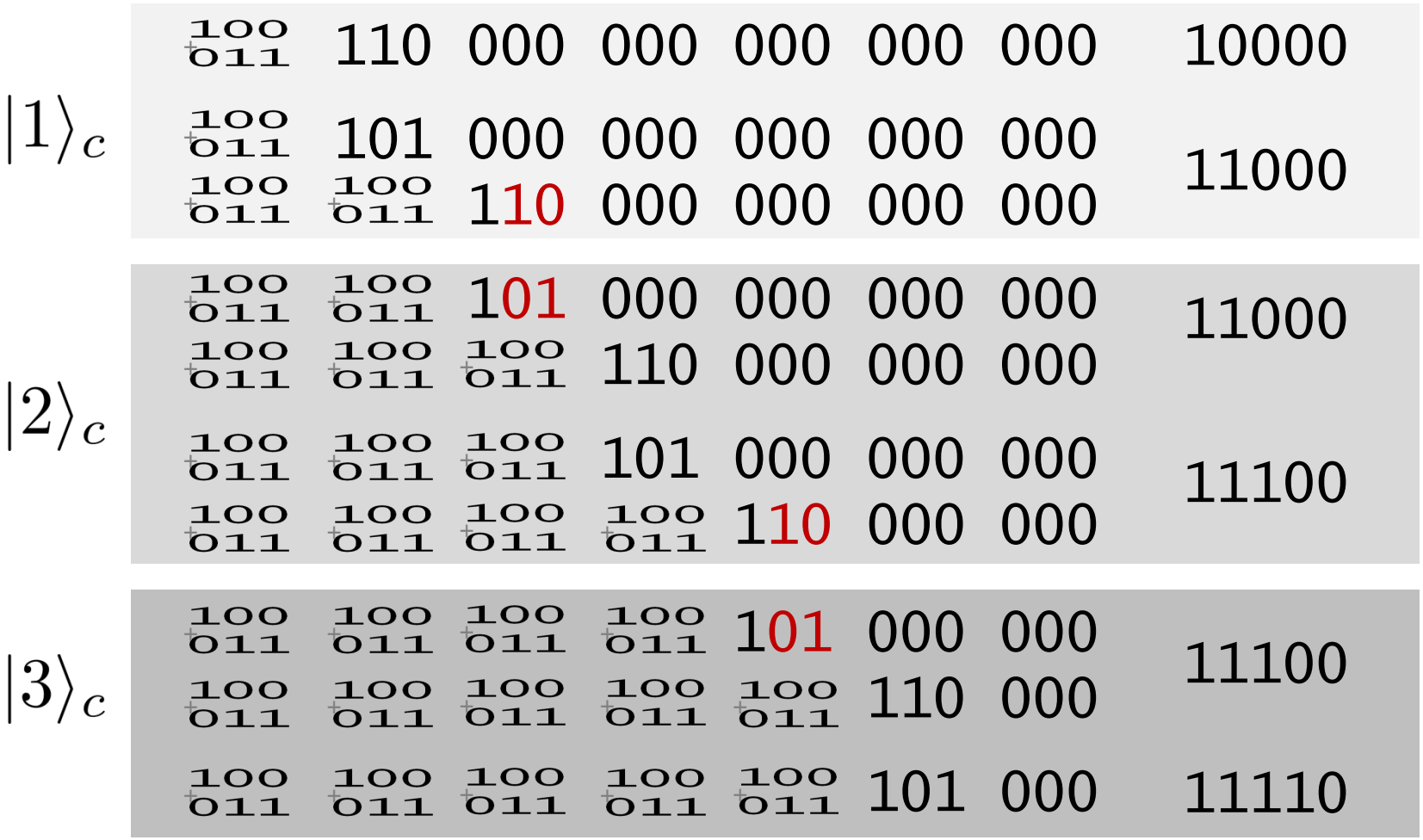
5 Is my time running out?

100 011	110	000	000	000	000	000	10000
100 011	101	000	000	000	000	000	11000
100 011	100 011	110	000	000	000	000	
100 011	100 011	101	000	000	000	000	11000
100 011	100 011	100 011	110	000	000	000	
100 011	100 011	100 011	101	000	000	000	11100
100 011	100 011	100 011	100 011	110	000	000	
100 011	100 011	100 011	100 011	101	000	000	11100
100 011	100 011	100 011	100 011	100 011	110	000	
100 011	100 011	100 011	100 011	100 011	101	000	11110

5 Is my time running out?

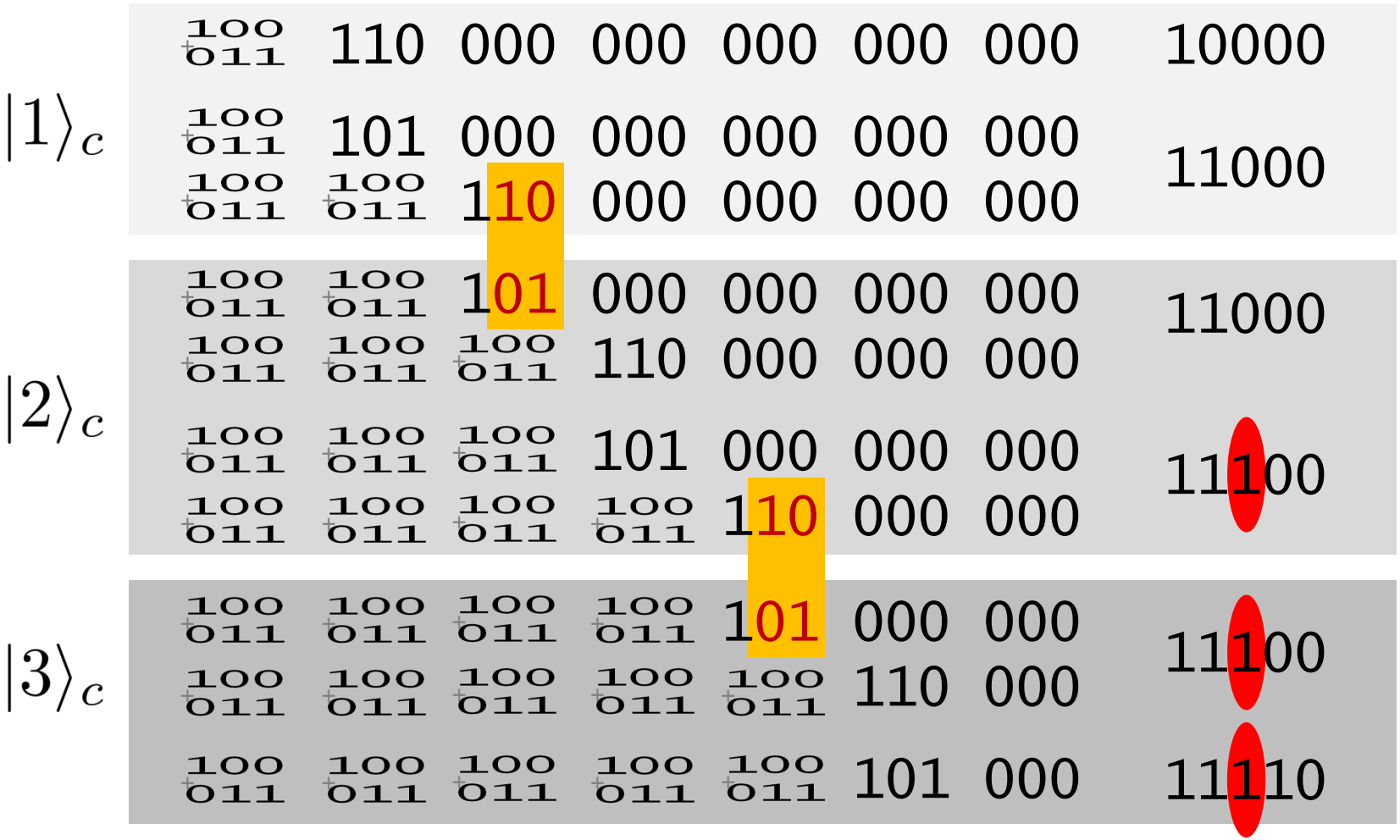


5 Is my time running out?



- a single active spot, a domain wall structure

5 Is my time running out? This clock can tell **1-locally**.



- 2-local connections, tell time 1-locally

5 Constructing clocks

- finding out I'm late

1-locally

- advance the clock

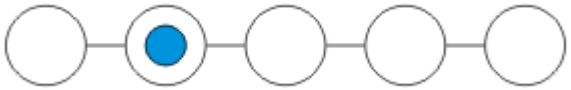
2-locally

- apply a 2-qubit gate

3-locally?

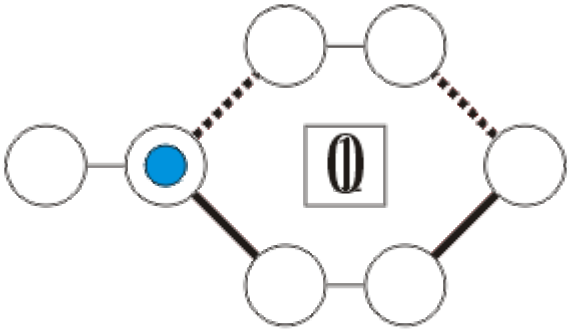
5 Applying 2-qubit gates 3-locally

- the railroad switch



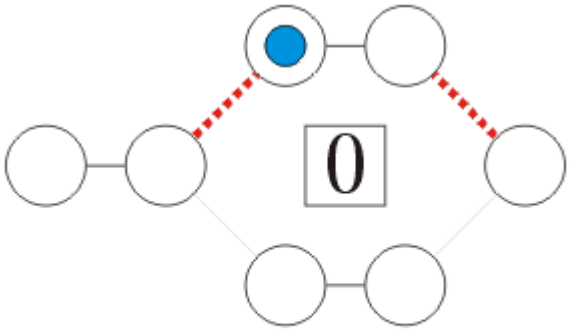
5 Applying 2-qubit gates 3-locally

- the railroad switch



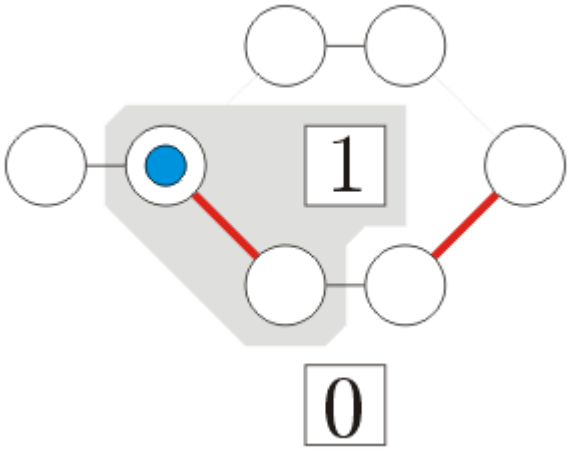
5 Applying 2-qubit gates 3-locally

- the railroad switch



5 Applying 2-qubit gates 3-locally

- the railroad switch

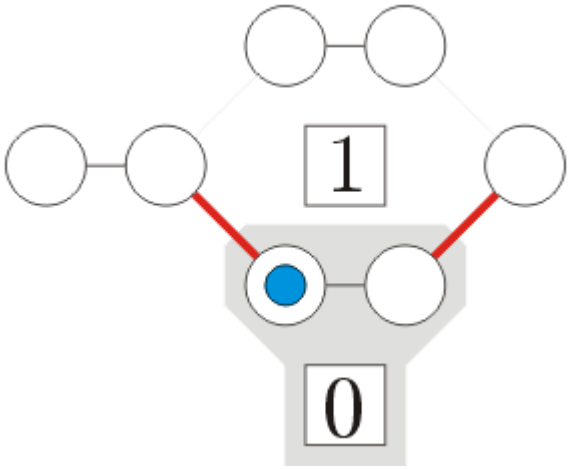


CNOT: 3-local

needs initialization

5 Applying 2-qubit gates 3-locally

- the railroad switch

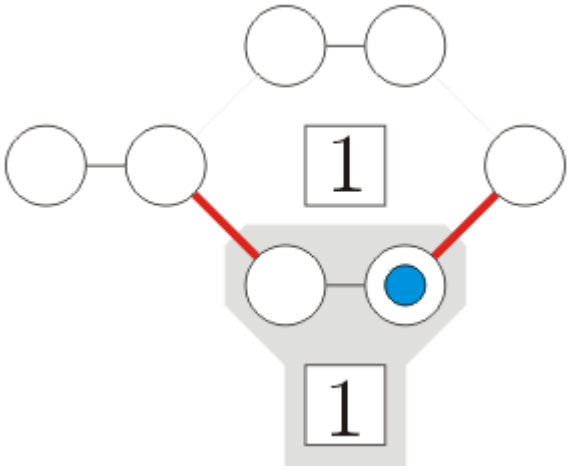


CNOT: 3-local

needs initialization

5 Applying 2-qubit gates 3-locally

- the railroad switch

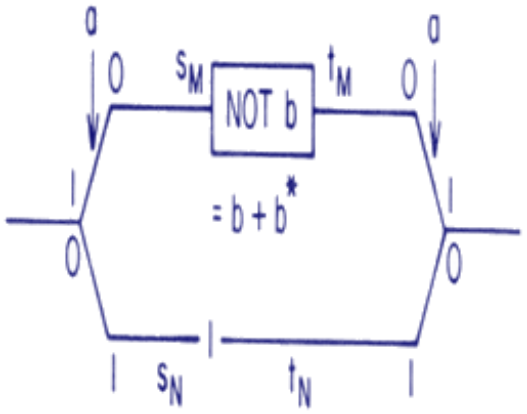


CNOT: 3-local

needs initialization

5 Applying 2-qubit gates 3-locally

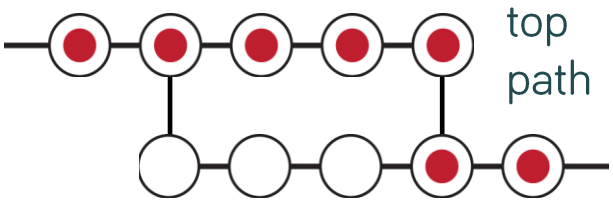
- the railroad switch

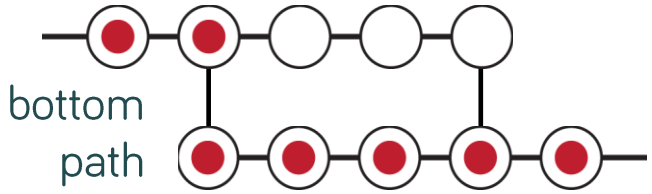


CNOT: 3-local

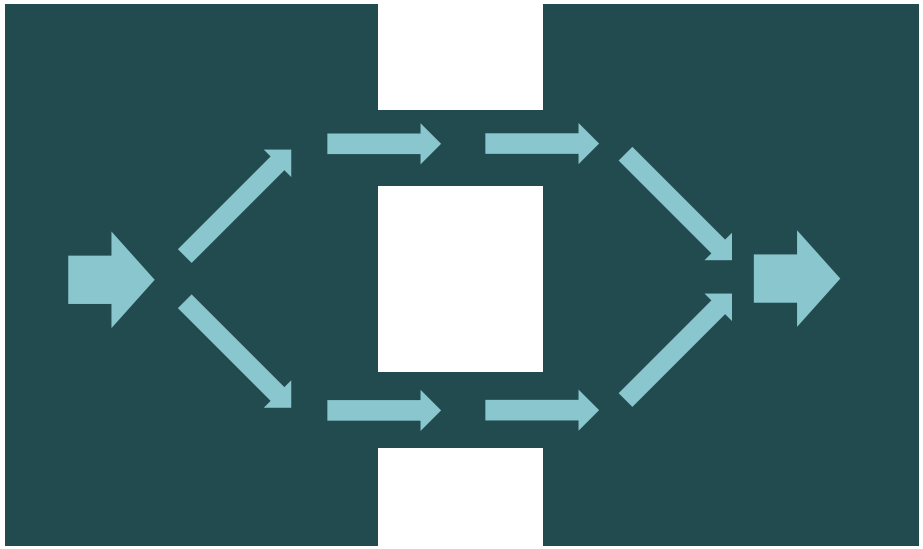
needs initialization

- a domain wall switch?

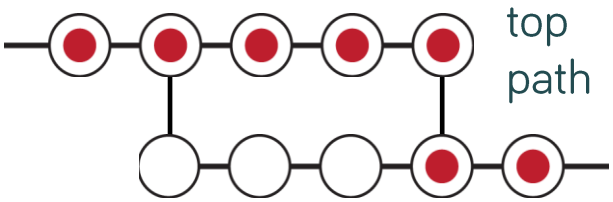


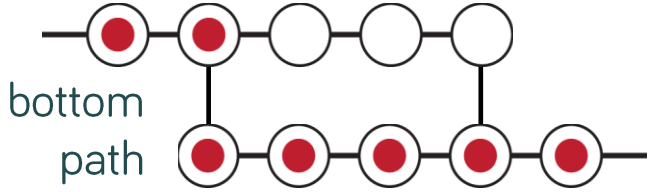


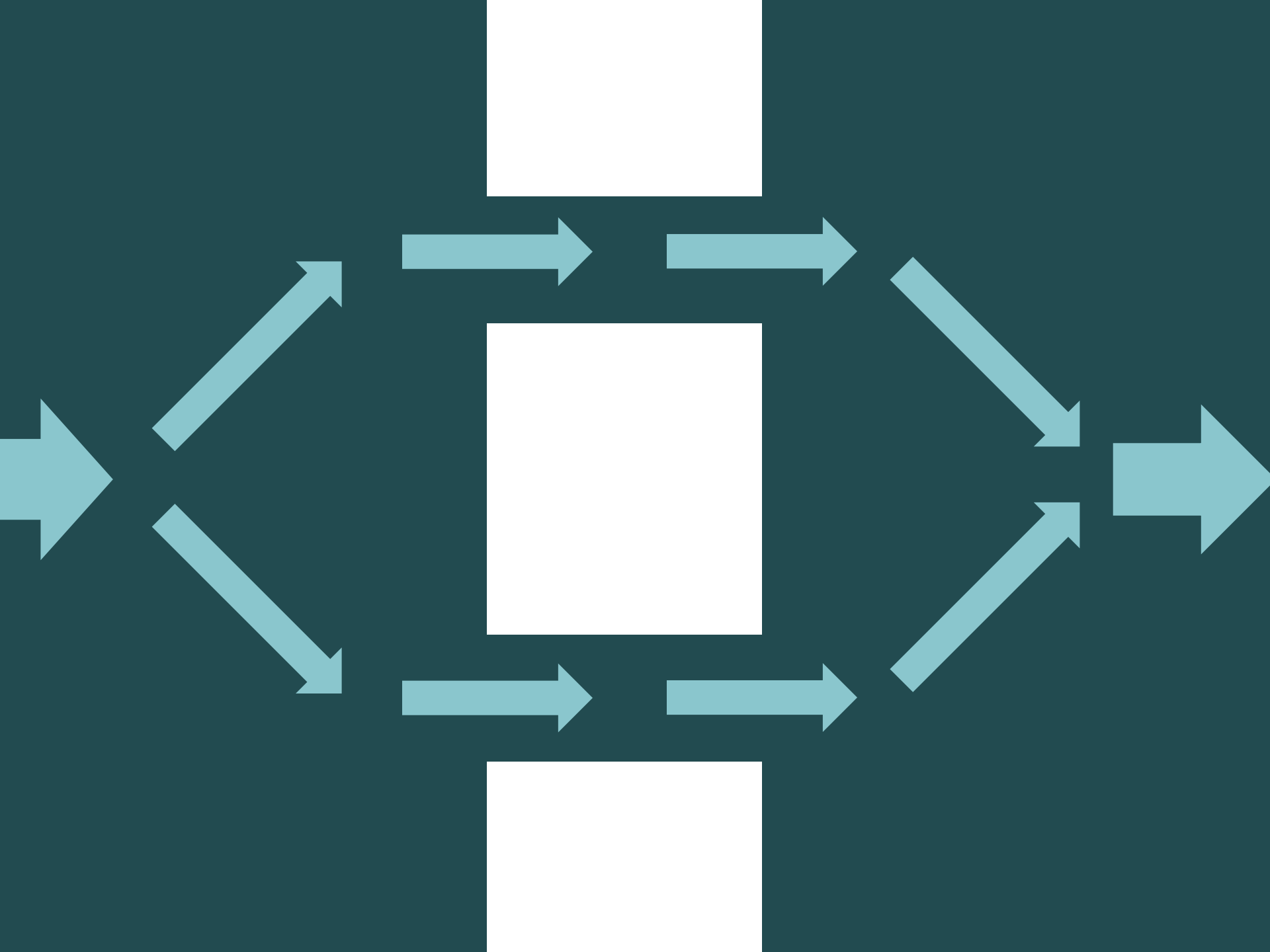
5 A train switch with a domain wall?

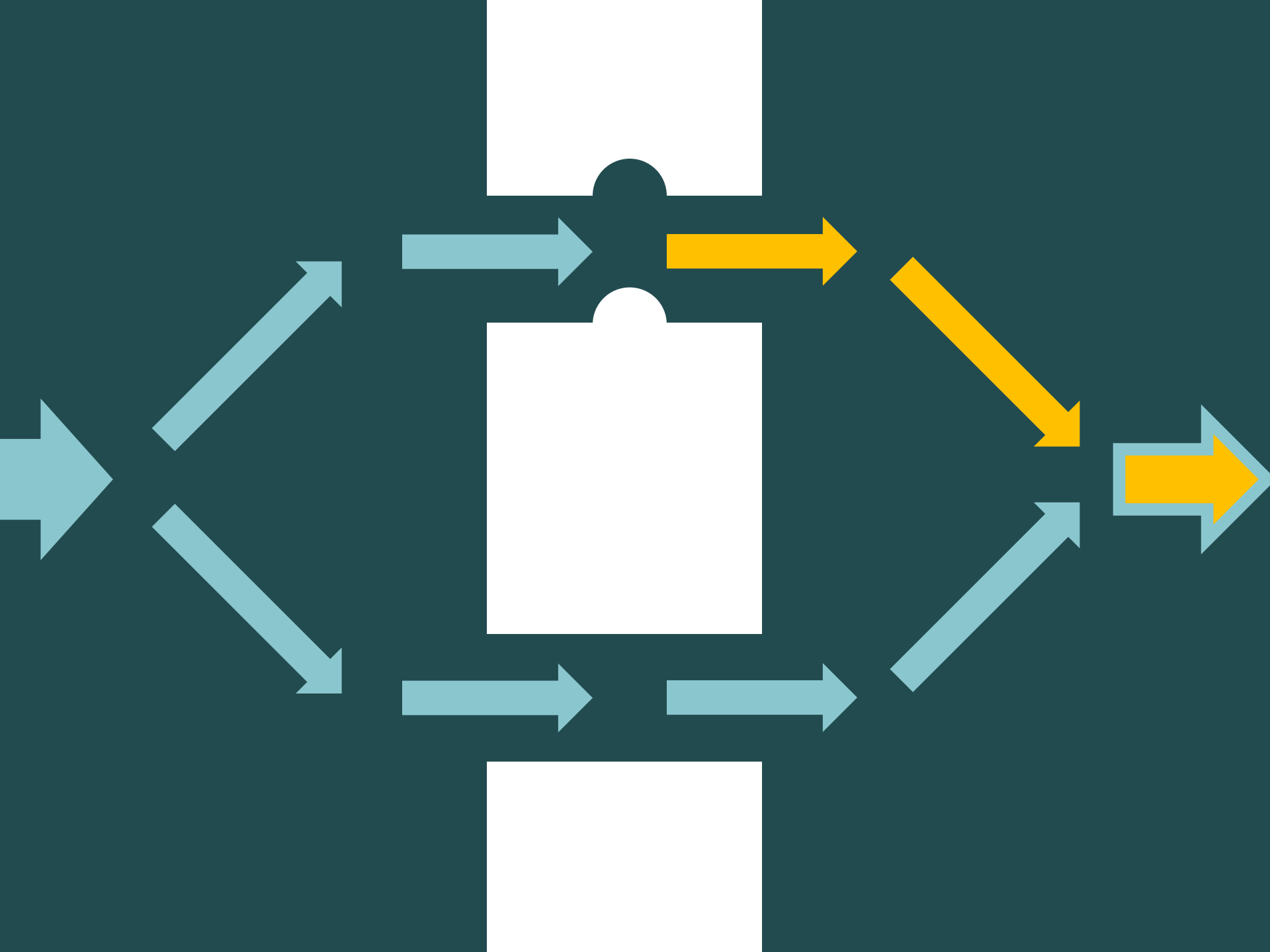


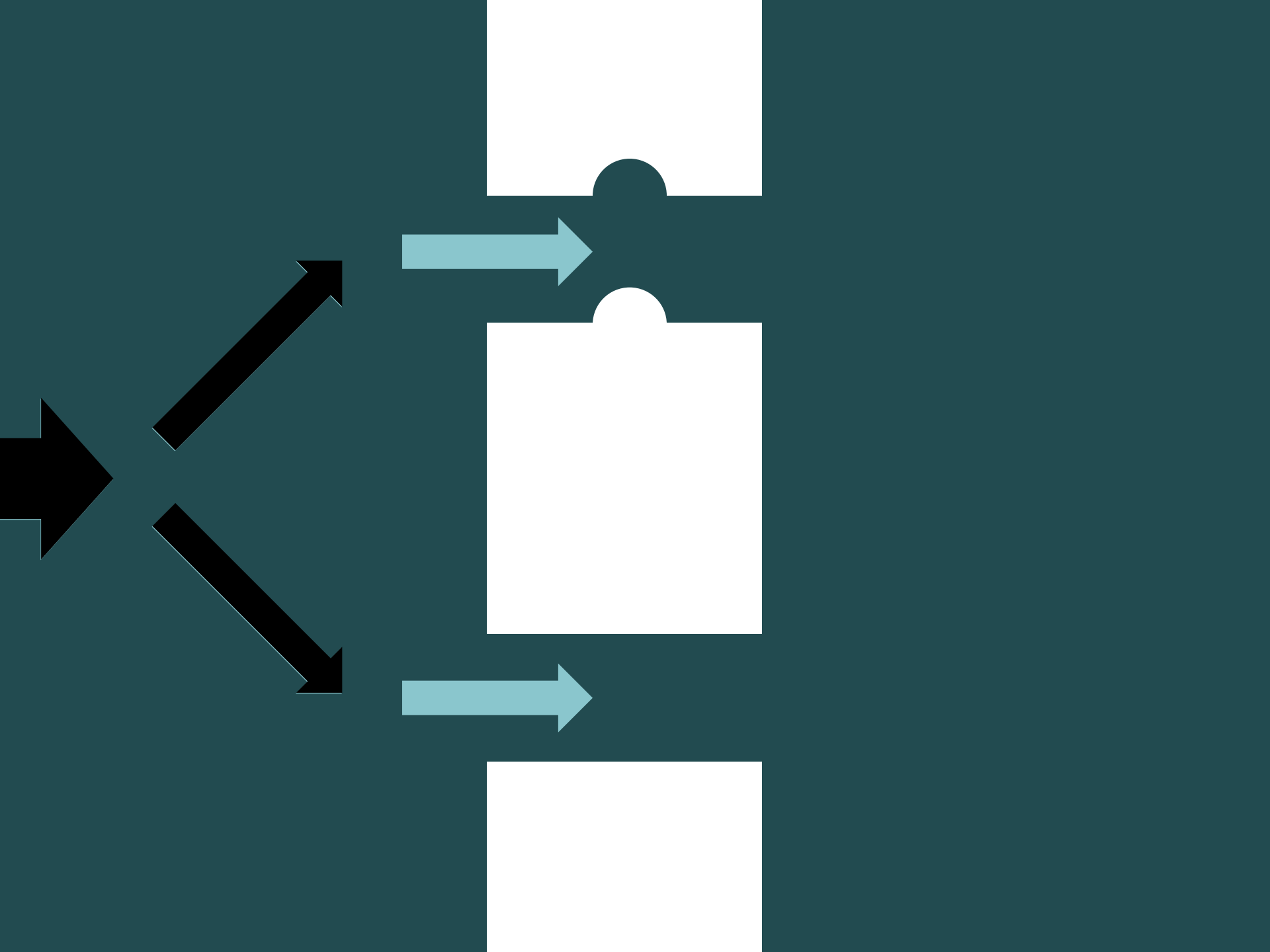
- a domain wall switch?

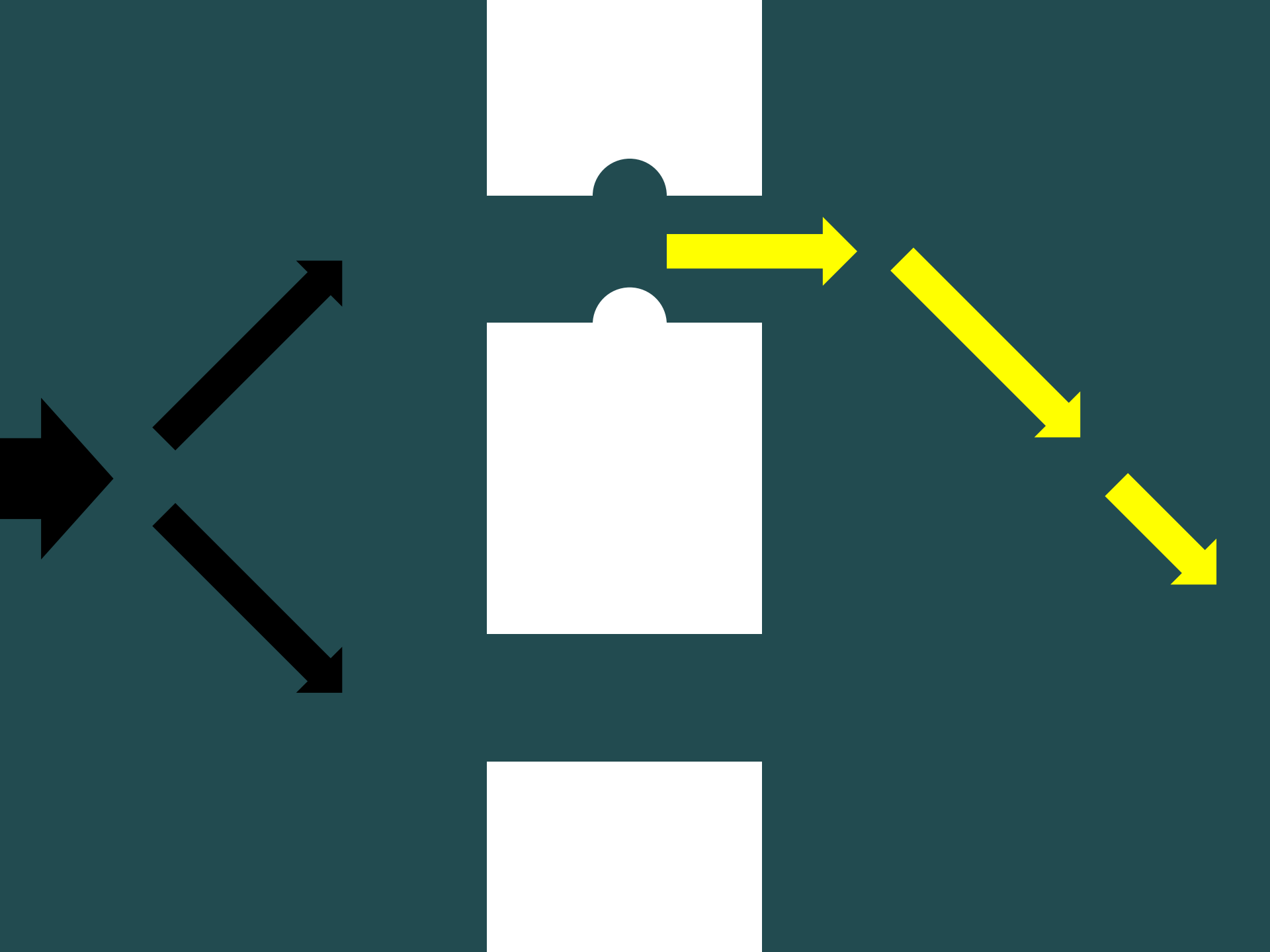




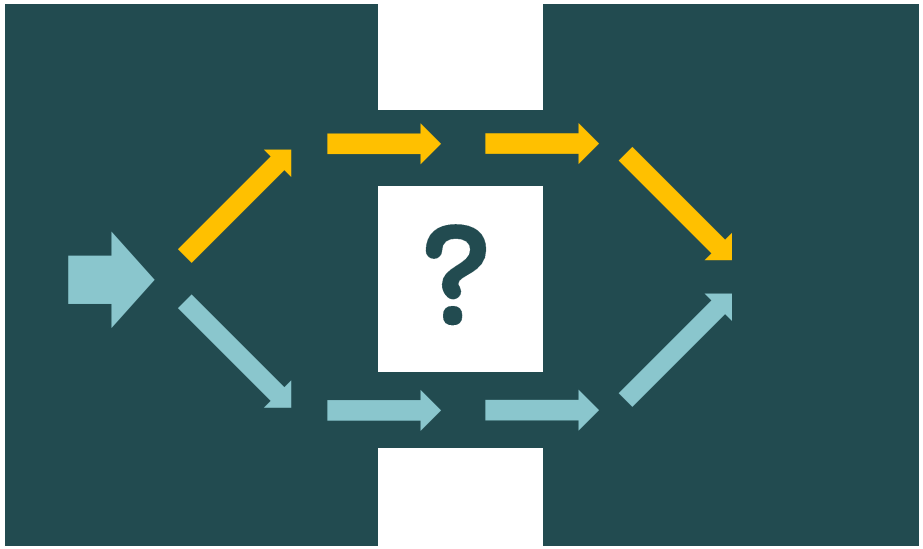




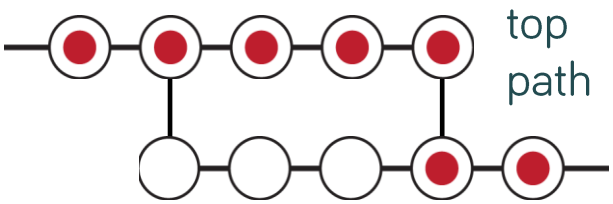


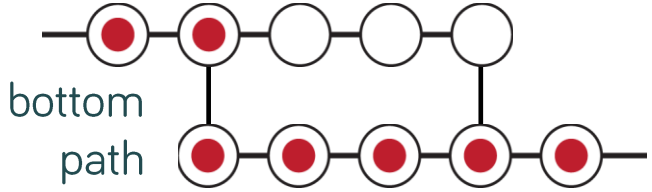


5 Combining a railroad switch with a domain wall?



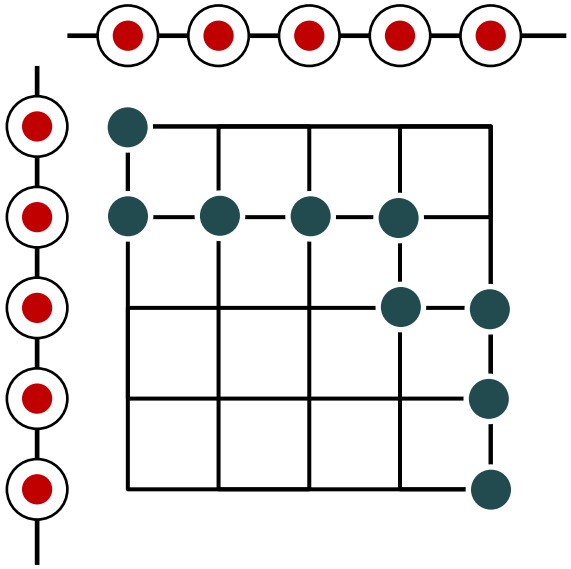
- a domain wall switch?





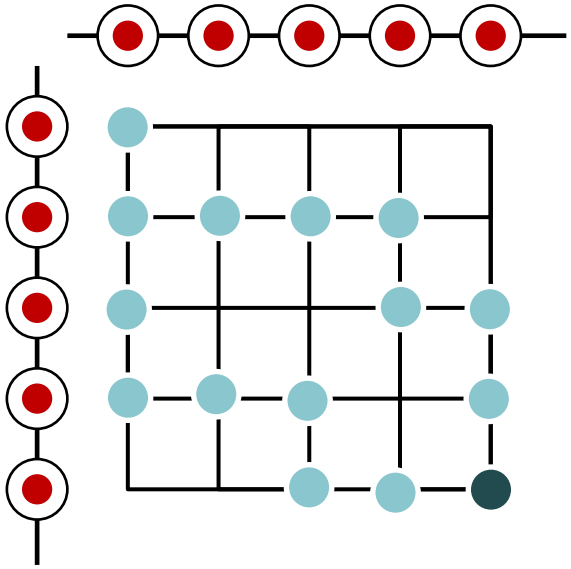
5 2D clocks (with two registers)

- two clocks



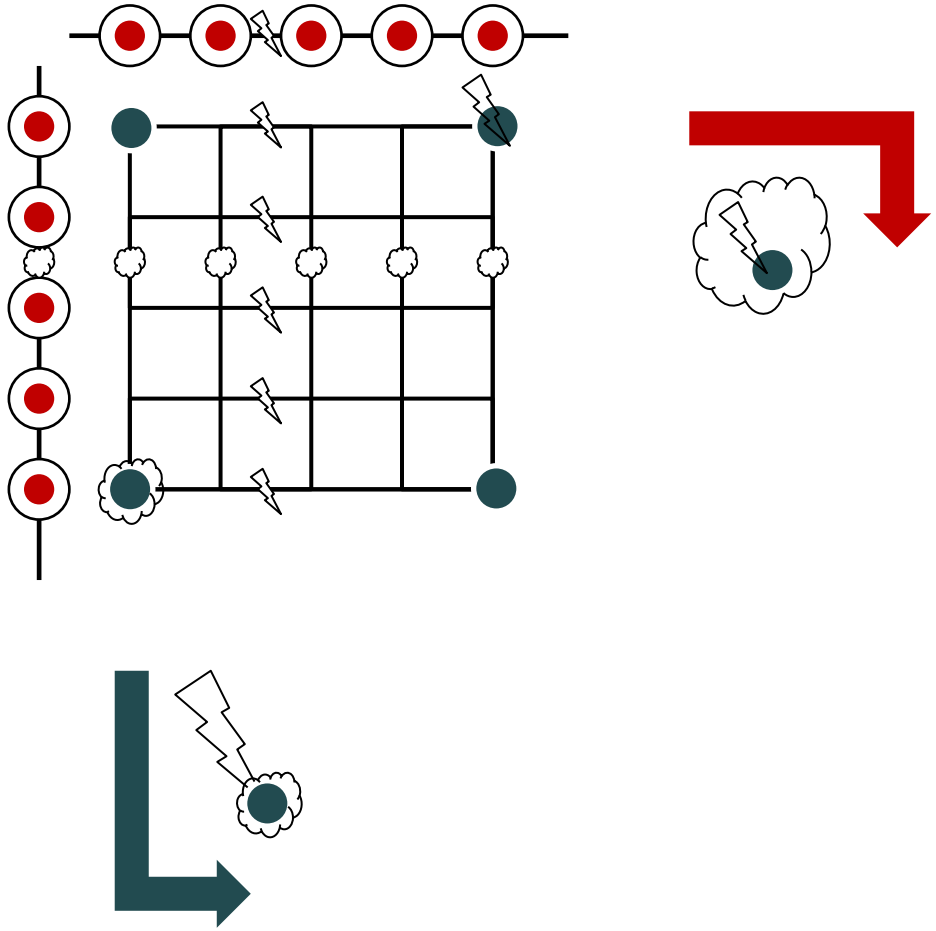
5 2D clocks (with two registers)

- two clocks



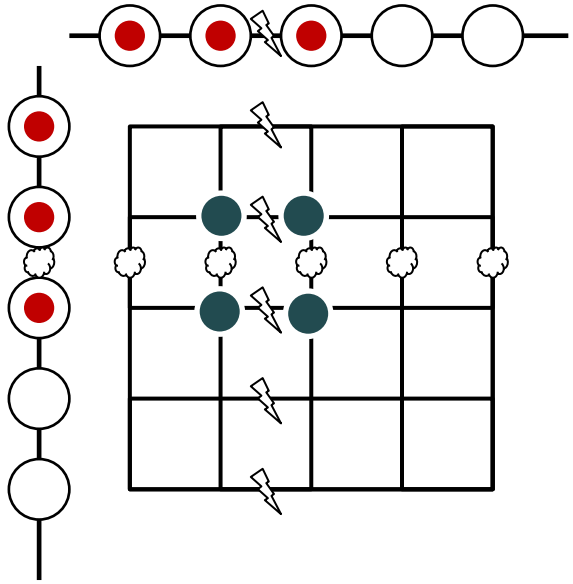
5 2D clocks (with two registers)

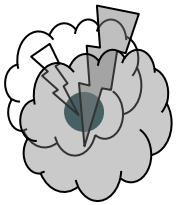
- add non-commuting (data) operations



5 2D clocks (with two registers)

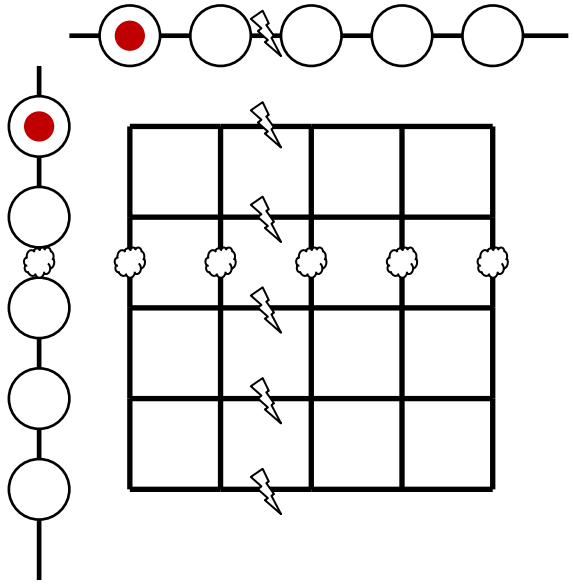
- add non-commuting (data) operations





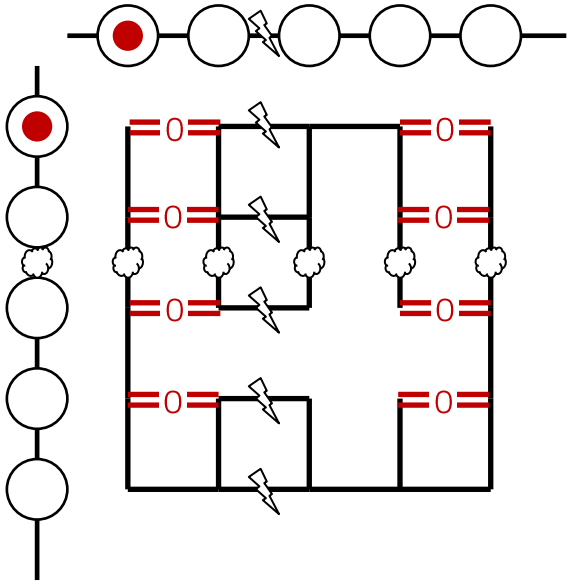
5 2D clocks (with two registers): fixing the mess

- remove some transitions



5 2D clocks (with two registers): fixing the mess

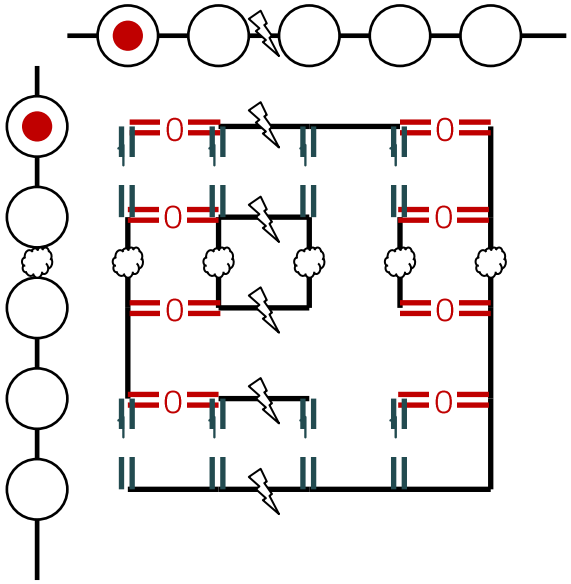
- condition transitions on a data (control) qubit



control: **0**

5 2D clocks (with two registers): fixing the mess

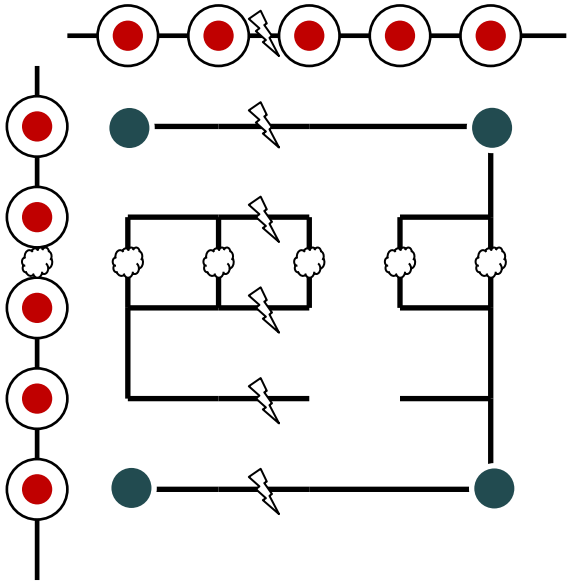
- condition transitions on a data (control) qubit

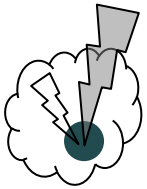


control: **0**,**1**

5 2D clocks (with two registers)

- condition transitions on a data (control) qubit

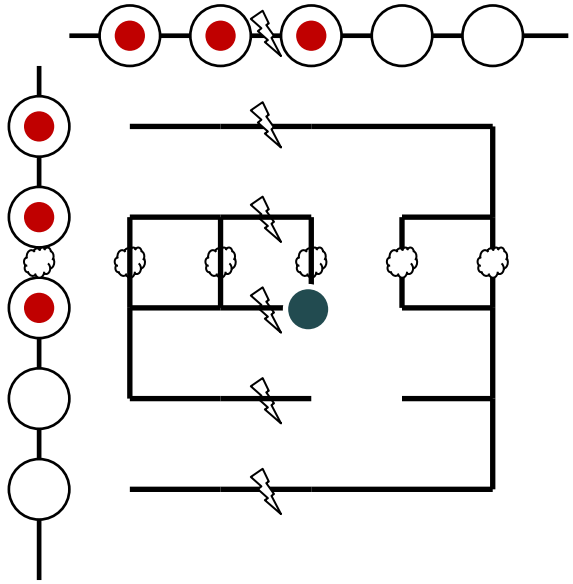


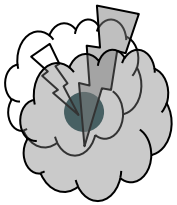


control: **0**

5 2D clocks (with two registers)

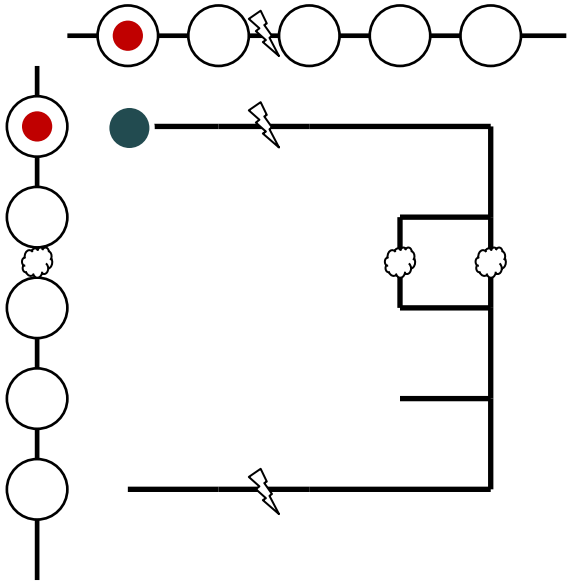
- there is no “bound” ground state (, don’t commute)

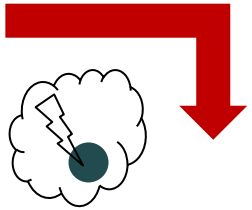




control: **0**

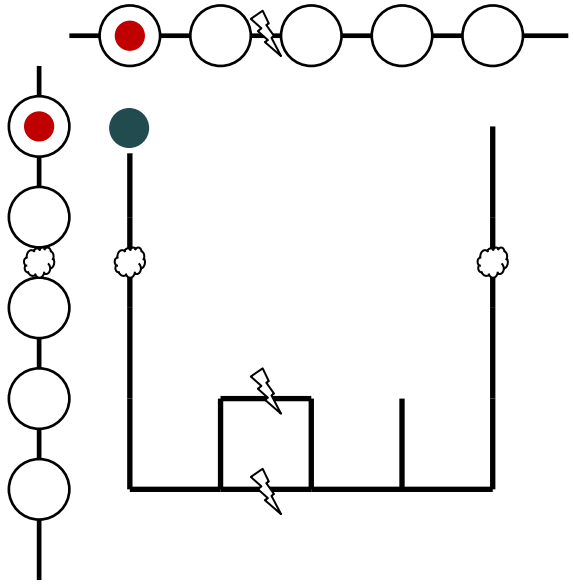
5 2D clocks (with two registers)

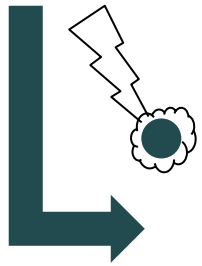




control: **0**

5 2D clocks (with two registers)

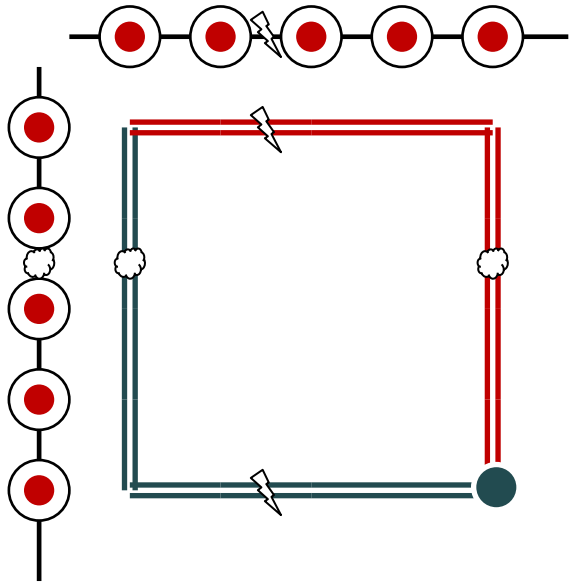


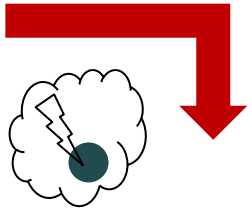


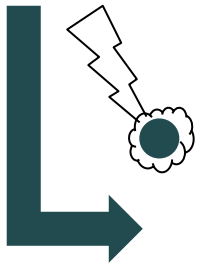
control: 1

5 2D clocks (with two registers)

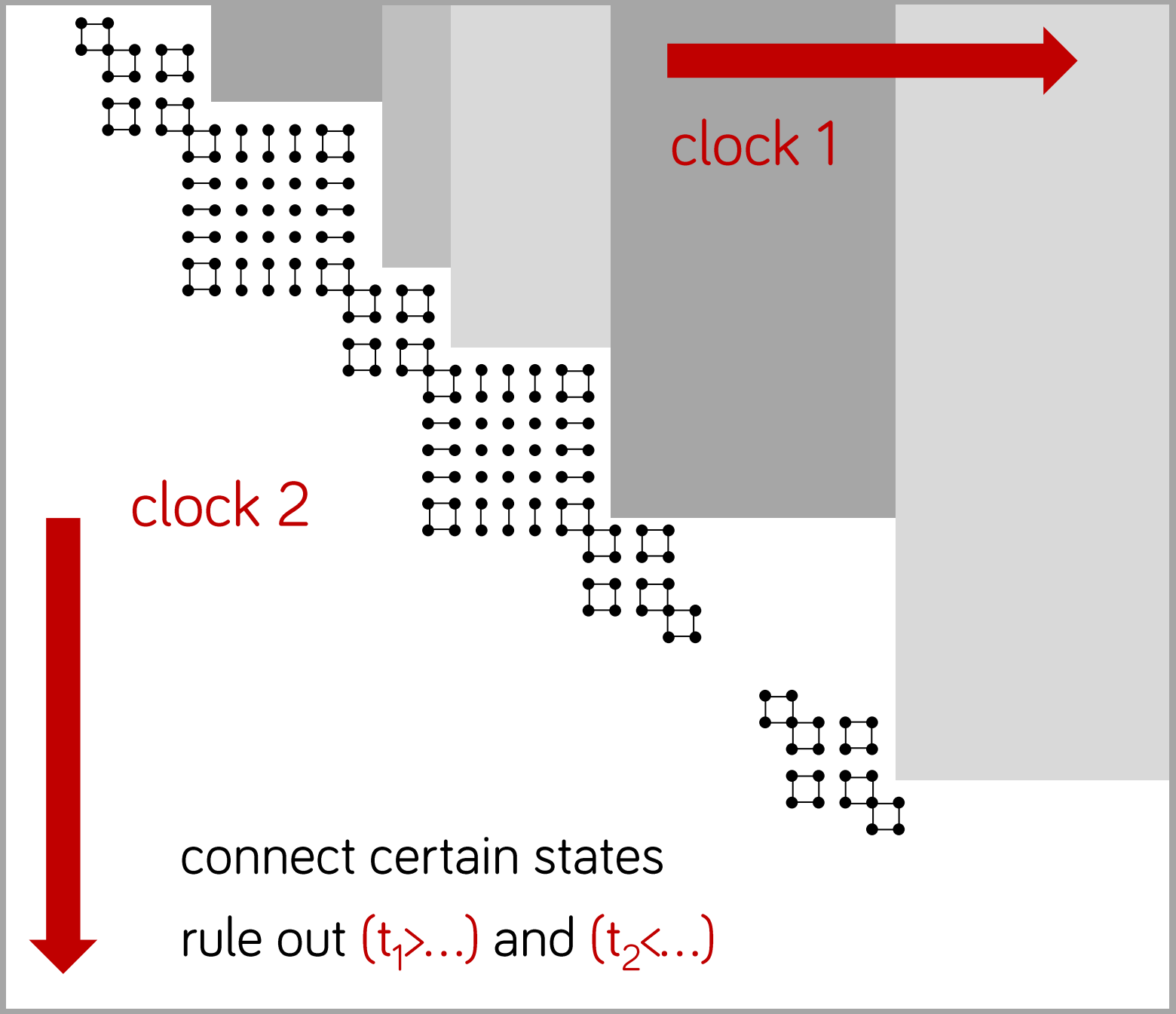
- like a railroad switch… with a single active site ensured

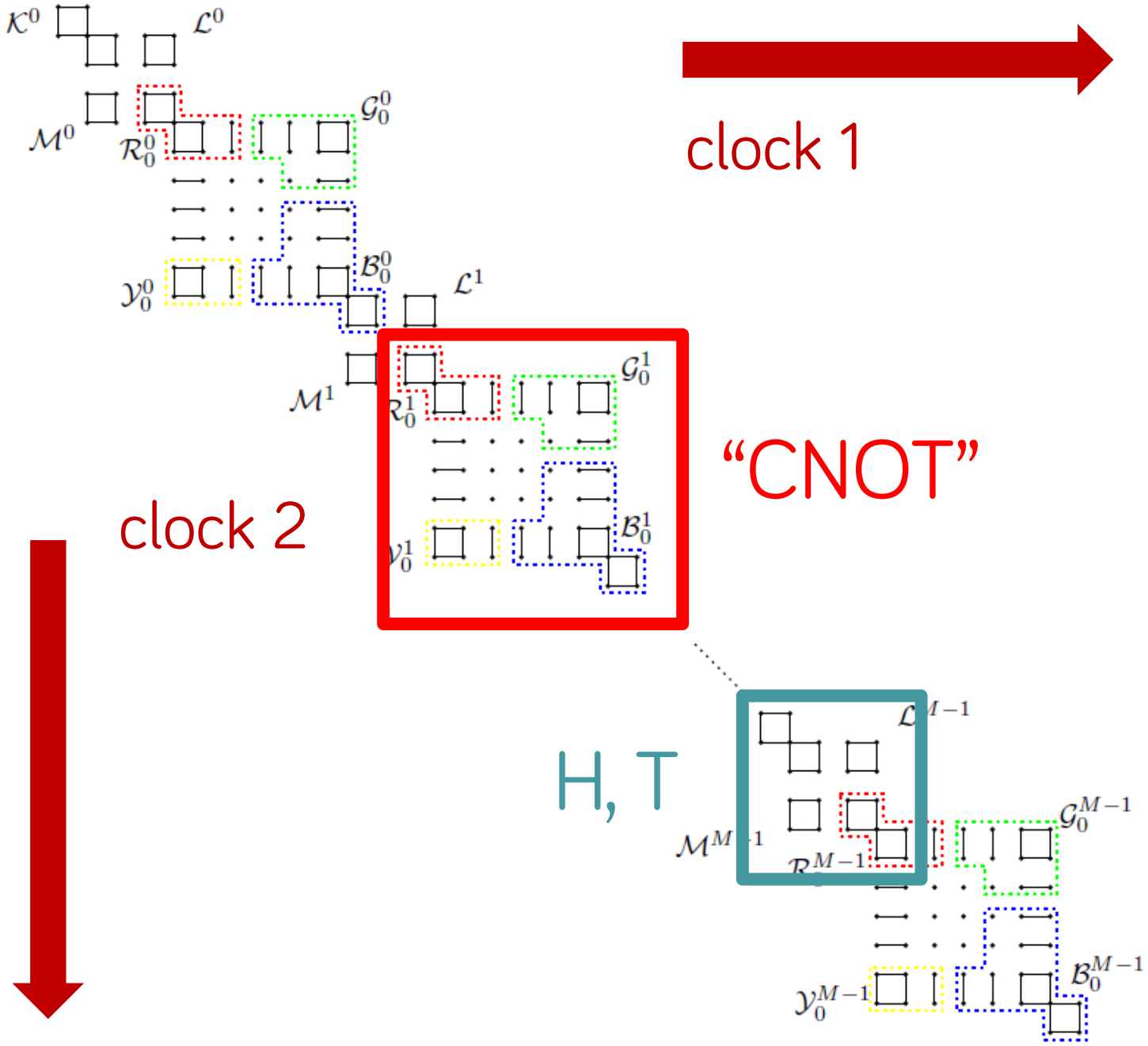


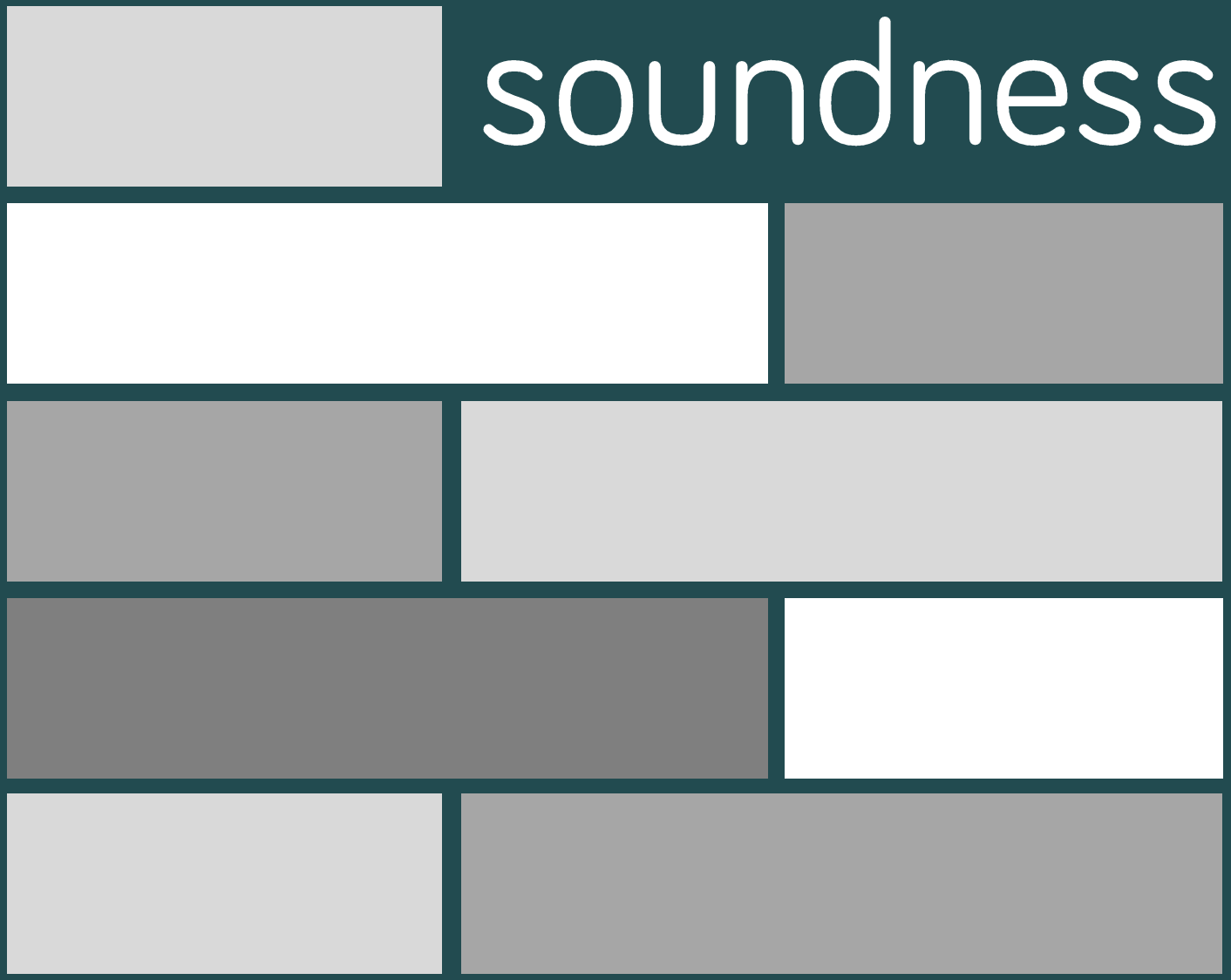




control: 0,1


clock 1
clock 2
connect certain states
rule out (t_1>...) and (t_2<...)


clock 1
clock 2
"CNOT"
H, T


soundness

projection lemma

projection lemma

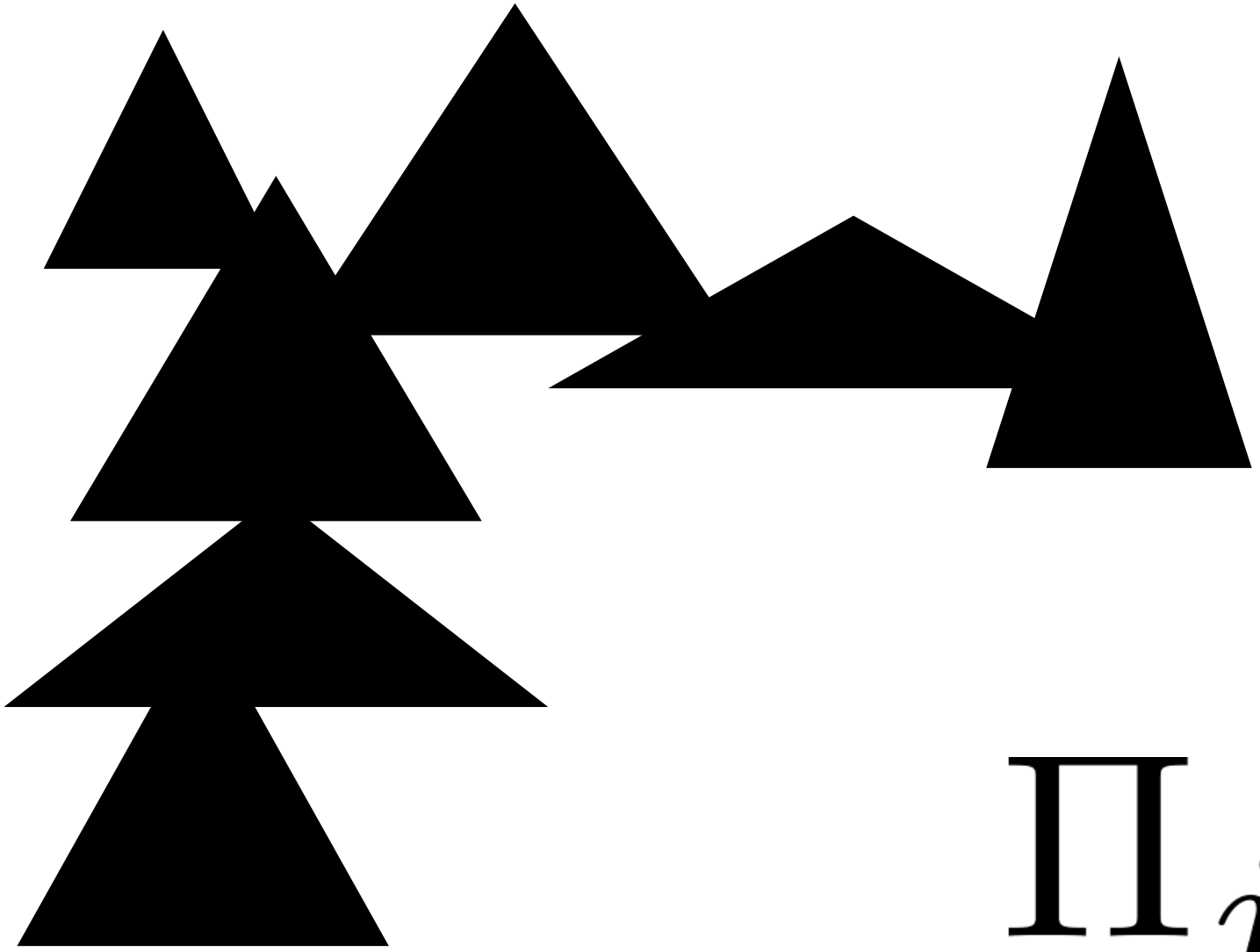
no solution?
all states have
a high energy

quantum 3-SAT
is QMA_1-complete

[Gosset, N. ’13]

random q-SAT

6 Random QSAT



$$\Pi_j$$

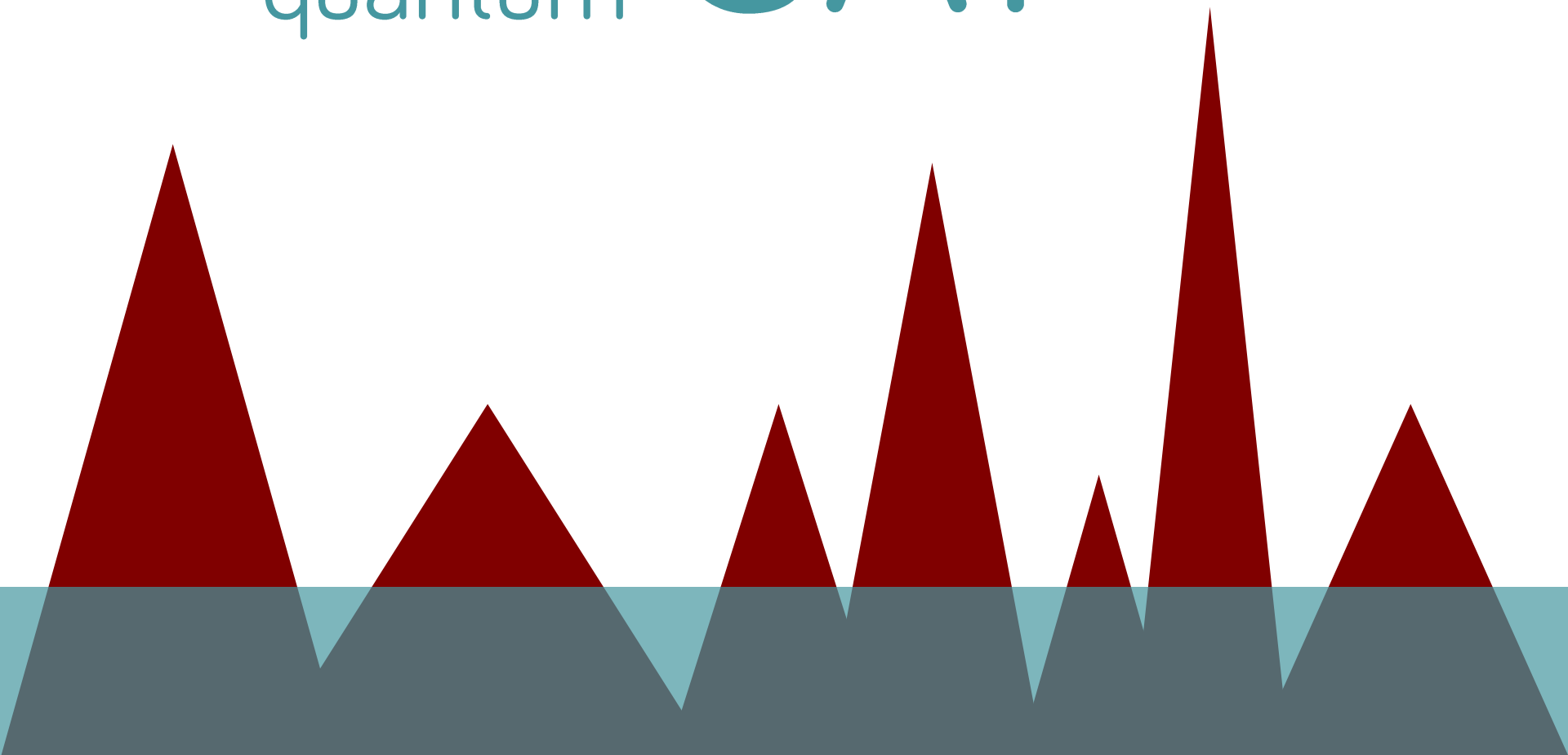

SAT


SAT


UNSAT

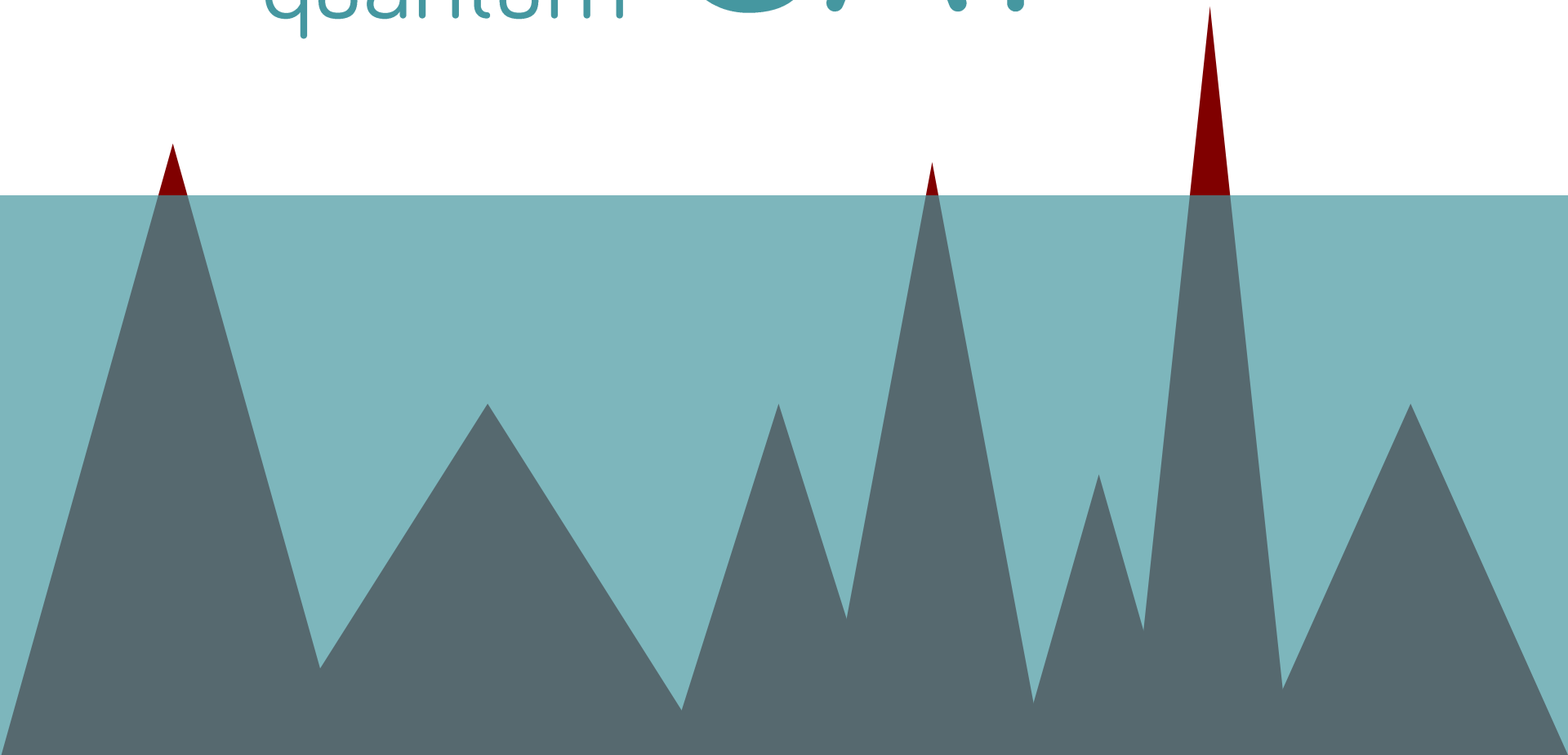
a fixed hypergraph of clauses

classical
quantum
SAT



a fixed hypergraph, more clauses

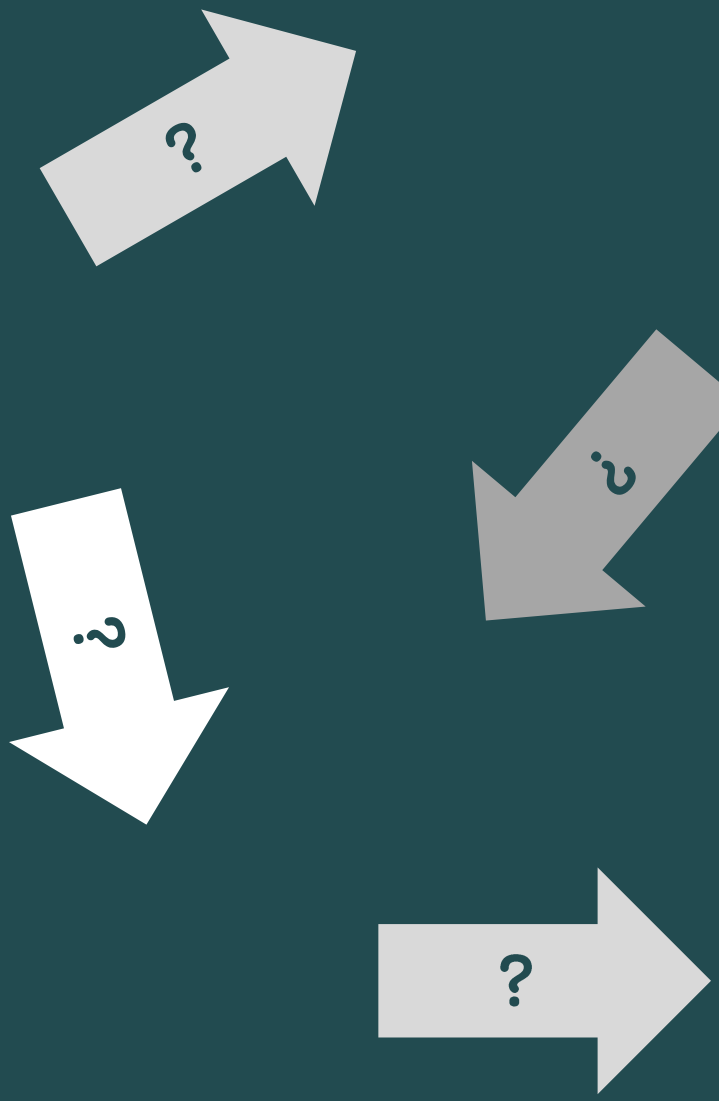
classical
quantum
SAT



a fixed hypergraph, worst clauses

adversary
quantum
SAT

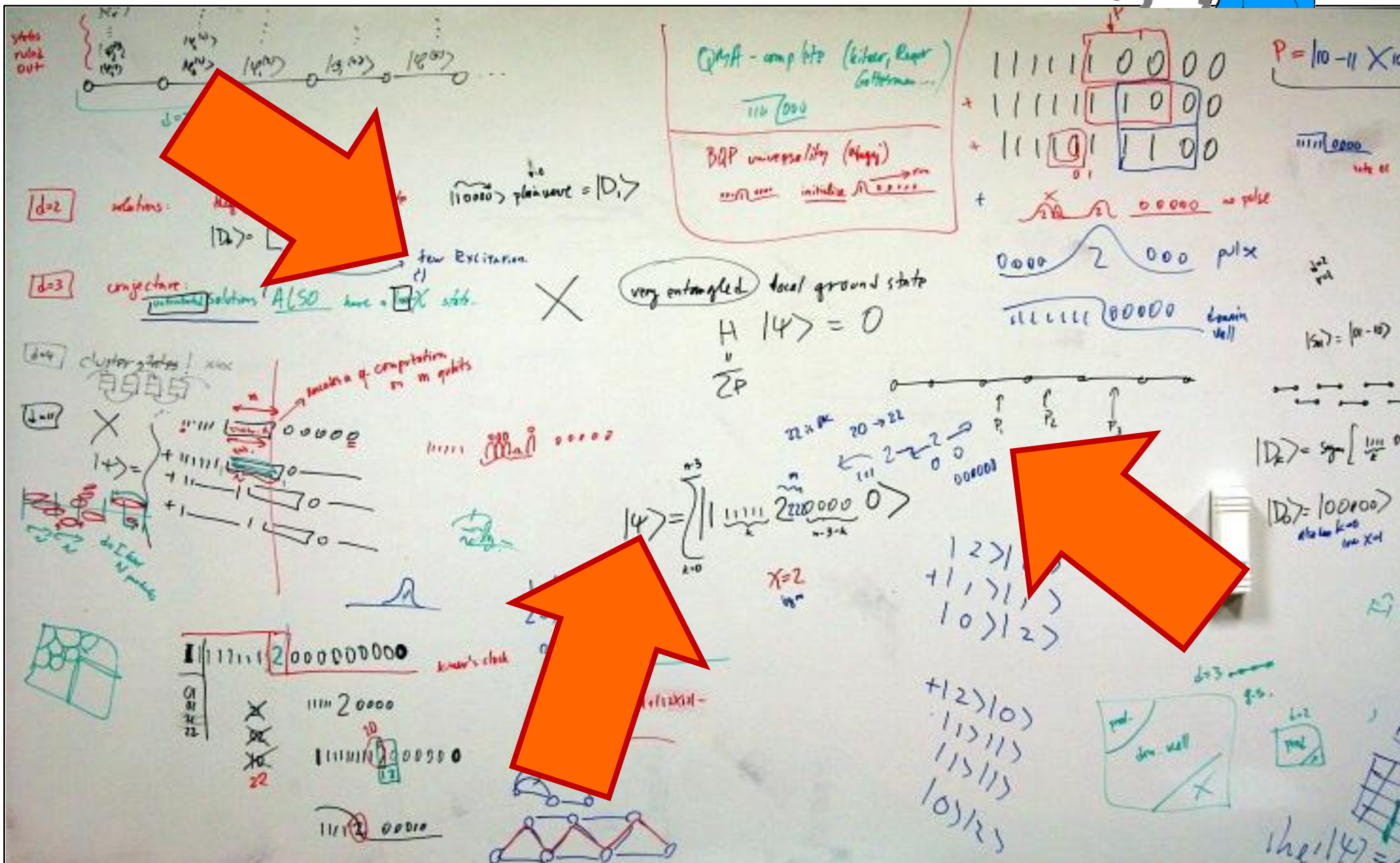
[Bardoscia, Scardicchio, N.]



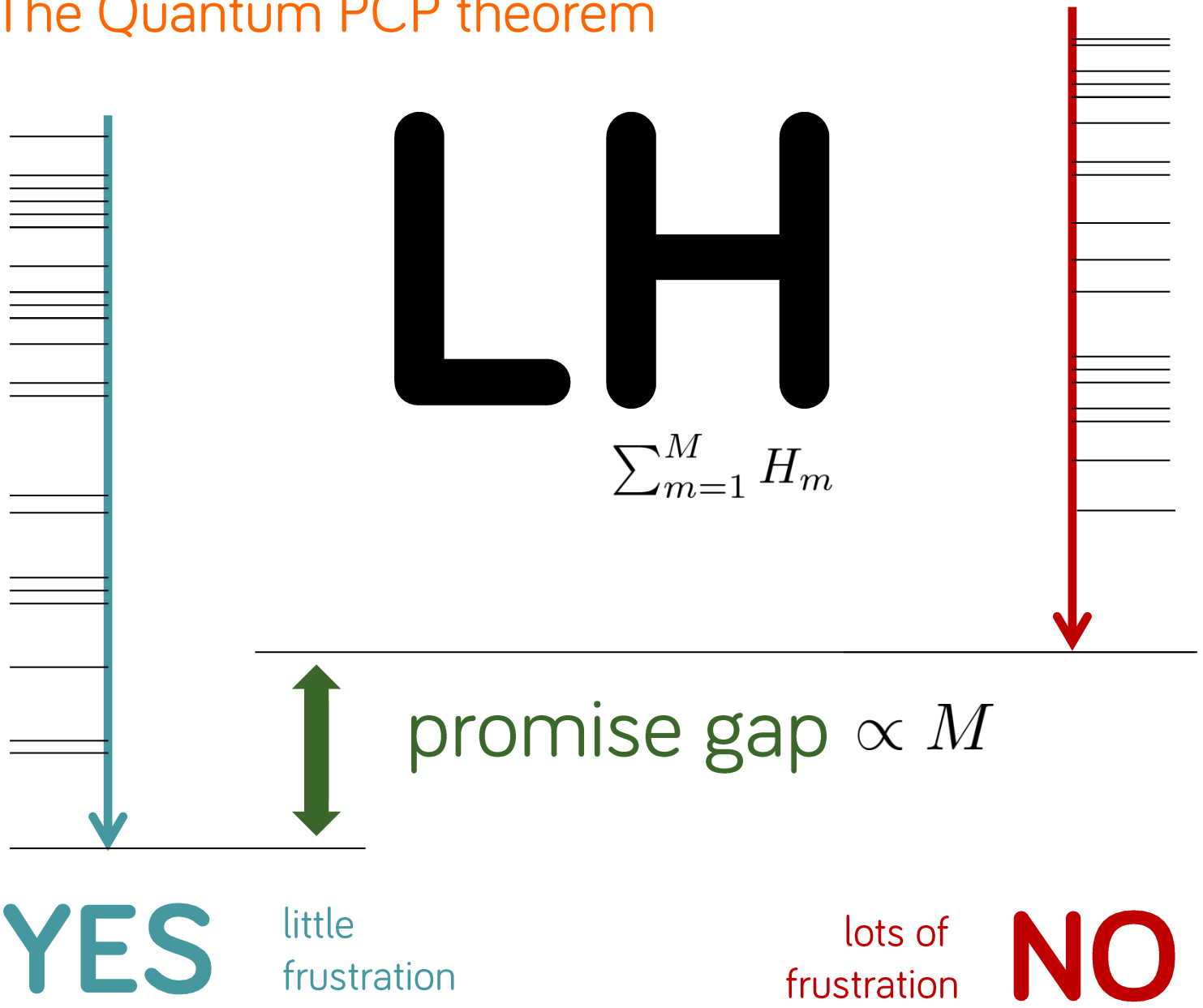
quantum

PCP

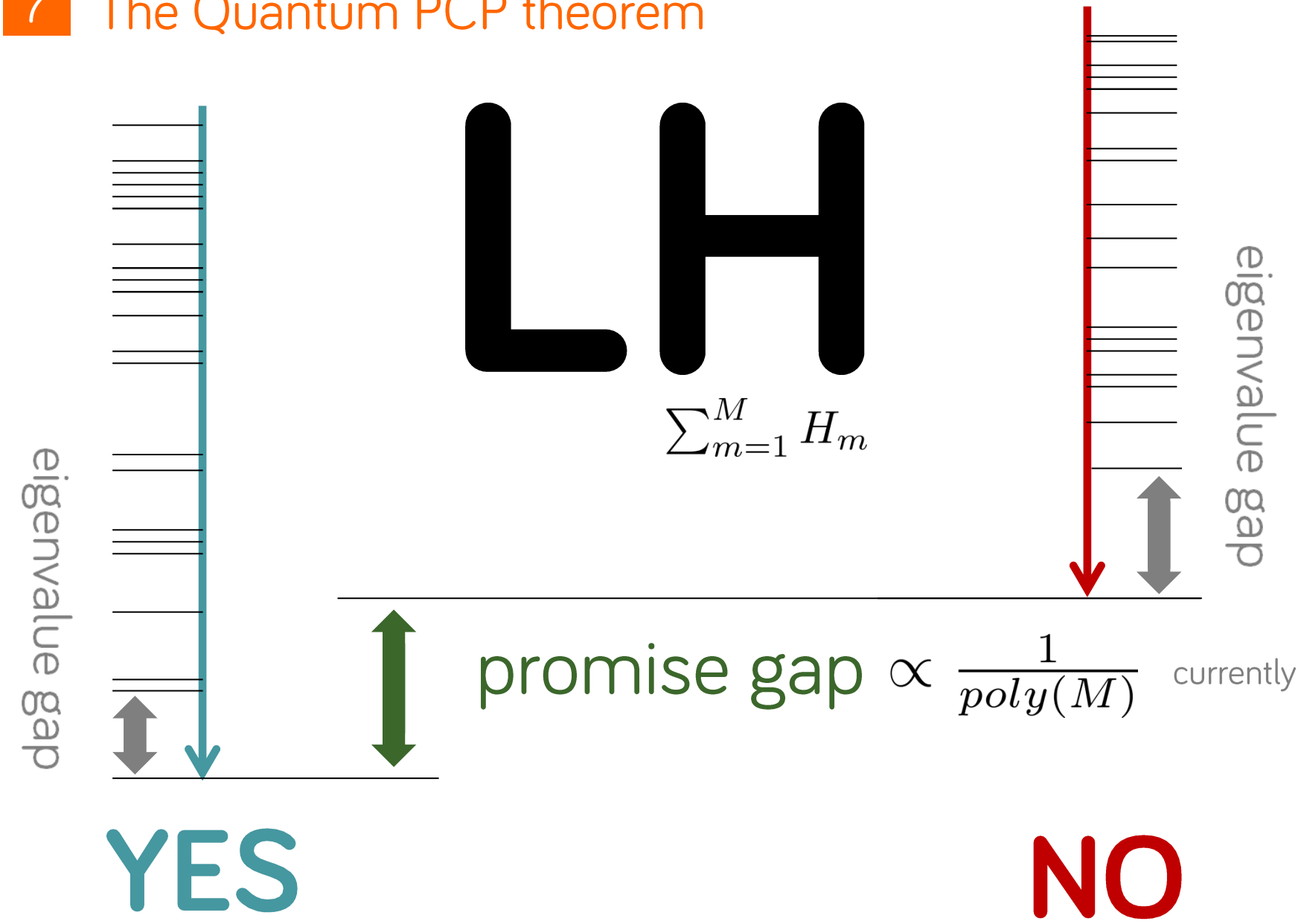
7 A randomly checked proof



7 The Quantum PCP theorem



7 The Quantum PCP theorem

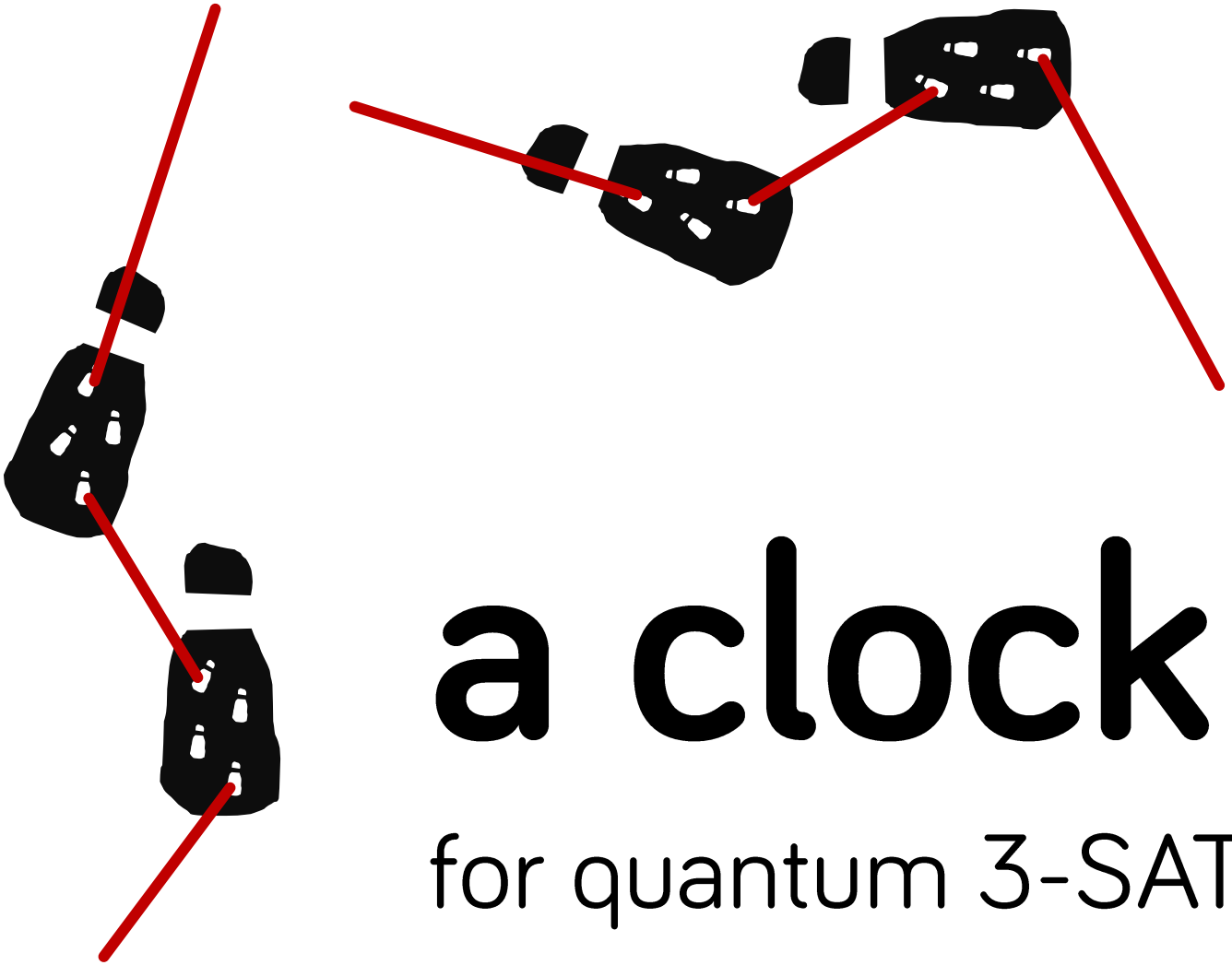


lots of fun **in 1D**

perfect verifiers

&

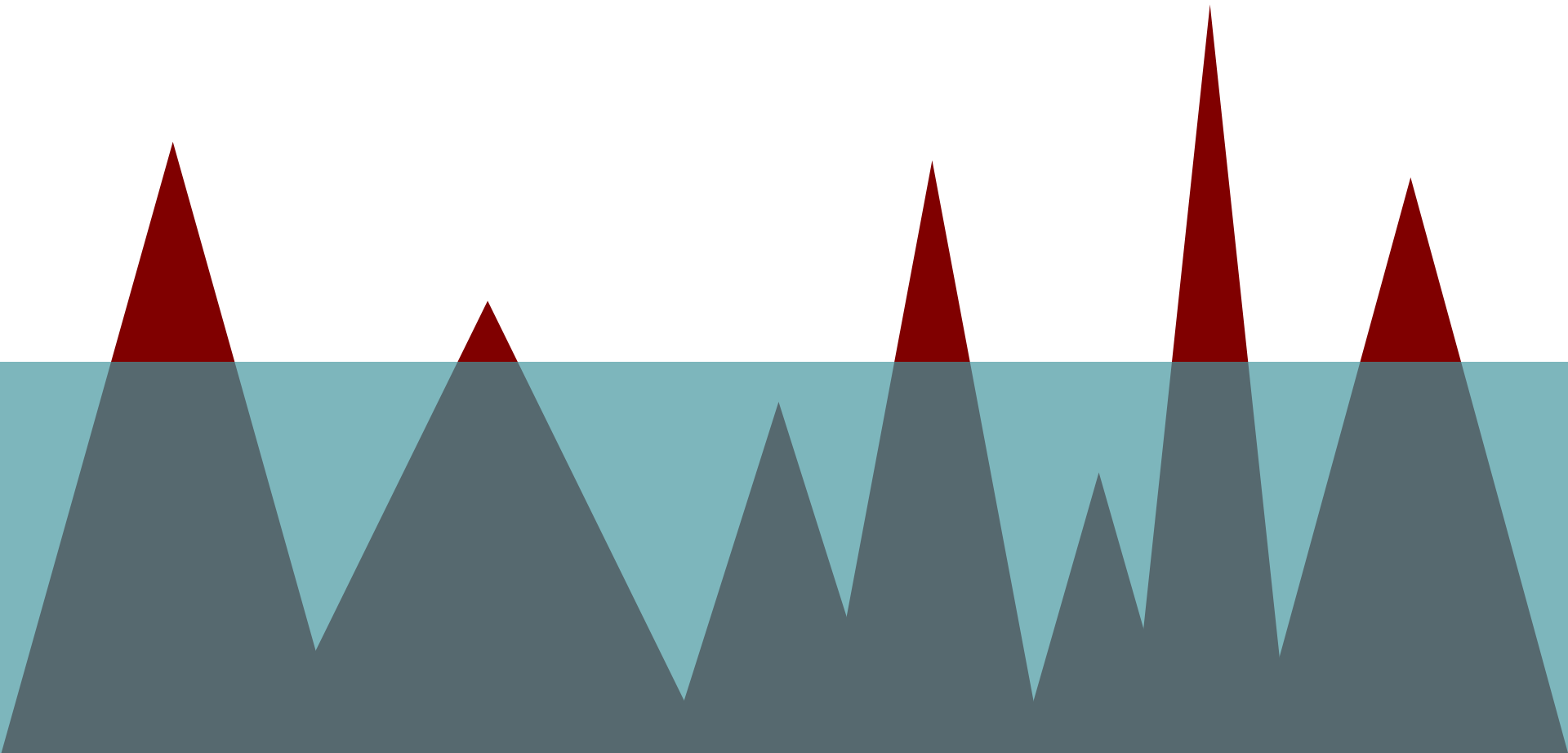
really convincing proofs

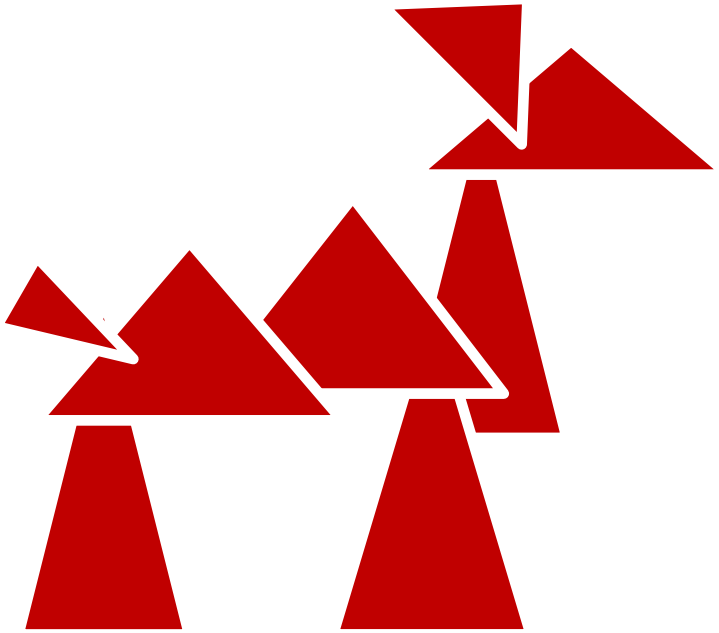


a clock

for quantum 3-SAT

random q-SAT





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Quantum Satisfiability

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Daniel Nagaj

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