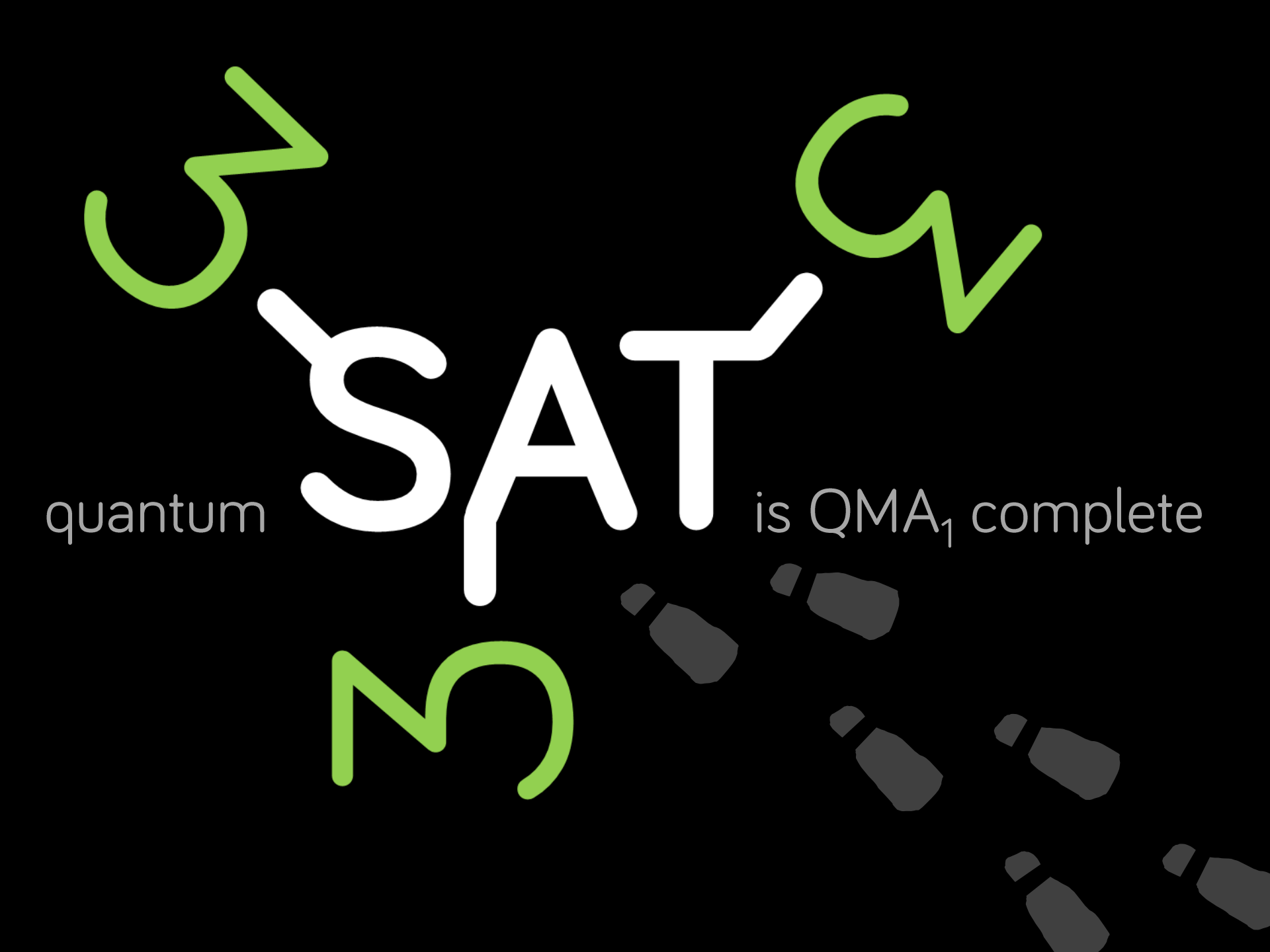
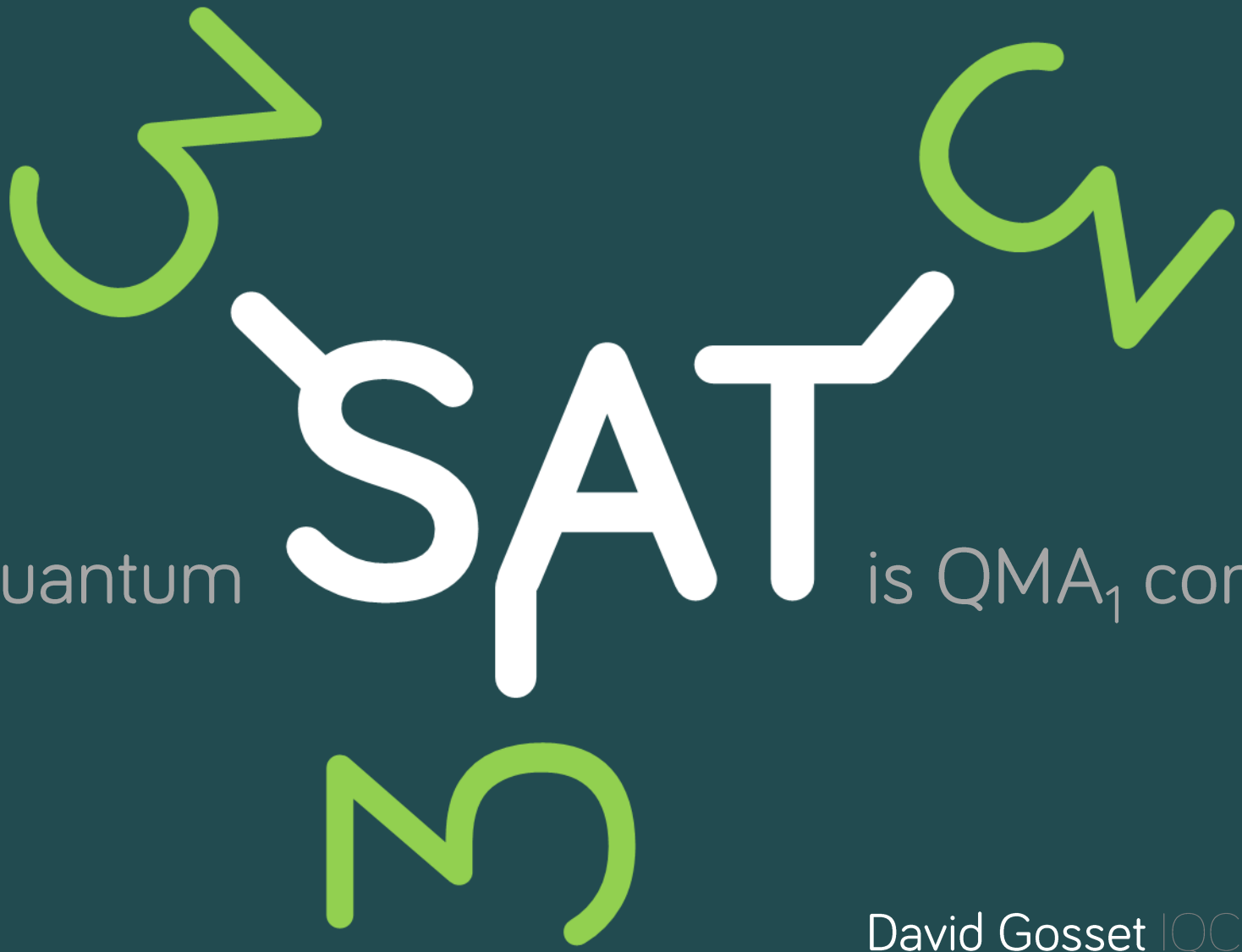

quantum SAT is QMA_1 complete



David Gosset IQC Waterloo

Daniel Nagaj UniWien

CEQIP Valtice | June 8, 2013 | 1302.0290

why QMA & QMA_1
puzzles and (quantum) proofs

what can we hide in ground states
local Hamiltonians & Kitaev

when clocks tick
pulses, walls, surfers & switches

which path will we take?
both, but the order matters

where could we be tricked?
nowhere, it works soundly

the puzzles of QMA

1 A cryptarithmetic puzzle

```
  DID
+ DINOS
-------
 CROAK
```

[www.iread.it/cryptarithms.php]

1 A cryptarithmetic puzzle

$$\begin{array}{r} 595 \\ +\ 59842 \\ \hline 60437 \end{array}$$

1 The MA protocol

Did dinosaurs exist?

triceratops

1 The MA protocol

Did dinosaurs exist?

[wooden animals: Imagination Kids Toys]

1 The MA protocol

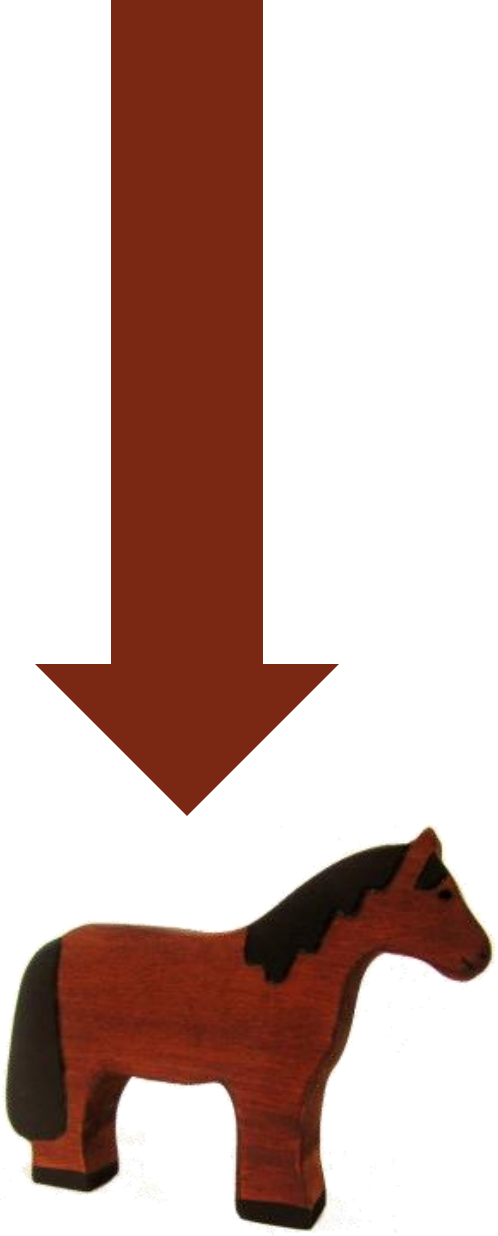
Did dinosaurs exist?

YES?
Eager to be convinced.

[magnifying glass: hillllllall]

1 The MA protocol

Recognizing fakes?



1 The MA protocol

Recognizing fakes?

NO?
Don't be
fooled
easily.

1 Probabilistic checks

Sometimes reject a genuine proof?

Accept a fake?

1 Perfect completeness

Never reject
a genuine proof?

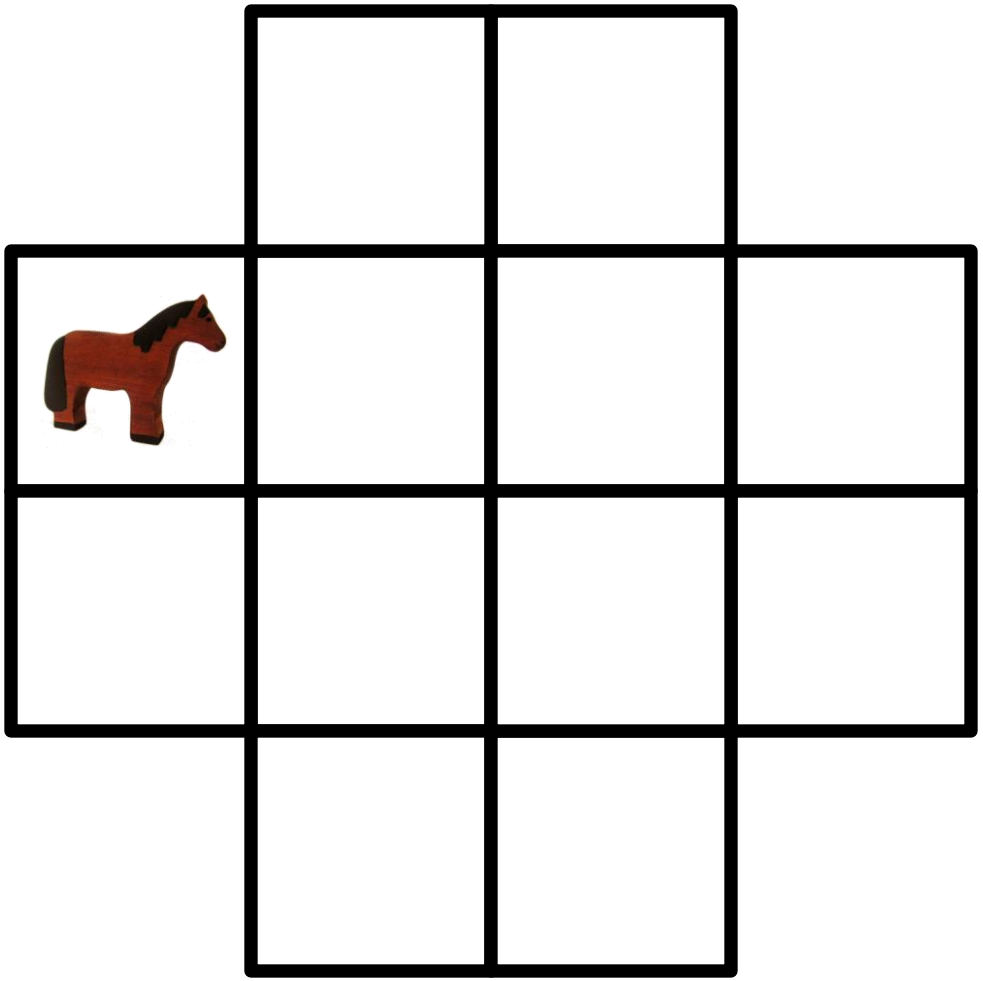
YES?
Accept some
proof without
any doubt.

NO?
Still don't get fooled easily.

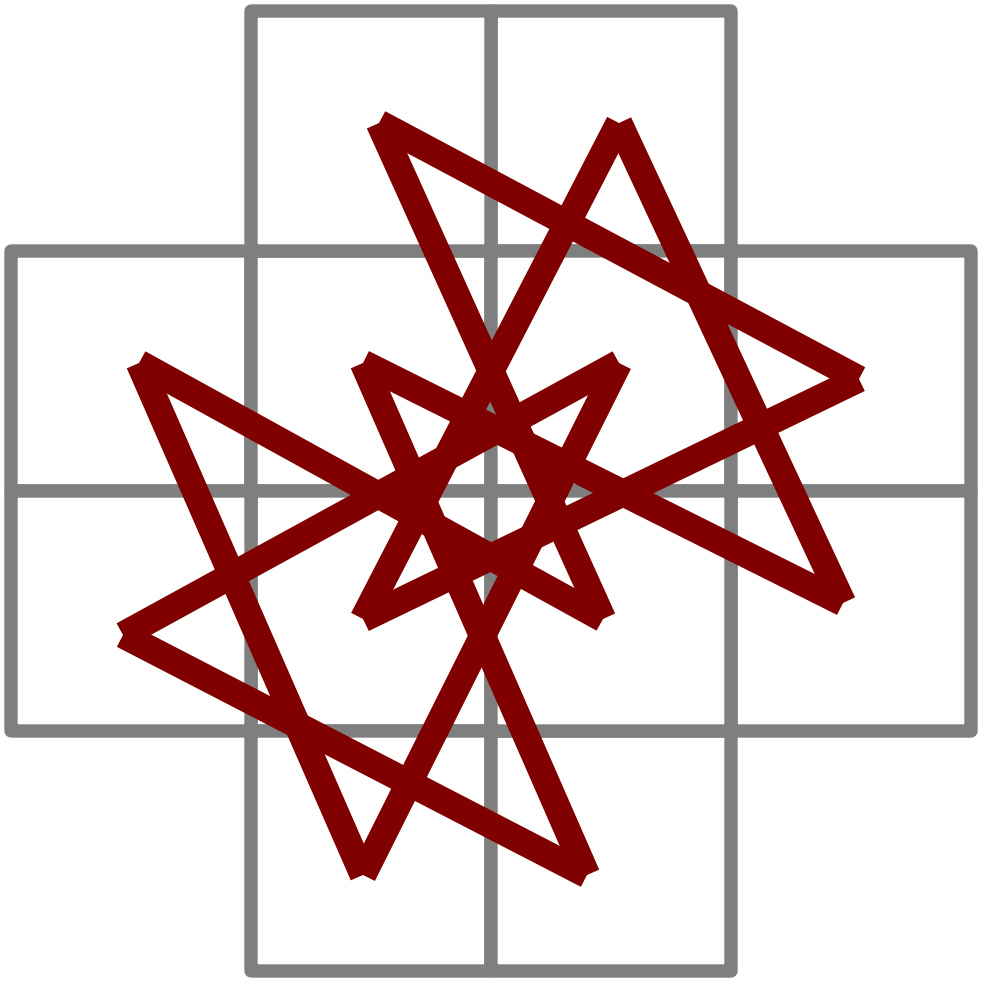
YES?
Accept some proof without any doubt.

MA_1

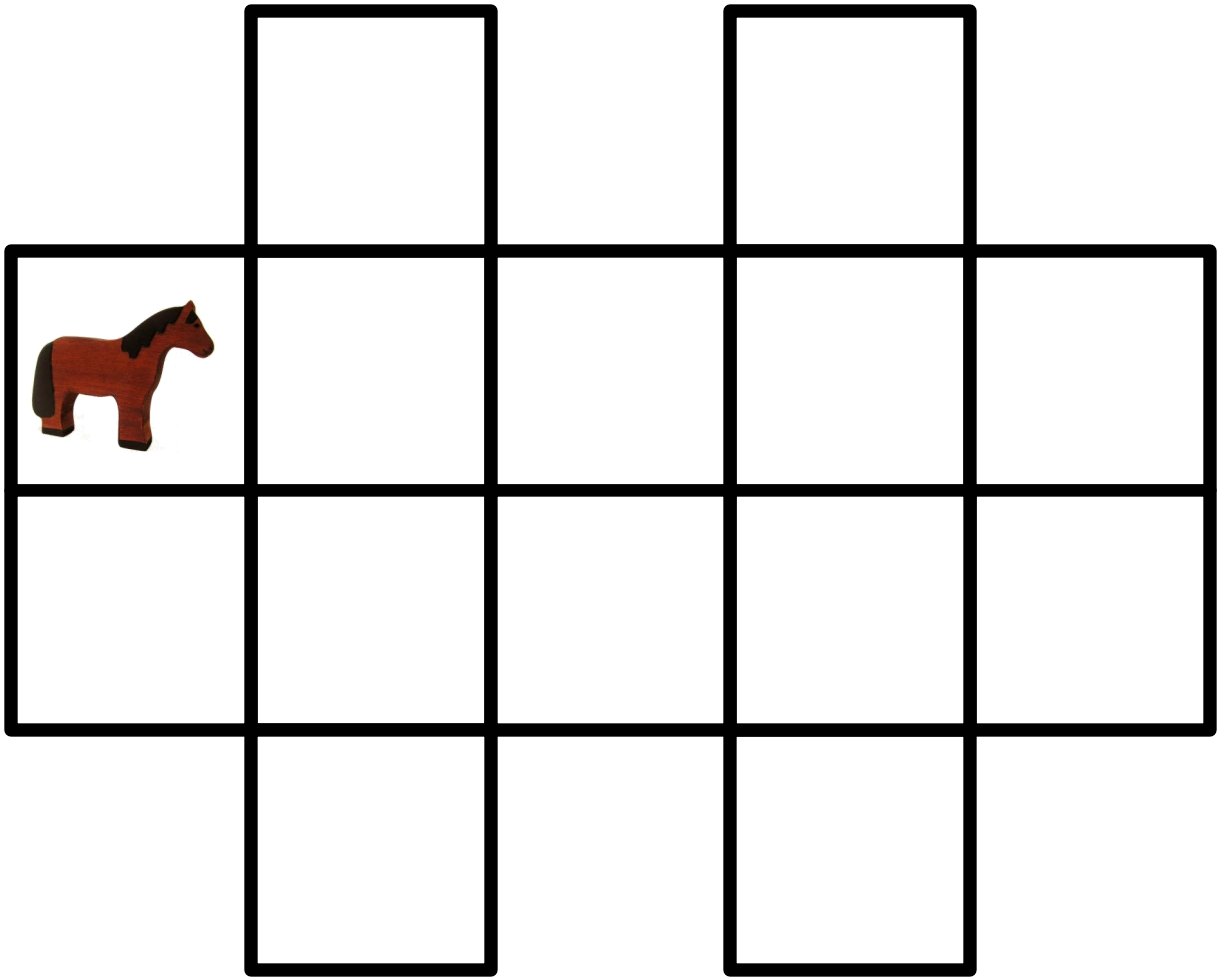
1 A Hamiltonian cycle problem



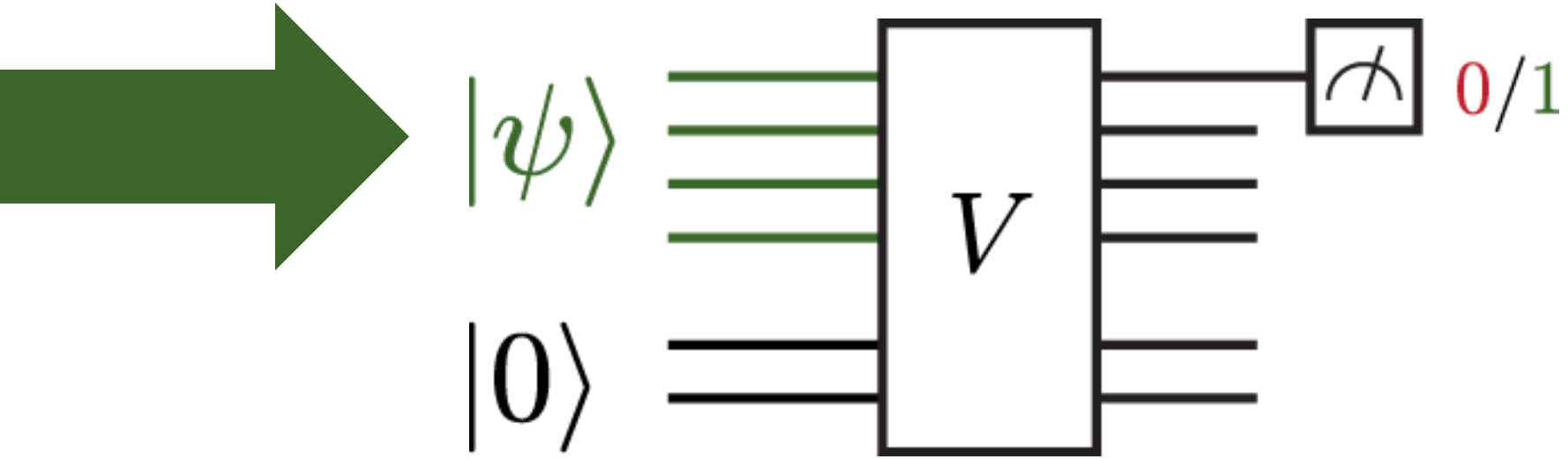
1 A Hamiltonian cycle problem



1 A Hamiltonian cycle problem



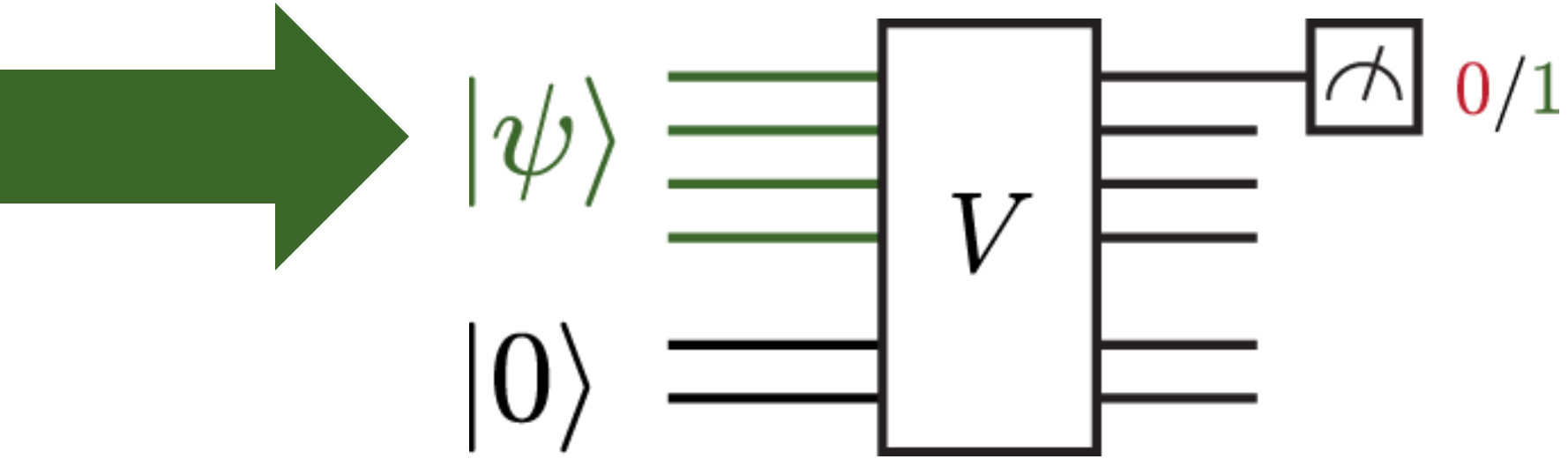
1 The QMA protocol



YES? Accept a good proof with $p > a$.

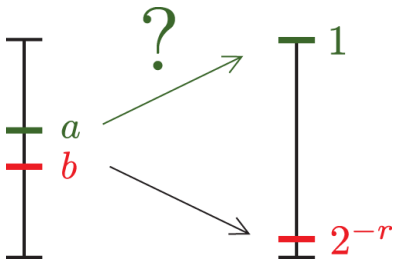
NO? Probability of accepting $p < b$.

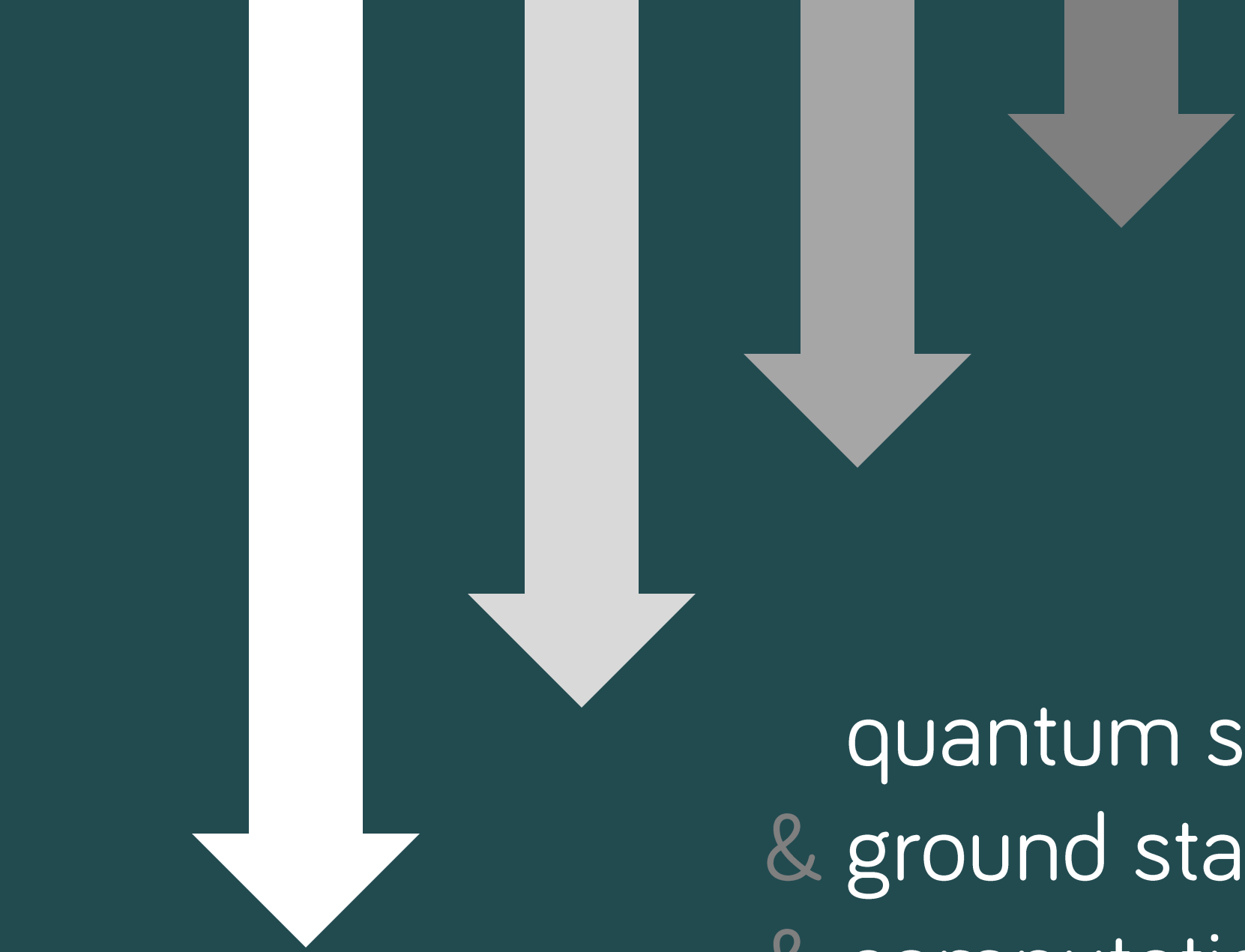
1 The QMA_1 protocol: perfect completeness



YES? Simply accept a good proof.

NO? Probability of accepting $p < b$.

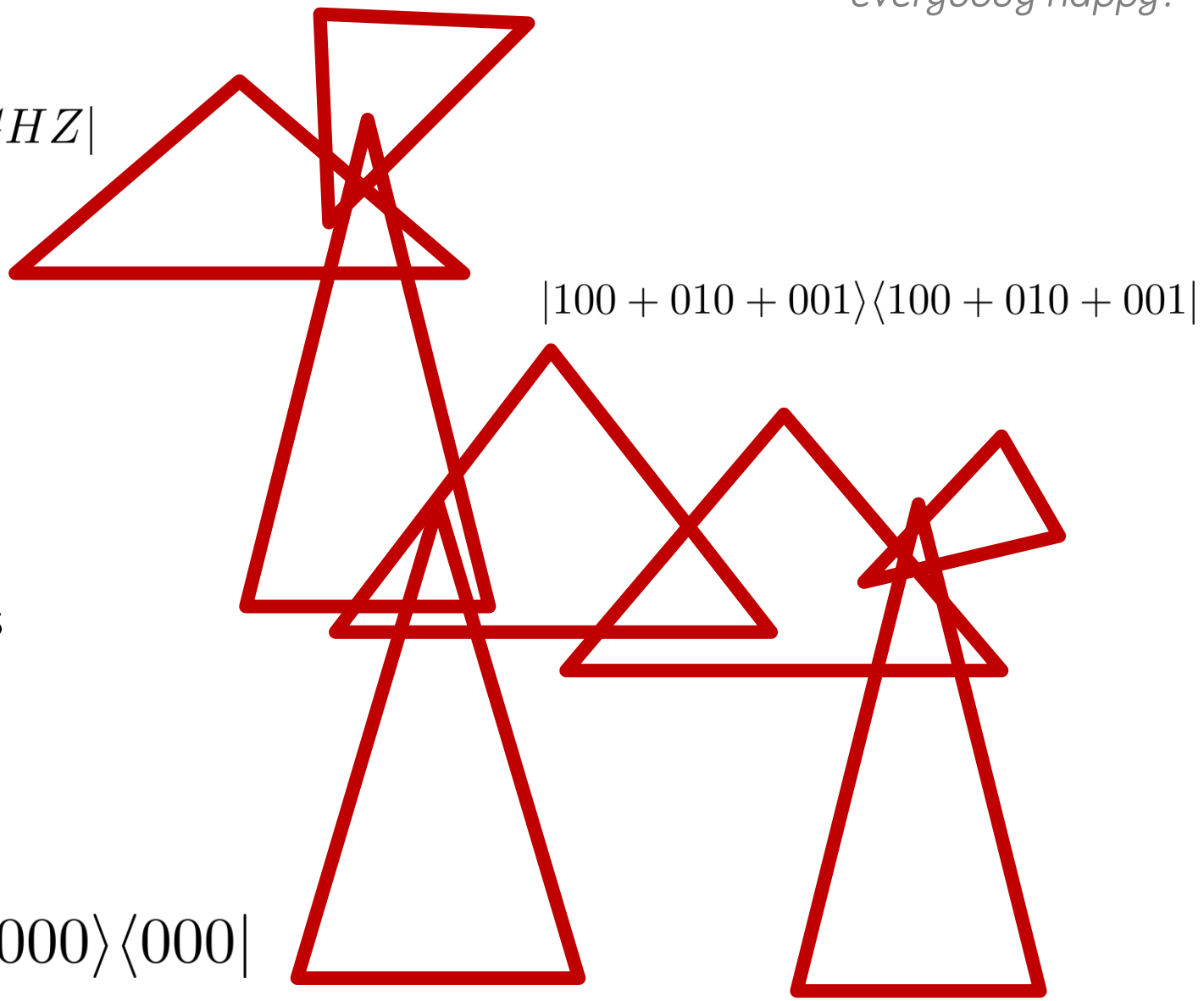




quantum sat
& ground states
& computation

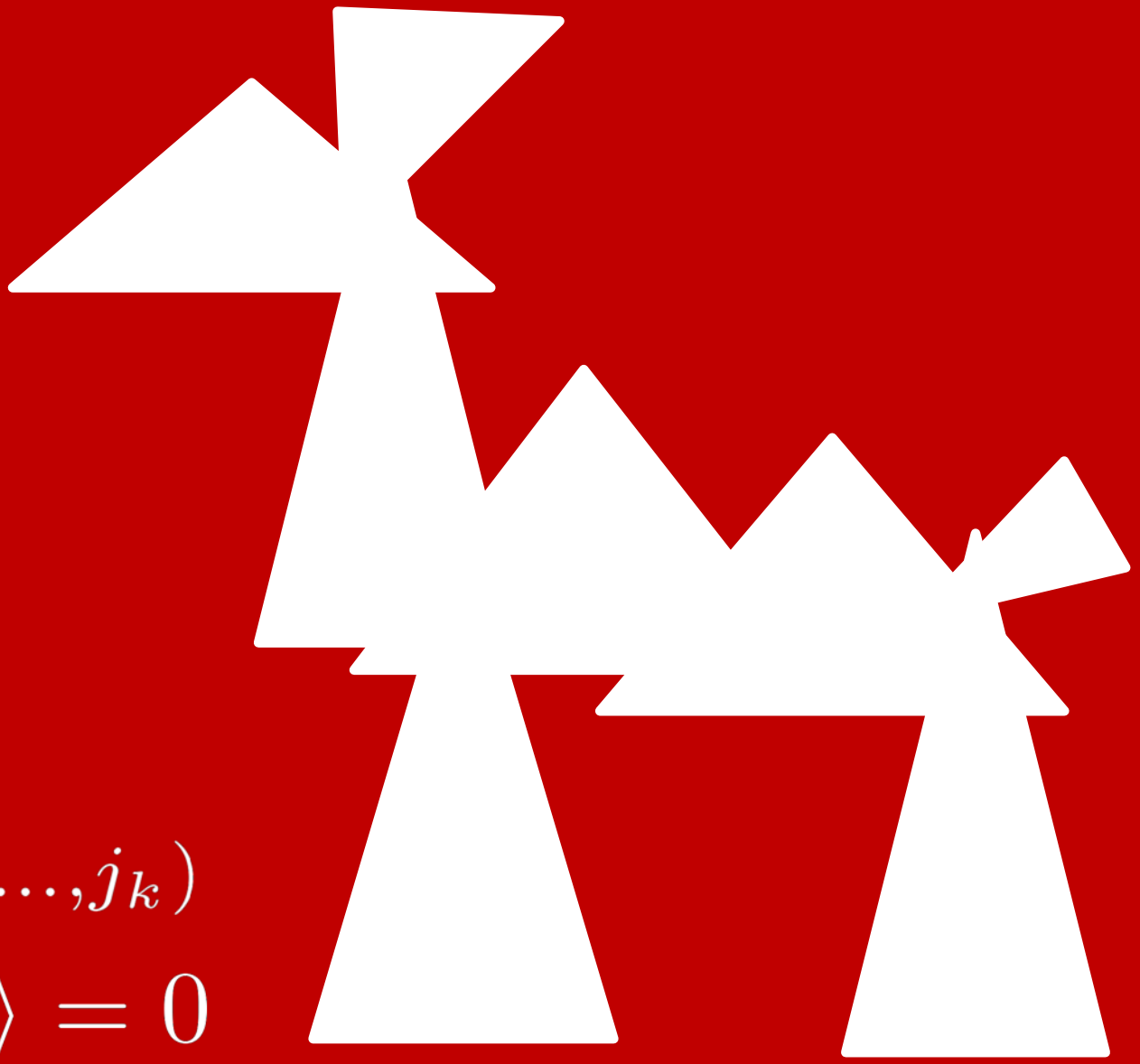
2 Quantum *k*-SAT

Can we make everybody happy?



- k-local projectors

Quantum *k*-SAT

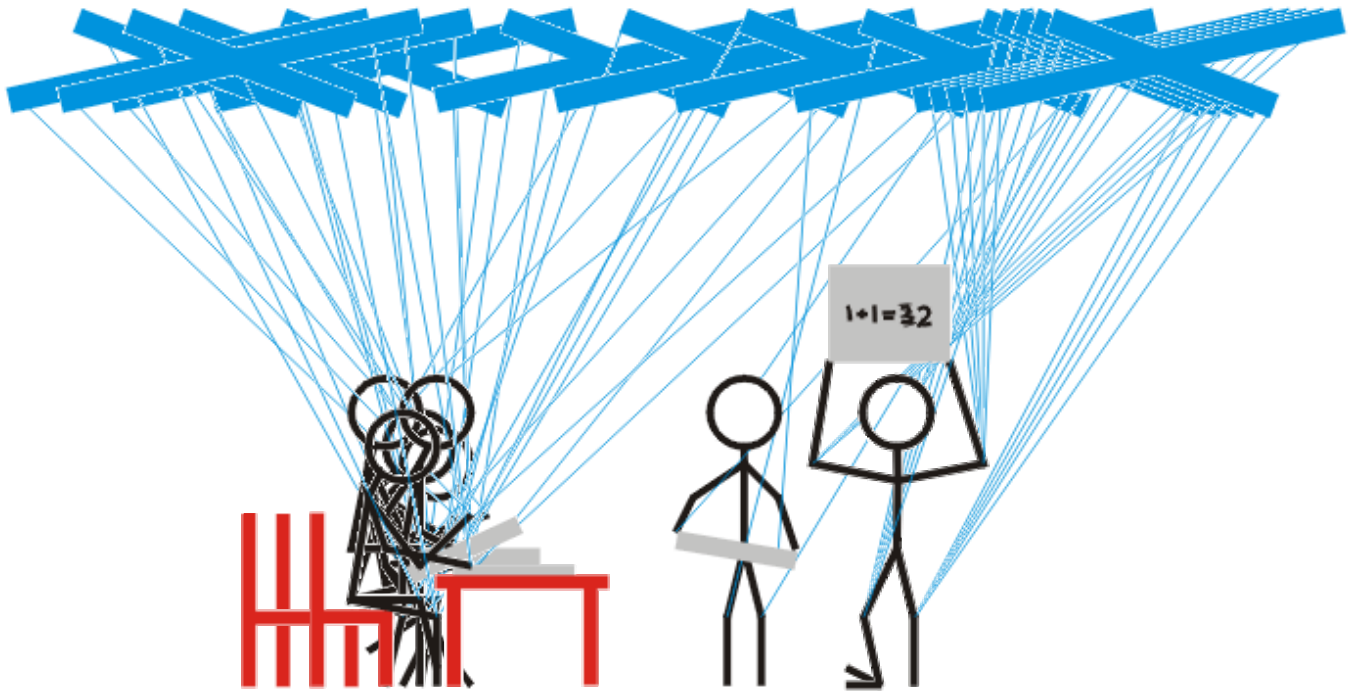


$$\Pi^j_{(j_1,\ldots,j_k)}$$

$$\Pi^j|\psi\rangle = 0$$

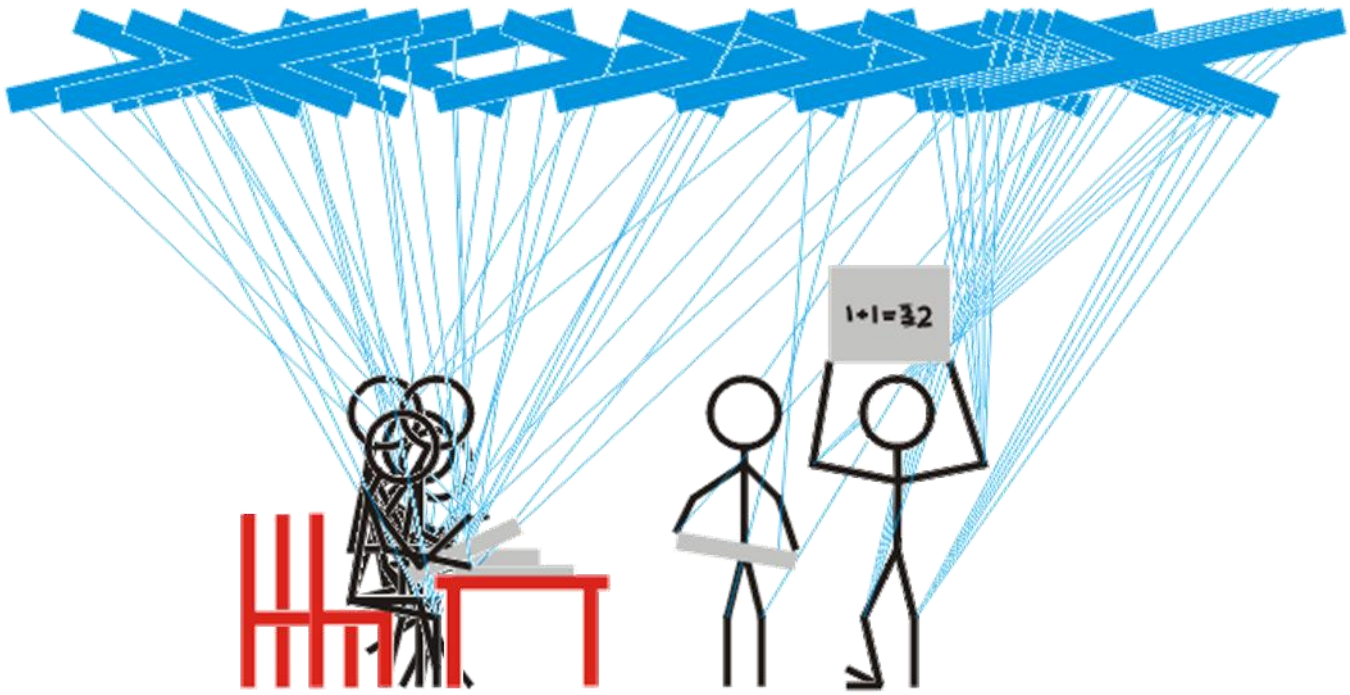
2 Computation histories

A pointer & data.



2 Computation histories

A pointer & data.



$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

$$U_t \cdots U_1 |\varphi_0\rangle$$

The history state is a ground state

k-local

c-o-n-d-i-t-i-o-n-s

clock encoding

state progression

initialization

$$|\cdots 000\cdots 0\rangle \otimes |0\rangle$$

$$|\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

The history state is a ground state

k-local

c-o-n-d-i-t-i-o-n-s

clock encoding

state progression

output

$$|\cdots 1\rangle \otimes |T\rangle$$

$$|\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

2 Checking proper computation

Antisymmetry checks.

- uniform superpositions: zero-energy eigenstates

$$H_t = \tfrac{1}{2}\left(|t+1\rangle\langle t+1| + |t\rangle\langle t|\right) - \tfrac{1}{2}\left(U_{t+1}\otimes|t+1\rangle\langle t| + U_{t+1}^{\dagger}|t\rangle\langle t+1|\right) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

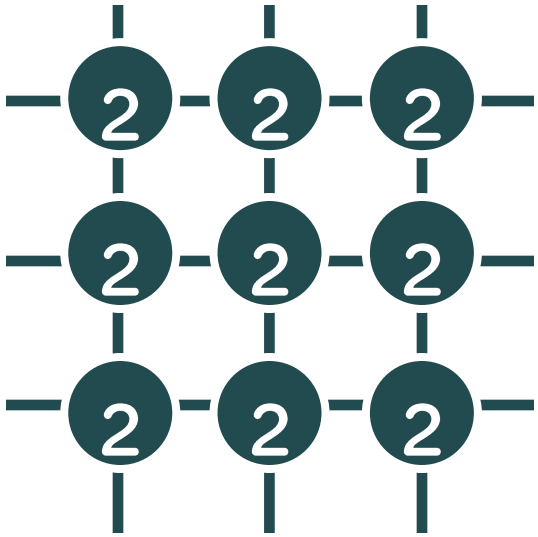
a projector

$$|\varphi_t\rangle \otimes |t\rangle$$

$$|\varphi_{t+1}\rangle \otimes |t+1\rangle$$

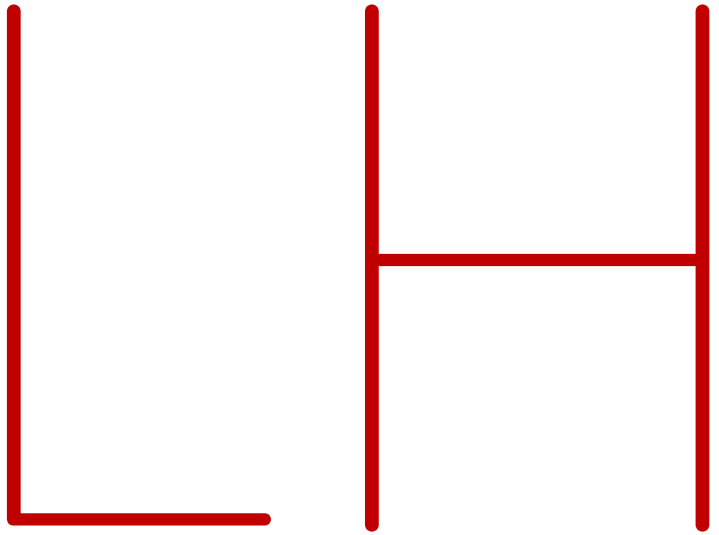
$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle$$

2 QMA-complete problems

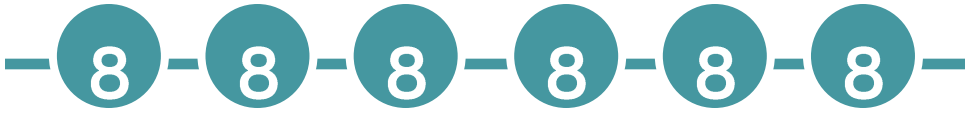


[Oliveira, Terhal '04]

a global minimum



$$\sum H_{jk}$$

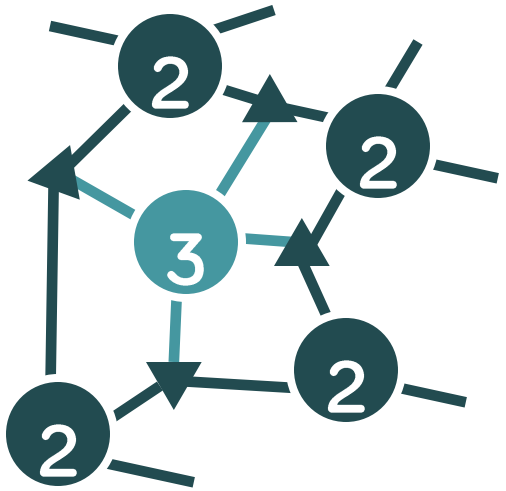


[Hallgren, Nagaj, Narayanaswami '13]


frustrated
FRUST
RATED

2 QMA_1-complete problems

[Nagaj '08]



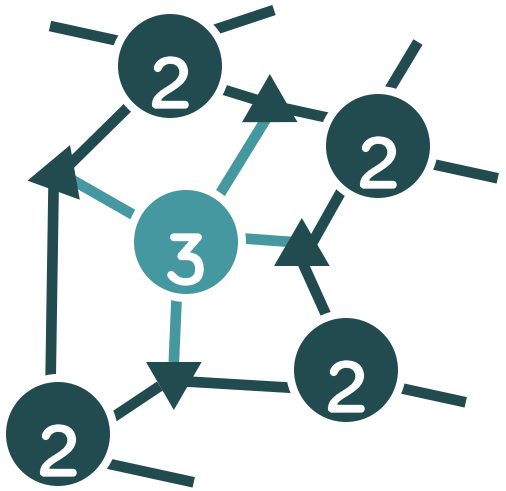
[Moses, Nagaj '07]

Π^j

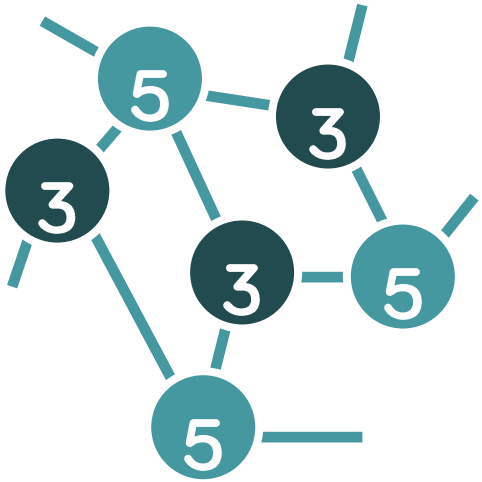
unfrustrated qSAT

2 QMA_1-complete problems

[Nagaj '08]



[Moses, Nagaj '07]



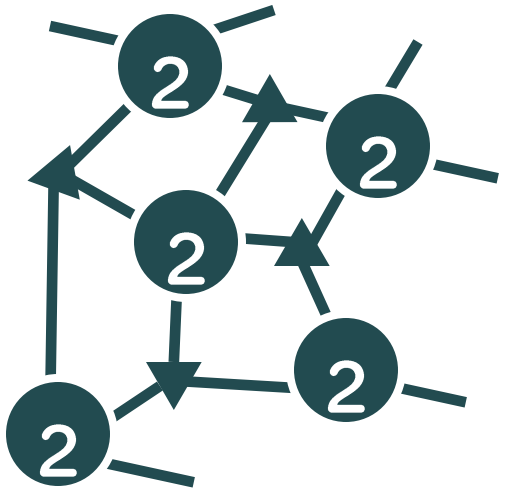
[Eldar, Regev '08]

unfrustrated

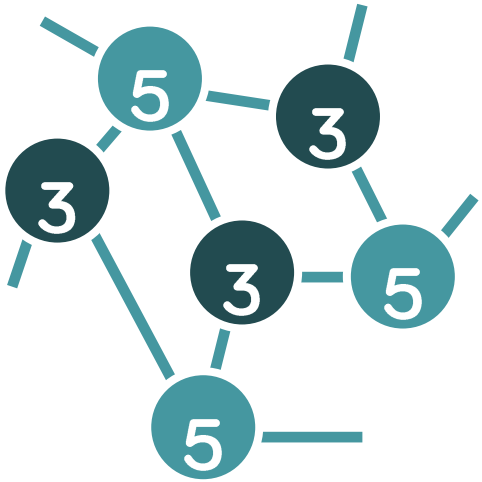
qSAT

2 QMA_1-complete problems

[Nagaj '08]



[Gosset, Nagaj '13]

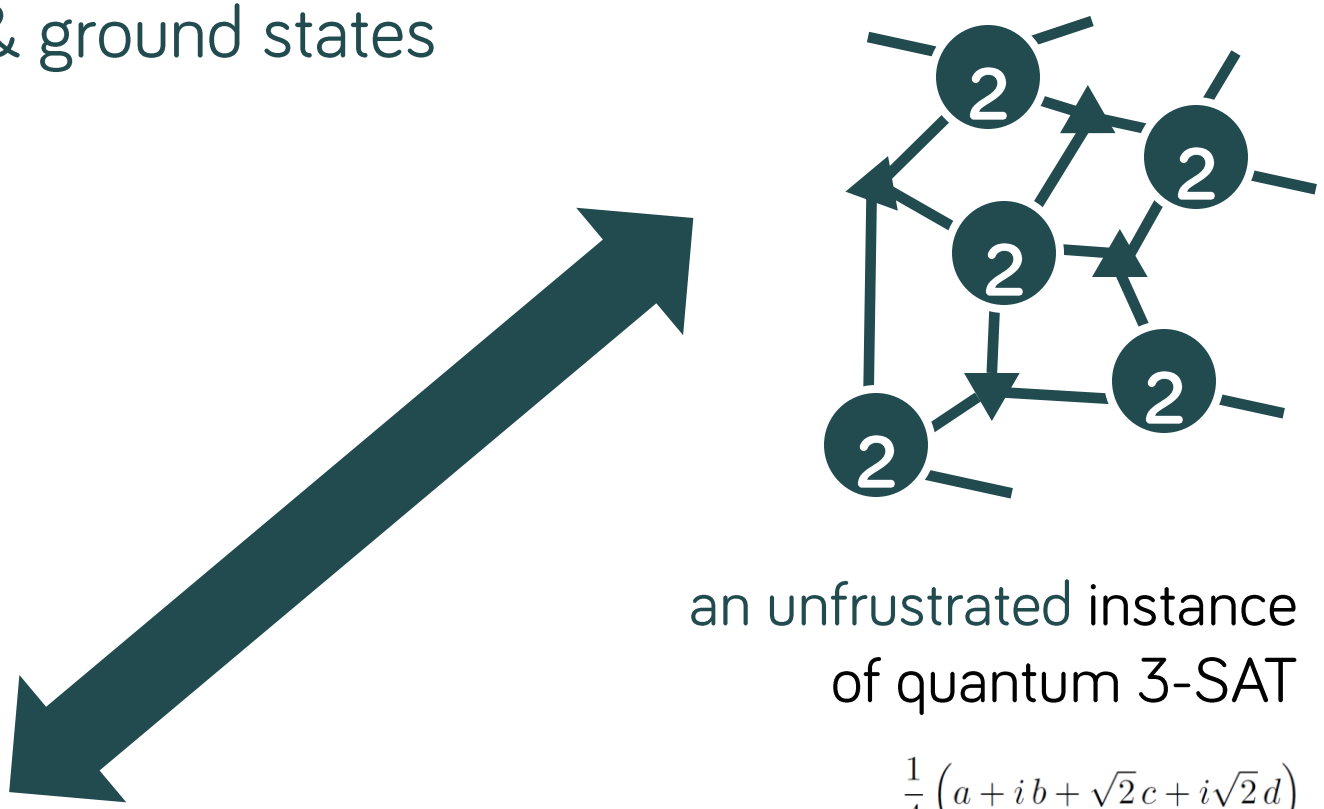


[Eldar, Regev '08]

unfrustrated

qSAT

2 Circuits & ground states



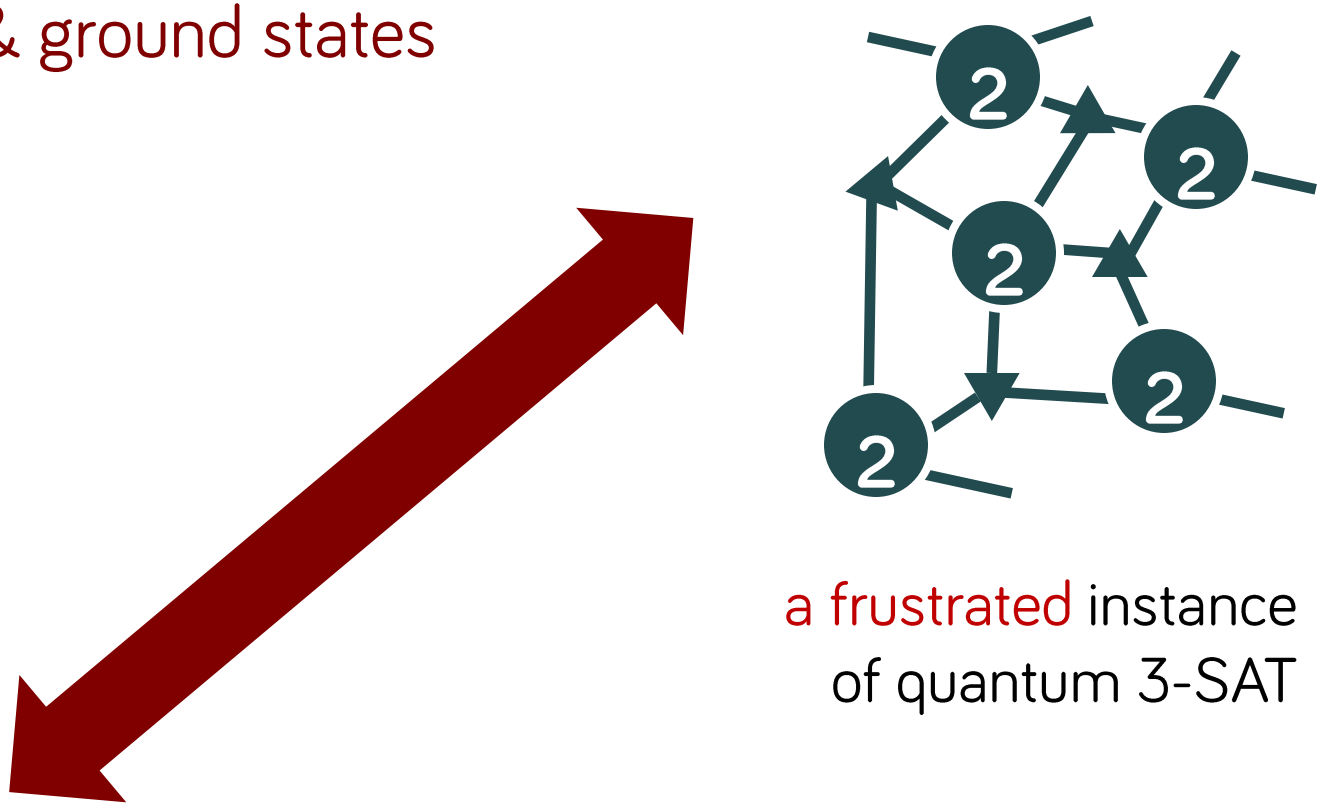
an unfrustrated instance
of quantum 3-SAT

$$\frac{1}{4}\left(a + i\,b + \sqrt{2}\,c + i\sqrt{2}\,d\right)$$

CNOT, H & T

a verifier circuit with a perfectly accepted witness

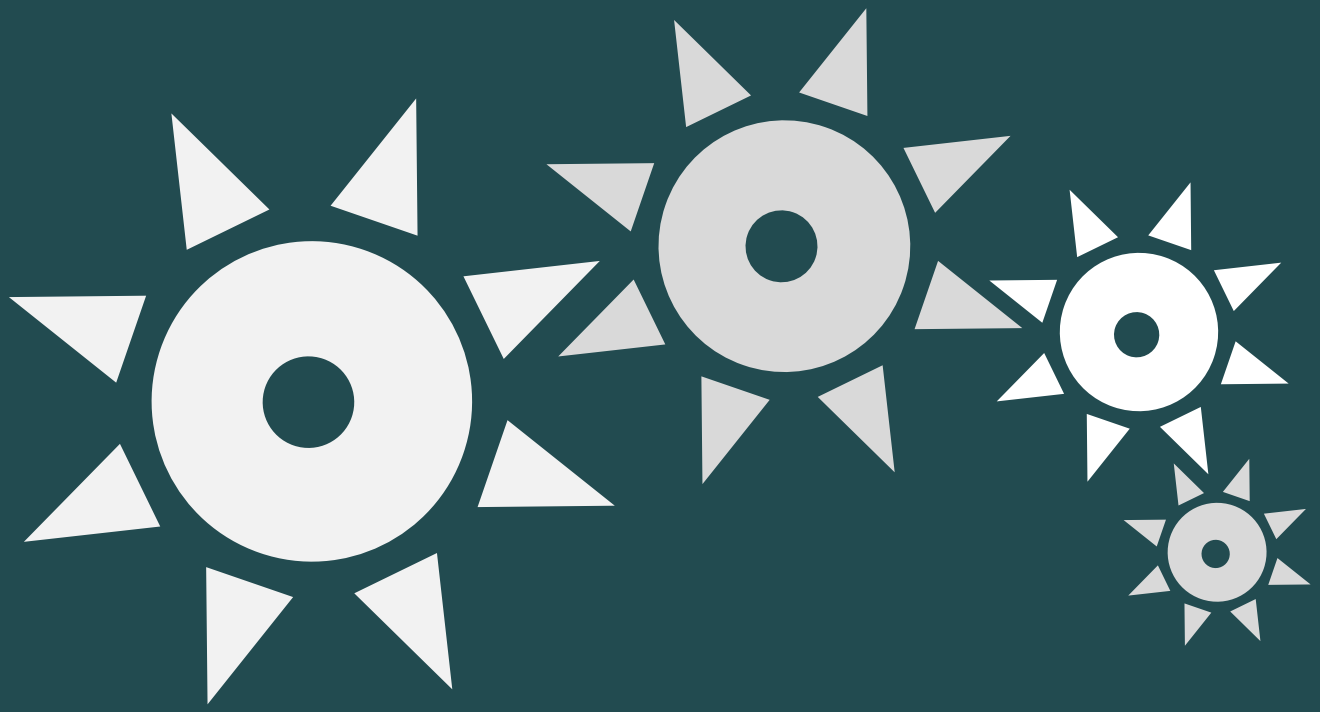
2 Circuits & ground states



a frustrated instance
of quantum 3-SAT

CNOT, H & T

a verifier circuit which doesn't like to accept anything



a clock workshop

3 Constructing clocks

- the domain wall 

$$|t\rangle = |2\rangle$$
$$= |11000\rangle$$

- the ground state of ? $|01\rangle\langle 01|$

3 Constructing clocks

- the domain wall 

$$|t\rangle = |5\rangle$$
$$= |11111\rangle$$

- the ground state of ? $|01\rangle\langle 01|$

3 Constructing clocks

- the domain wall 

transitions: 3-local

$$|t\rangle = |3\rangle$$
$$= |11000\rangle$$

- joining clock states by transitions?

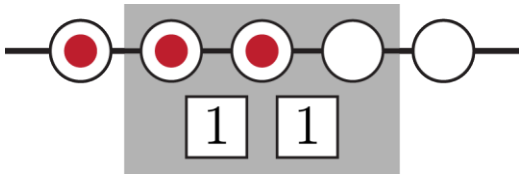
$$|100 - 110\rangle\langle 100 - 110|$$

- the ground state

$$\ldots + |2\rangle + |3\rangle + \ldots$$

3 Constructing clocks

- the domain wall



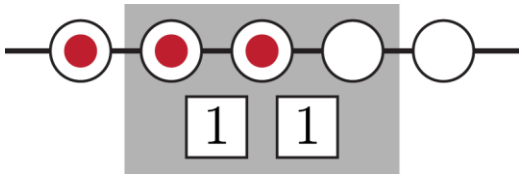
transitions: 3-local
2-qubit gates: 5-local

standard in QMA hardness proofs

$$H_t = \tfrac{1}{2}\left(|t+1\rangle\langle t+1| + |t\rangle\langle t|\right) - \tfrac{1}{2}\left(U_{t+1}\otimes|t+1\rangle\langle t| + U_{t+1}^{\dagger}|t\rangle\langle t+1|\right)$$

3 Constructing clocks

- the domain wall



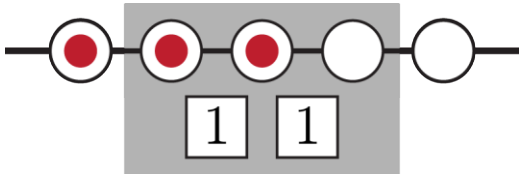
transitions: 3-local
2-qubit gates: 5-local

standard in QMA hardness proofs

- the pulse

3 Constructing clocks

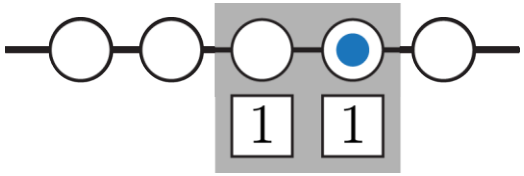
- the domain wall



transitions: 3-local
2-qubit gates: 5-local

standard in QMA hardness proofs

- the pulse



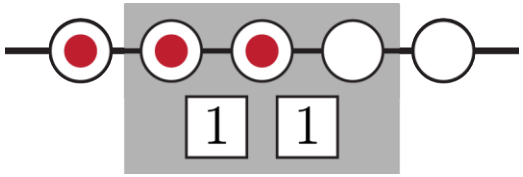
transitions: 2-local
2-qubit gates: 4-local

- joining the states by projectors

$$|01-10\rangle\langle 01-10|$$

3 Constructing clocks

- the domain wall



transitions: 3-local
2-qubit gates: 5-local

standard in QMA hardness proofs

- the pulse

transitions: 2-local
2-qubit gates: 4-local

needs initialization!

3 Constructing clocks

- the domain wall 

transitions: 3-local
2-qubit gates: 5-local

standard in QMA hardness proofs

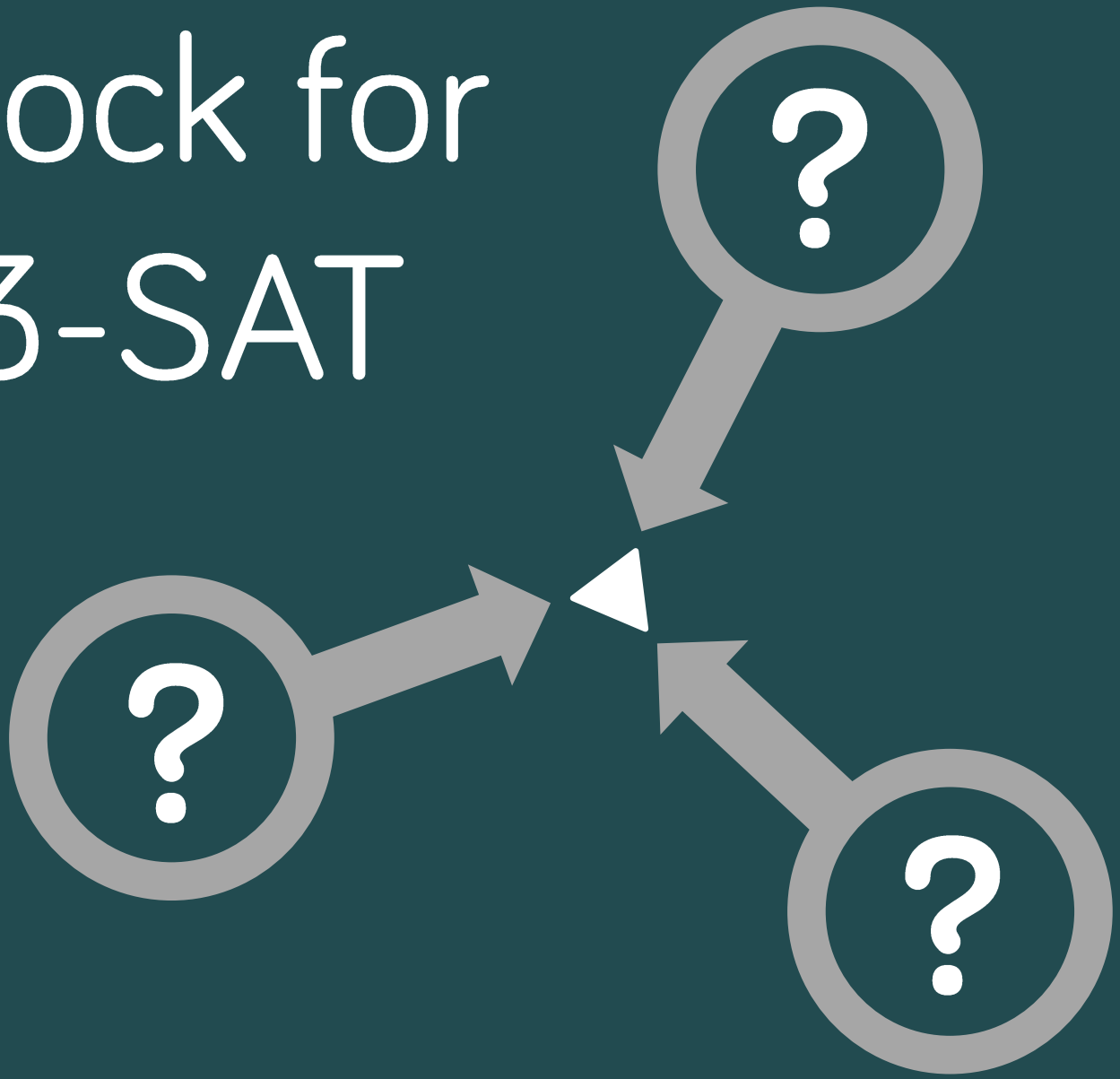
- the pulse 

transitions: 2-local
2-qubit gates: 4-local

eeds initialization)-:

- progress the clock *2-locally?*

a clock for Q-3-SAT



3 Constructing a composite clock with 2-local progress

1111000000000
1111100000000
1111110000000
1111111000000
1111111100000
1111111110000
1111111111000
1111111111100
1111111111110

3 Constructing a composite clock with 2-local progress

111 100 000 000
111 110 000 000
111 111 000 000
111 111 100 000
111 111 110 000
111 111 111 000
111 111 111 100
111 111 111 110

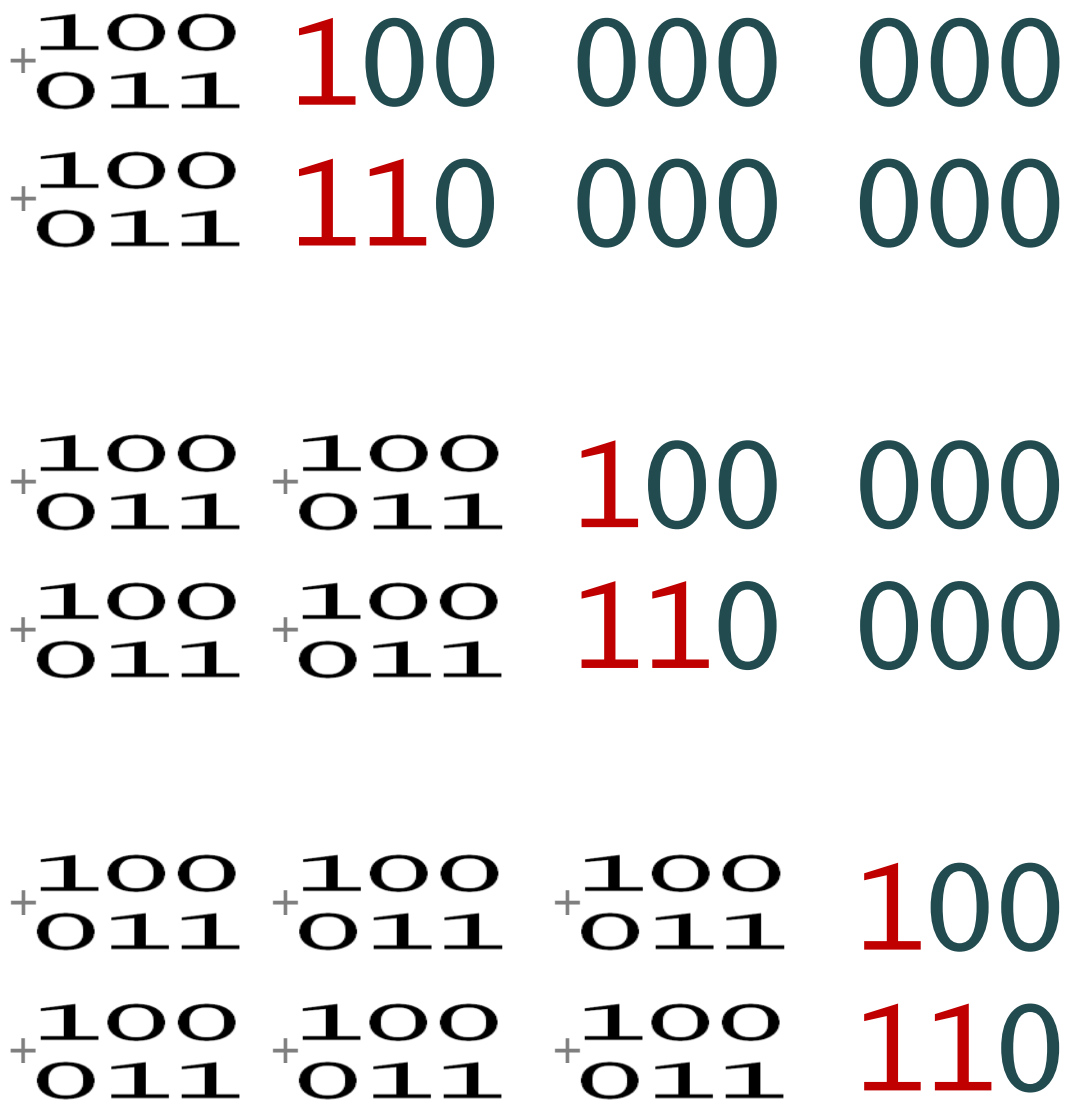
3 Constructing a composite clock with 2-local progress

111 100 000 000
111 110 000 000

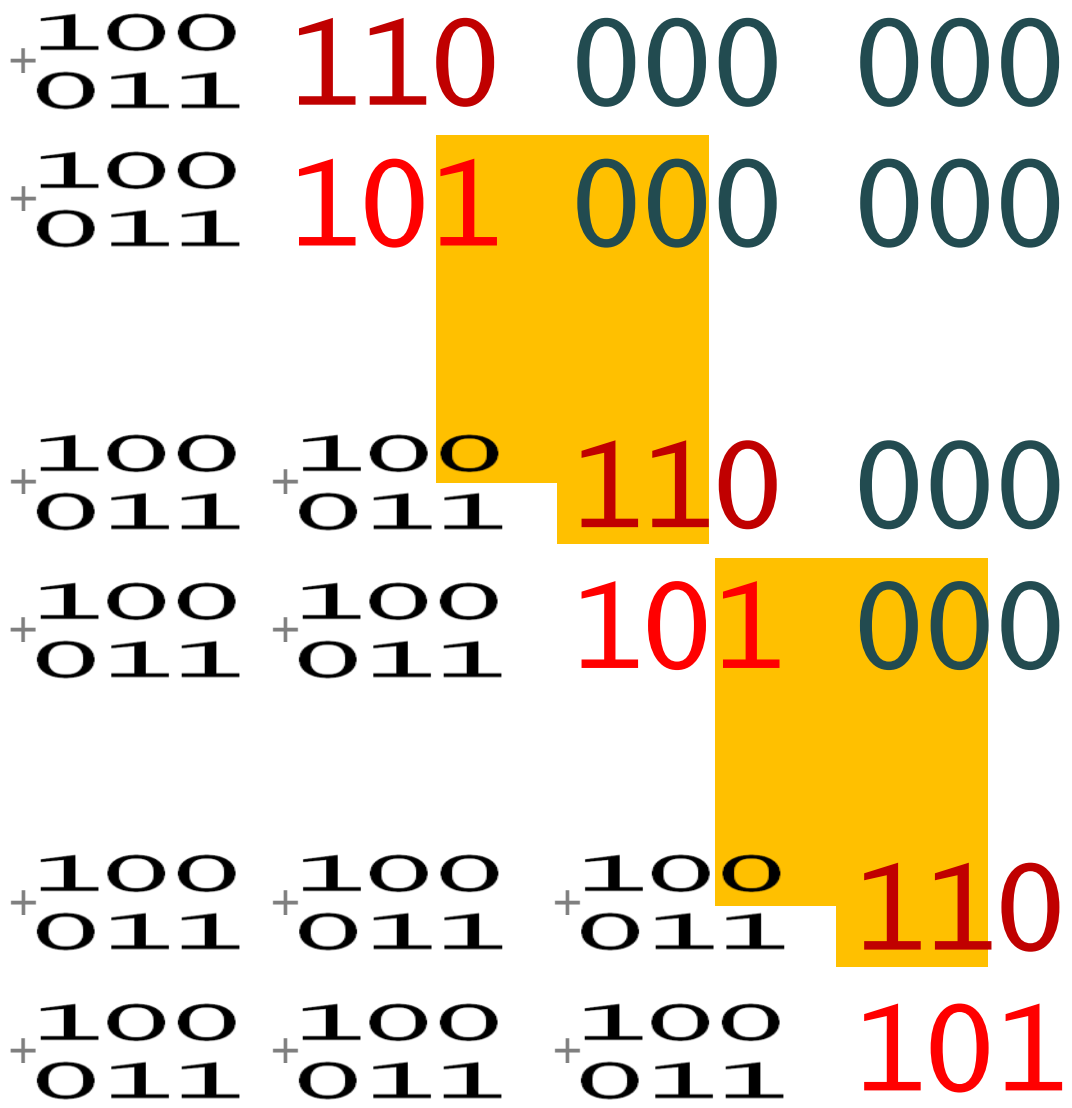
111 111 100 000
111 111 110 000

111 111 111 100
111 111 111 110

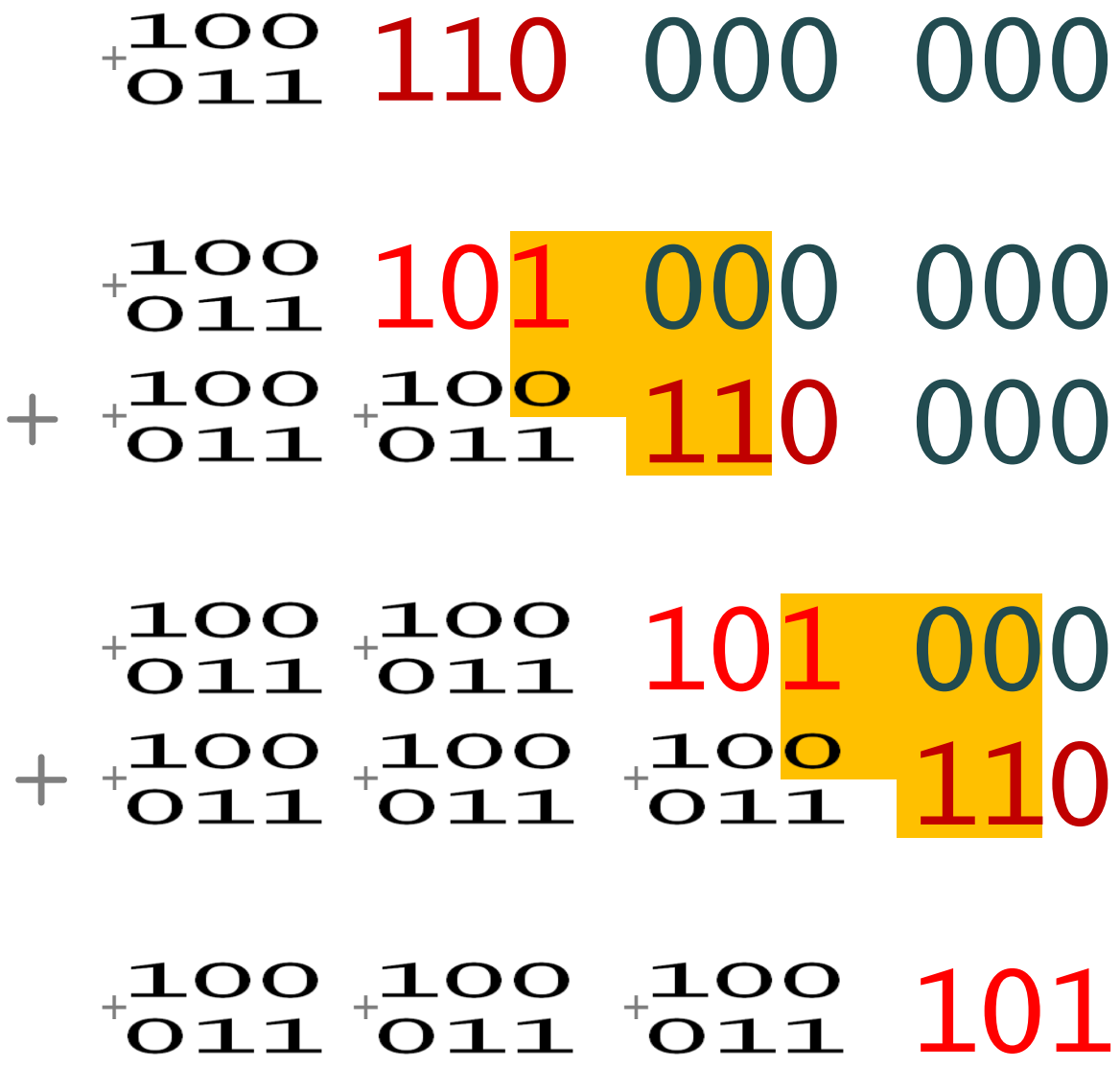
3 Constructing a composite clock with 2-local progress



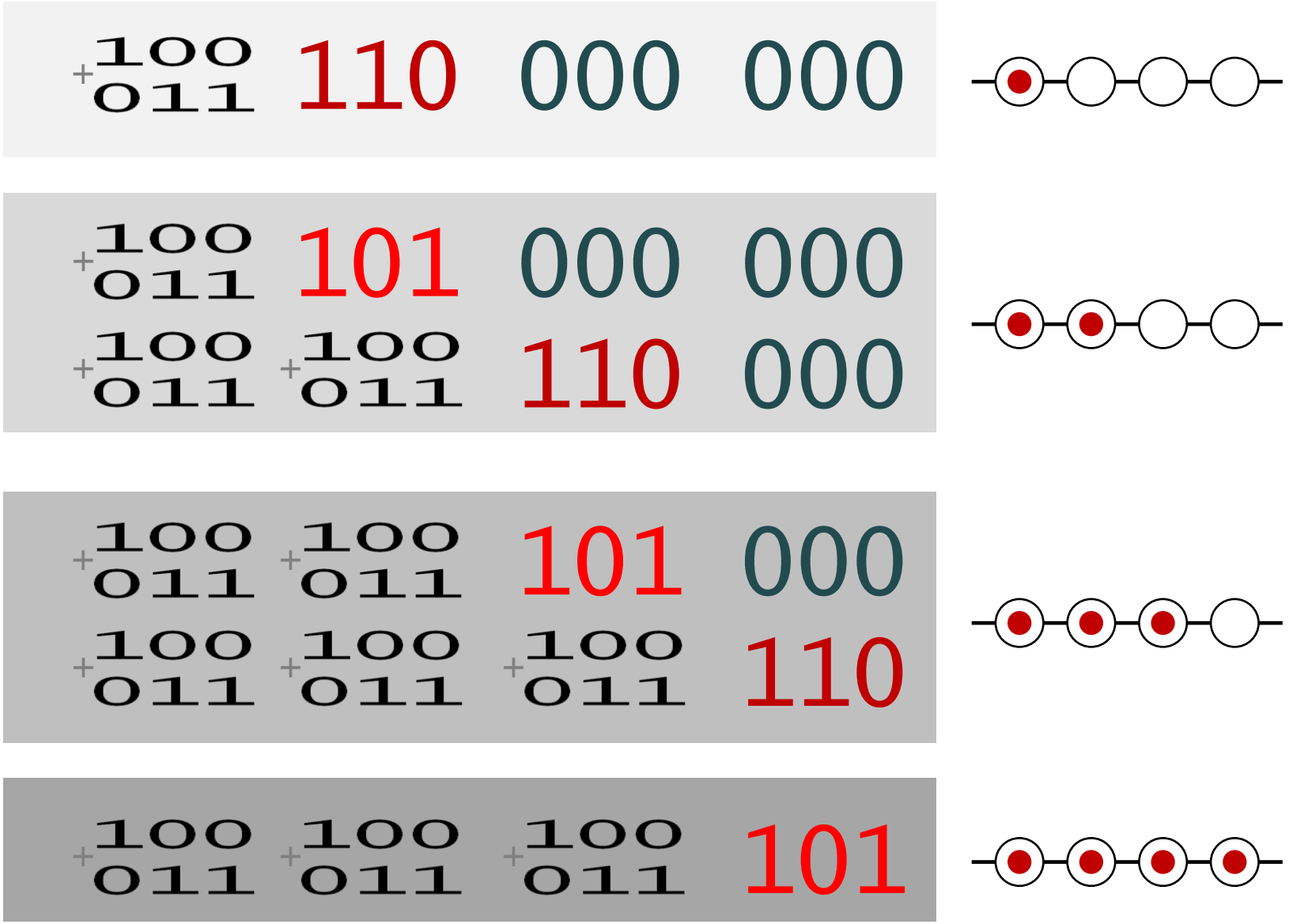
3 Constructing a composite clock with 2-local progress



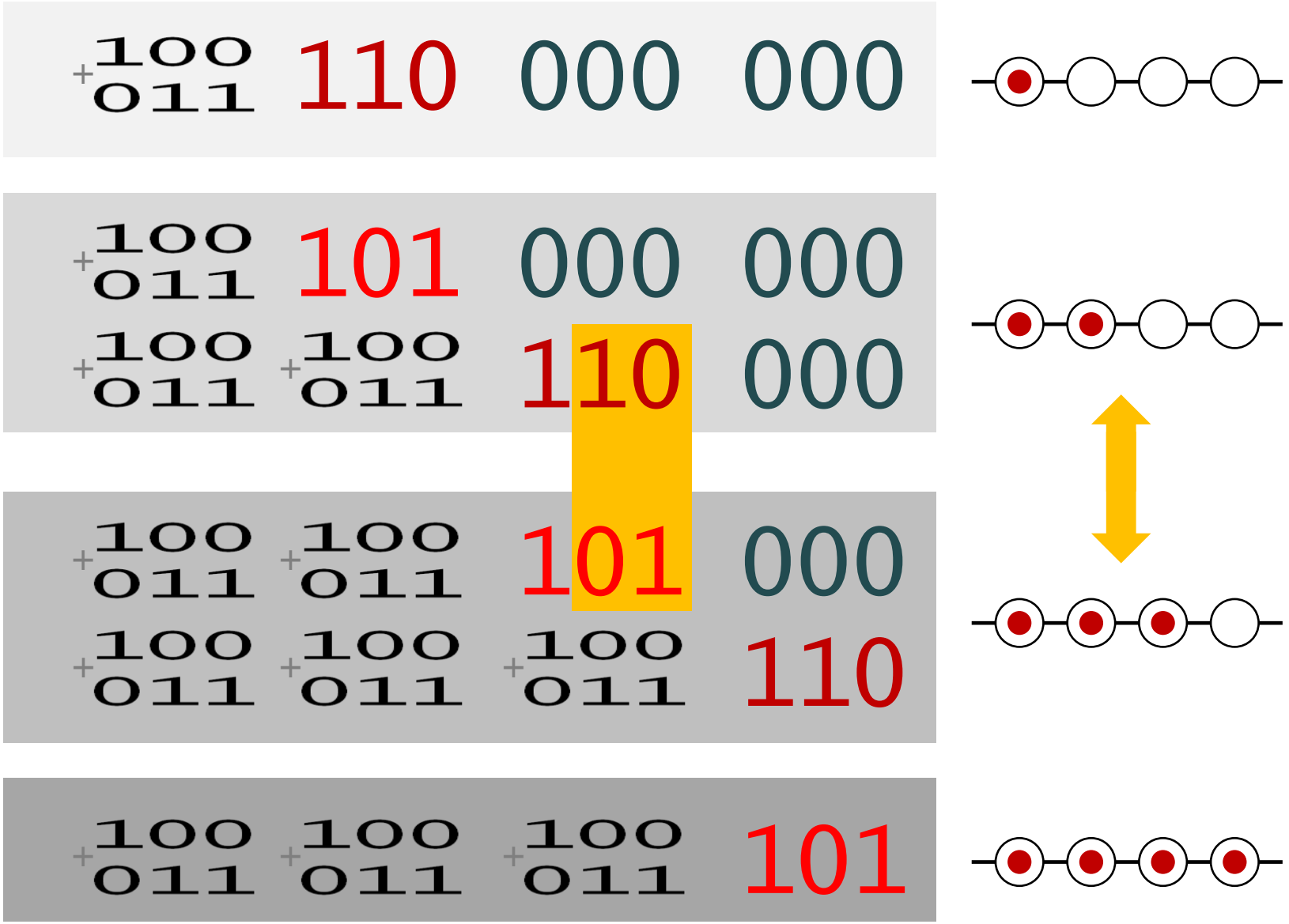
3 Constructing a composite clock with 2-local progress

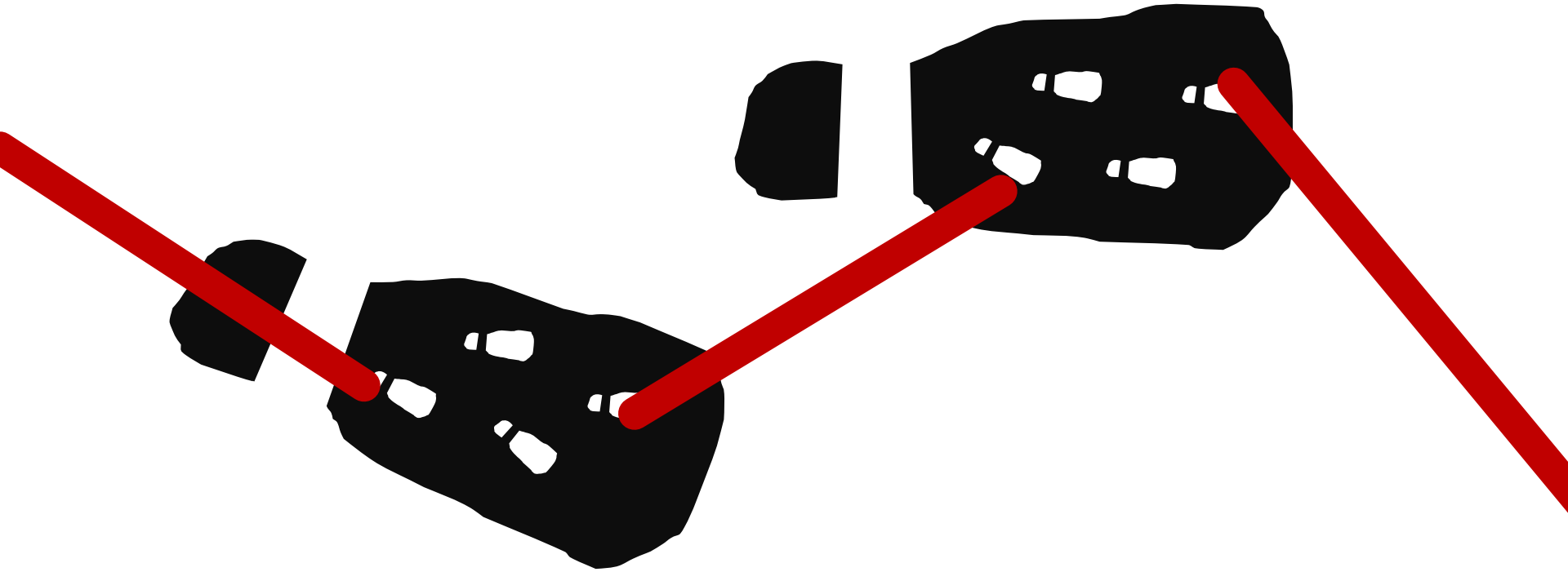


3 Constructing a composite clock with 2-local progress



3 Constructing a composite clock with **2-local progress**

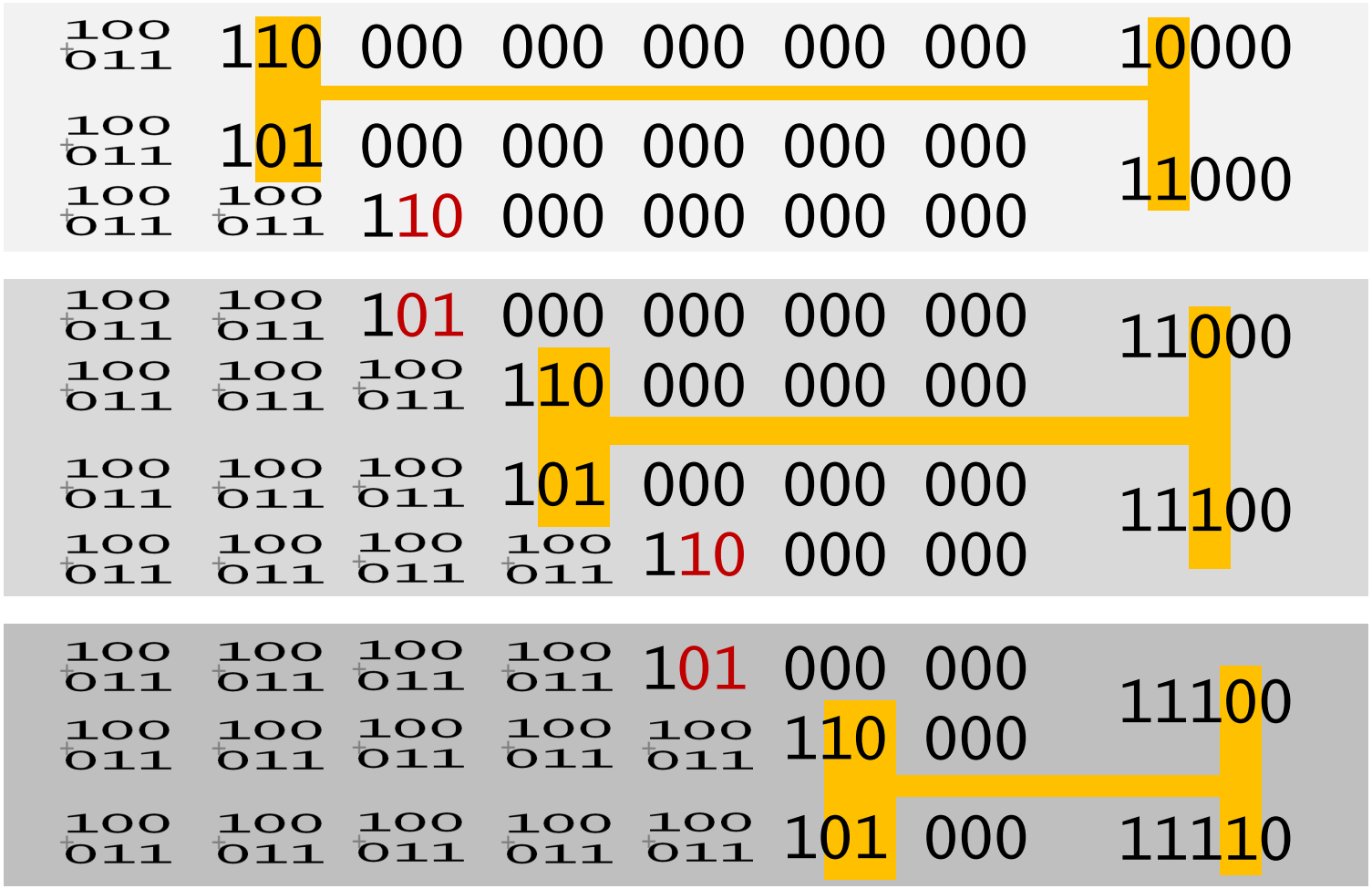




3 A composite clock with **2-local progress & 1-local id**

100 011	110	000	000	000	000	000	10000
100 011	101	000	000	000	000	000	11000
100 011	100 011	110	000	000	000	000	
100 011	100 011	101	000	000	000	000	11000
100 011	100 011	100 011	110	000	000	000	
100 011	100 011	100 011	101	000	000	000	11100
100 011	100 011	100 011	100 011	110	000	000	
100 011	100 011	100 011	100 011	101	000	000	11100
100 011	100 011	100 011	100 011	100 011	110	000	
100 011	100 011	100 011	100 011	100 011	101	000	11110

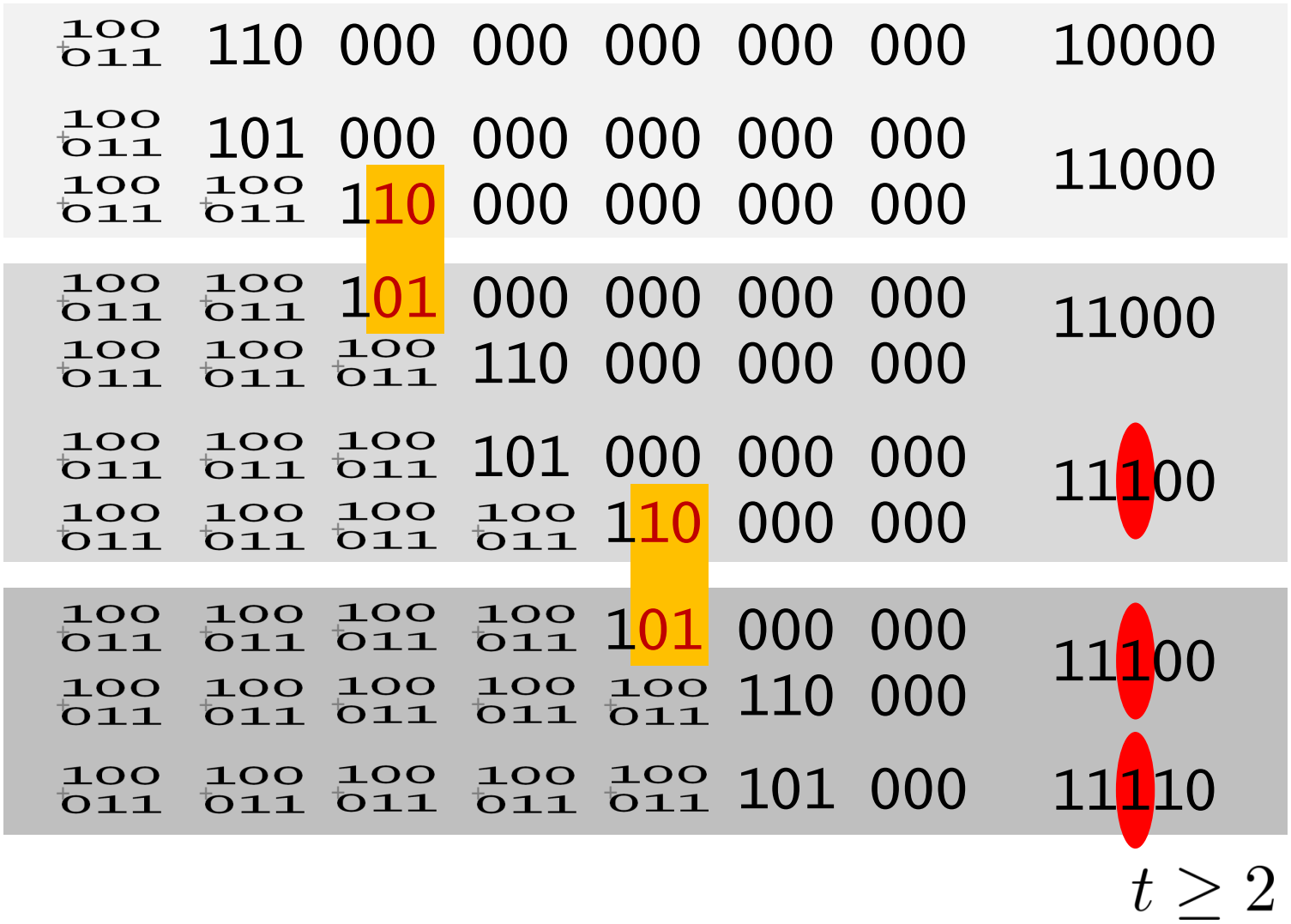
3 A composite clock with **2-local progress & 1-local id**



3 A composite clock with **2-local progress & 1-local id**

100 011	110	000	000	000	000	000	10000
100 011	101	000	000	000	000	000	11000
100 011	100 011	110	000	000	000	000	
100 011	100 011	101	000	000	000	000	11000
100 011	100 011	100 011	110	000	000	000	
100 011	100 011	100 011	101	000	000	000	11100
100 011	100 011	100 011	100 011	110	000	000	
100 011	100 011	100 011	100 011	101	000	000	11100
100 011	100 011	100 011	100 011	100 011	110	000	
100 011	100 011	100 011	100 011	100 011	101	000	11110

3 A composite clock with **2-local progress & 1-local id**



3 Constructing clocks

- telling what time it is

1-locally

- advance the clock

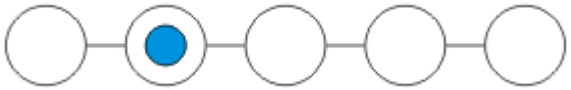
2-locally

- apply a 2-qubit gate

3-locally

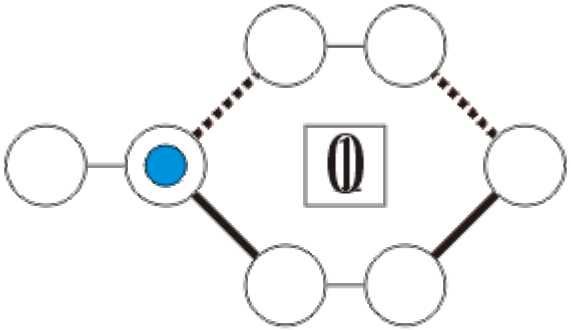
3 Applying 2-qubit gates 3-locally

- the railroad switch



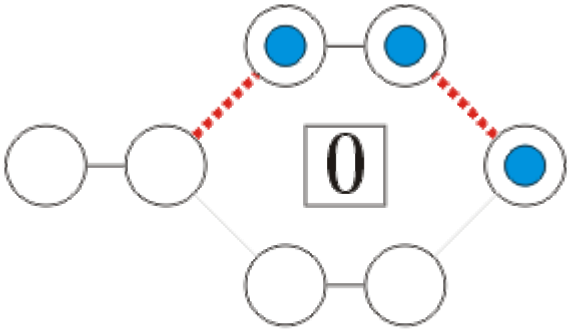
3 Applying 2-qubit gates 3-locally

- the railroad switch



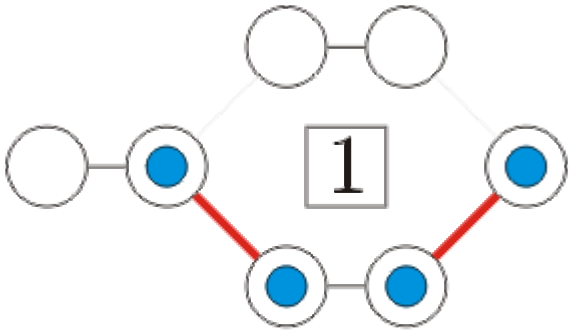
3 Applying 2-qubit gates 3-locally

- the railroad switch



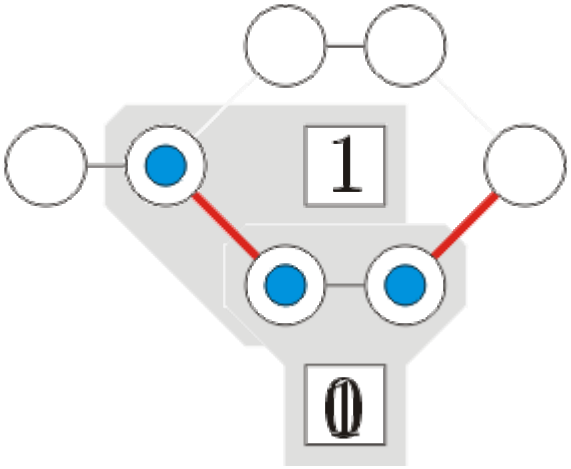
3 Applying 2-qubit gates 3-locally

- the railroad switch



3 Applying 2-qubit gates 3-locally

- the railroad switch

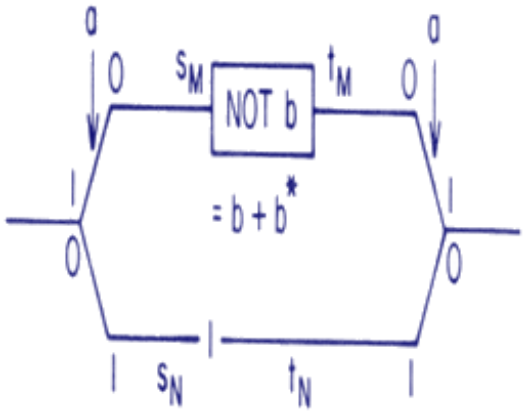


CNOT: 3-local

needs initialization

3 Applying 2-qubit gates 3-locally

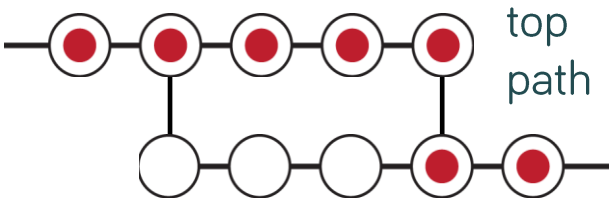
- the railroad switch

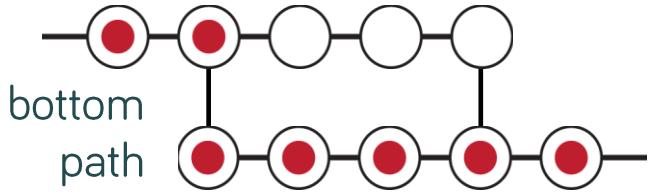


CNOT: 3-local

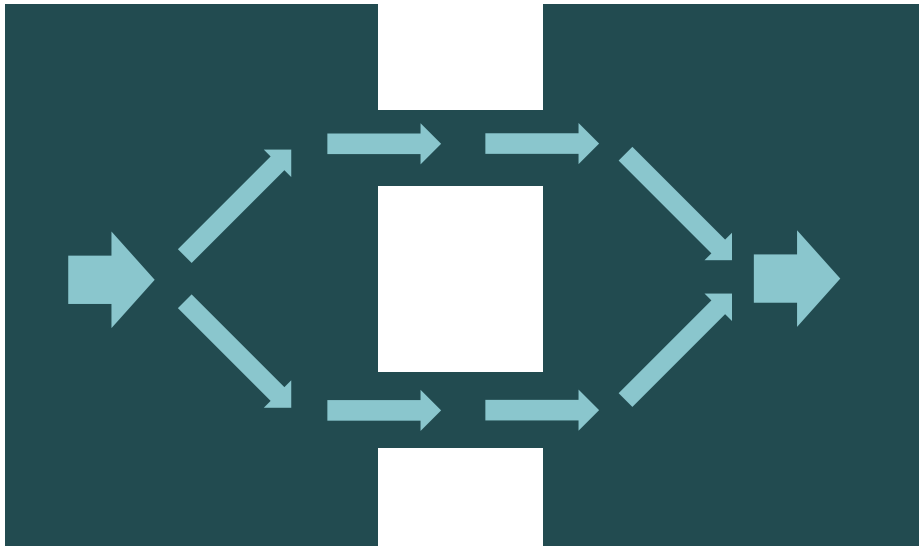
needs initialization

- a domain wall switch?

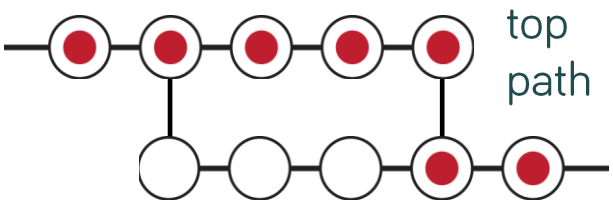


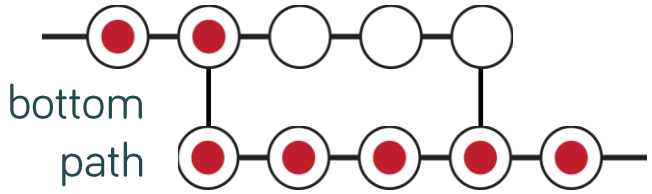


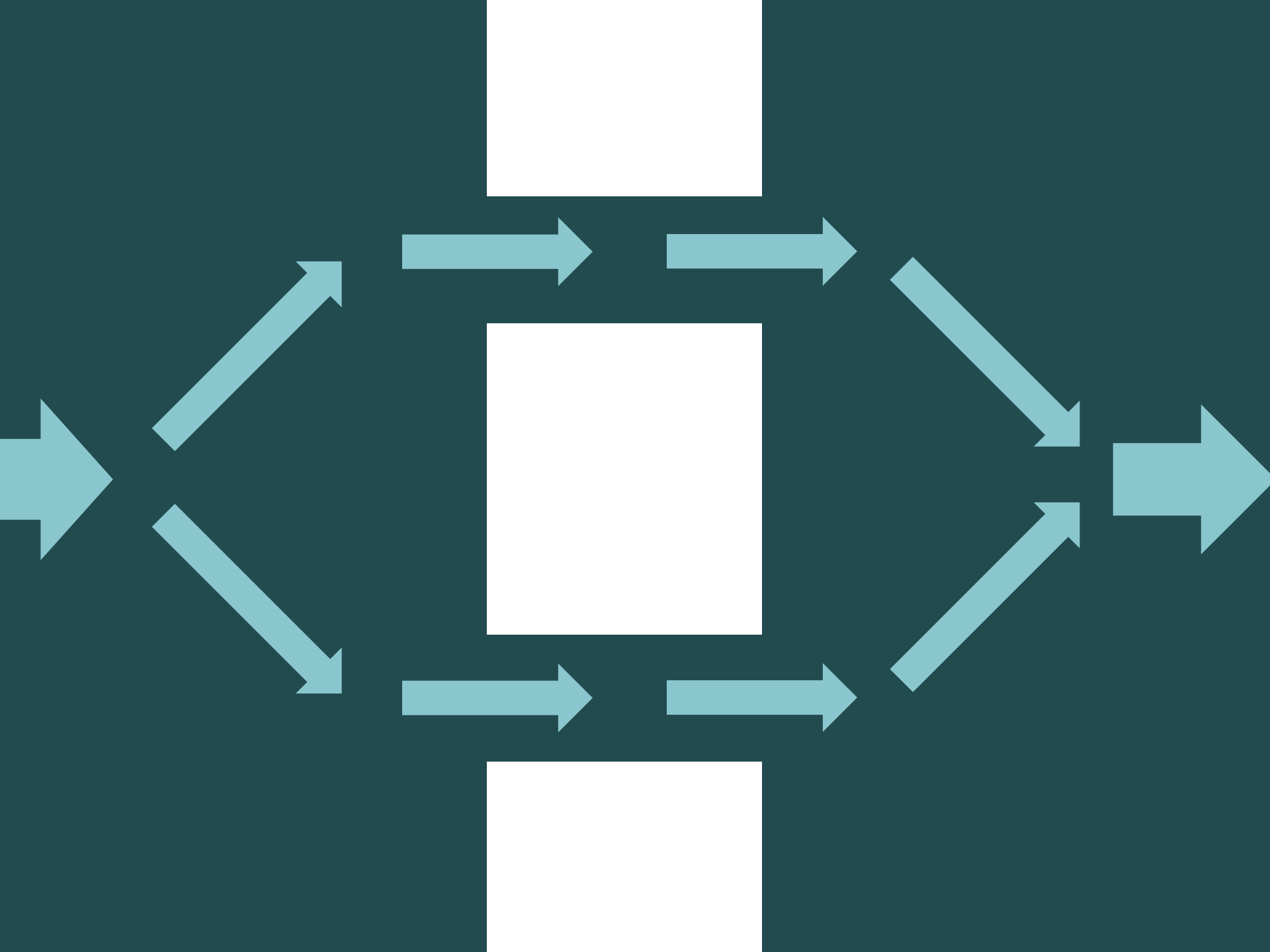
3 A train switch with a domain wall?

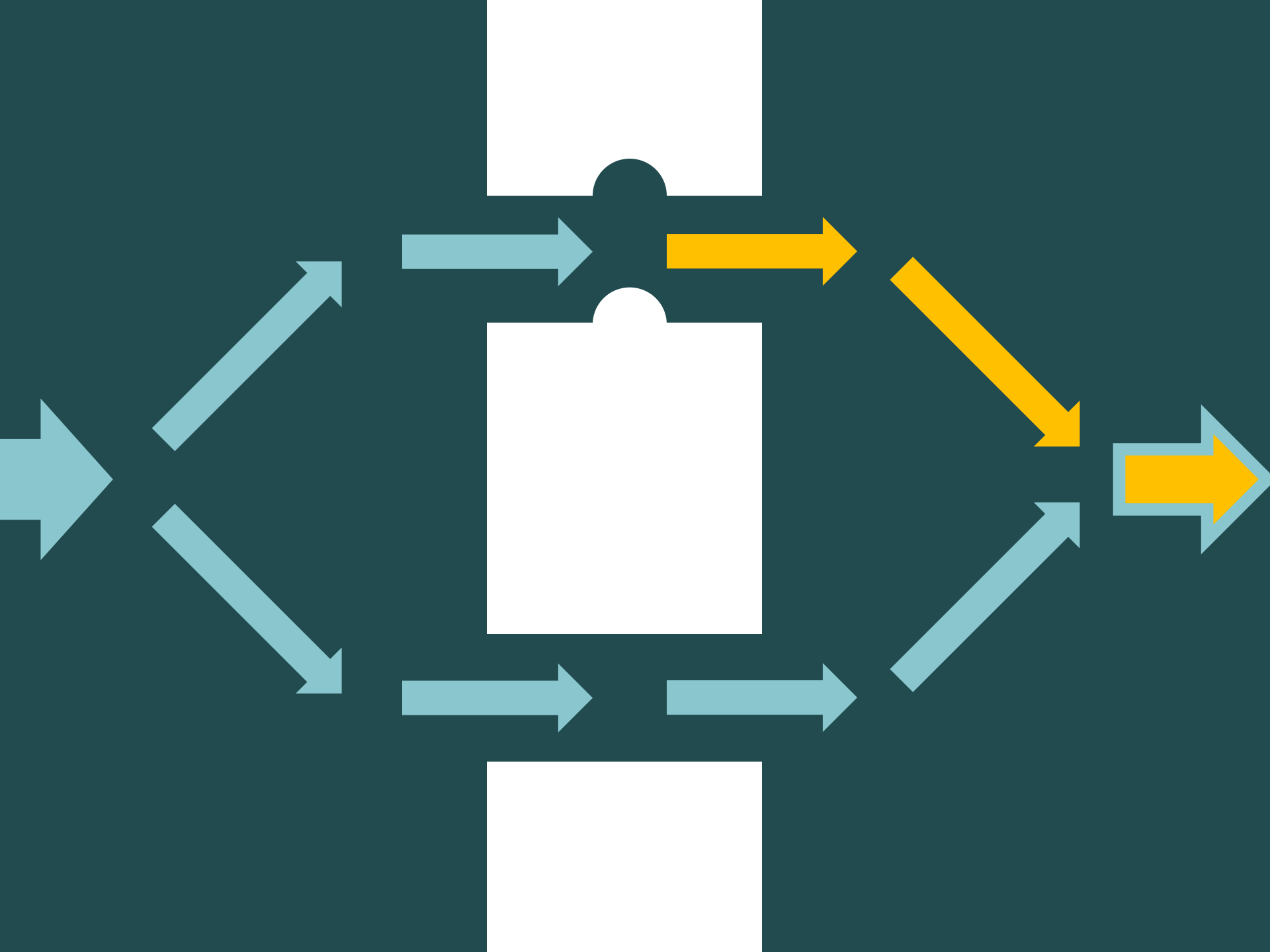


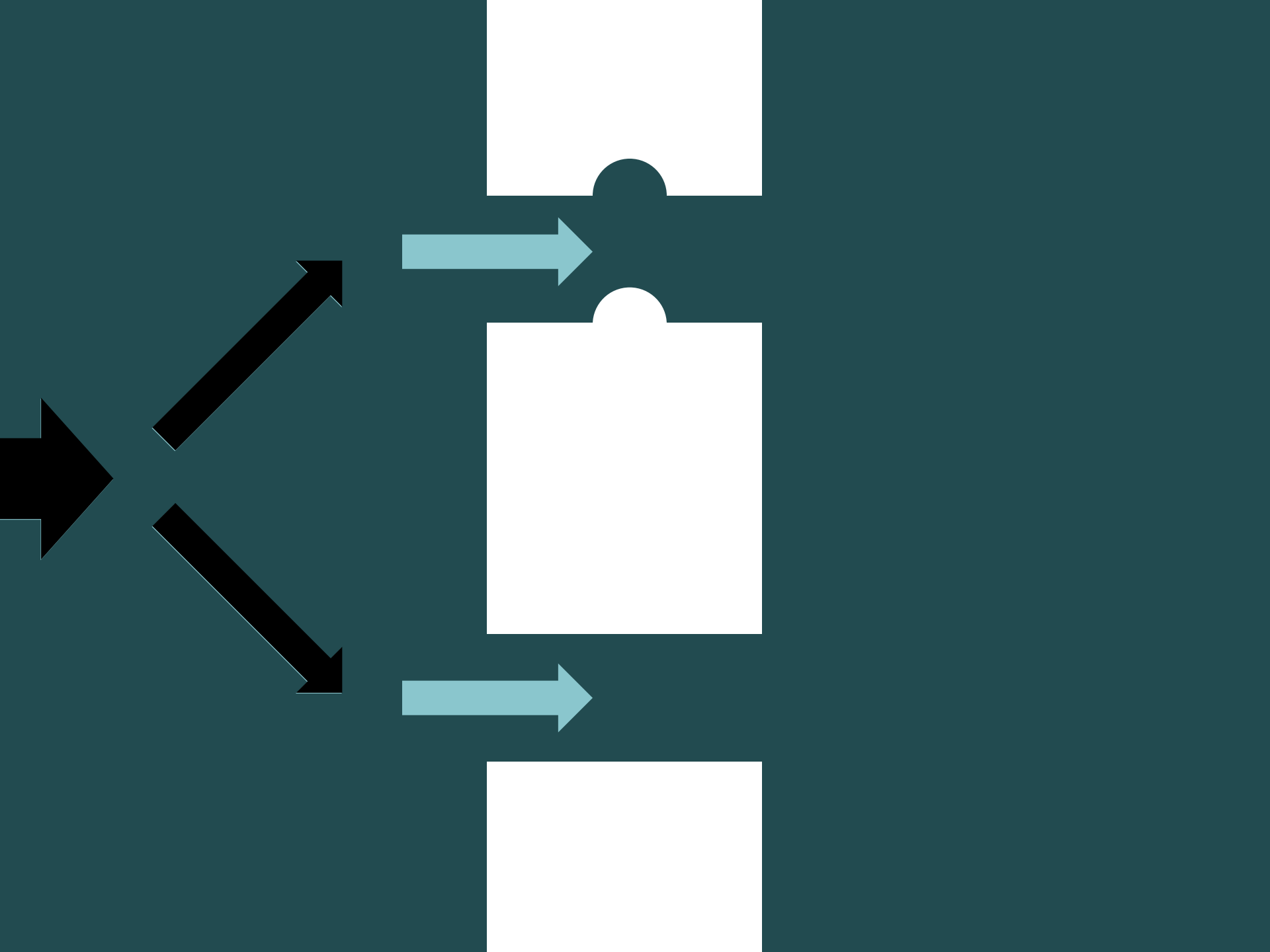
- a domain wall switch?



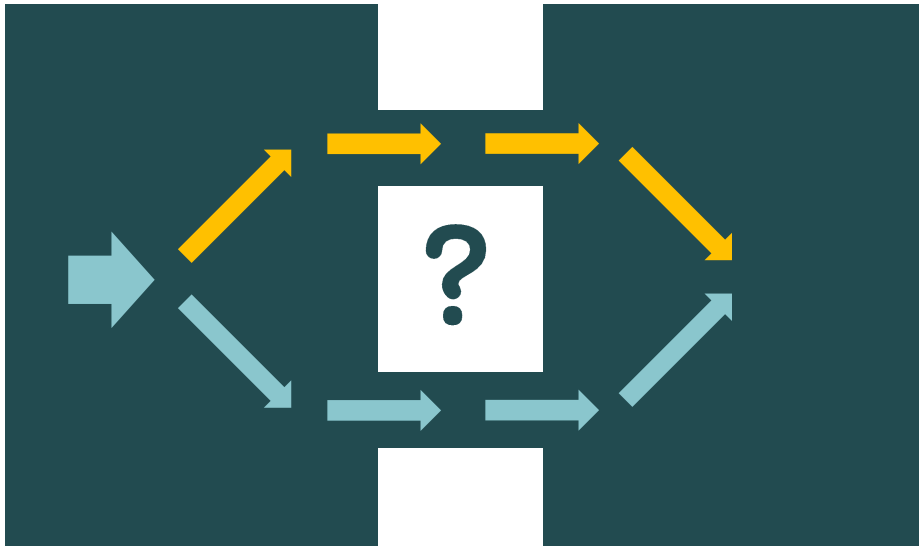




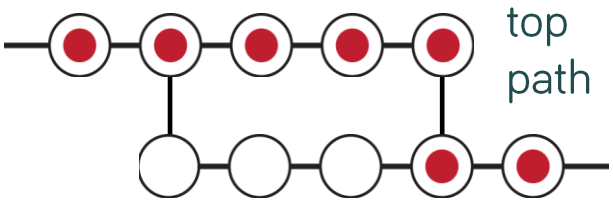


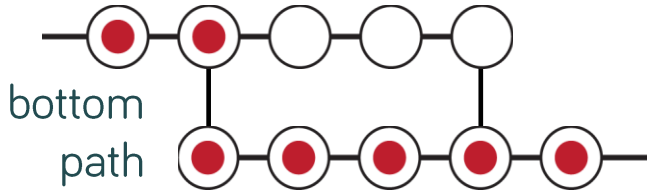


3 Combining a railroad switch with a domain wall?



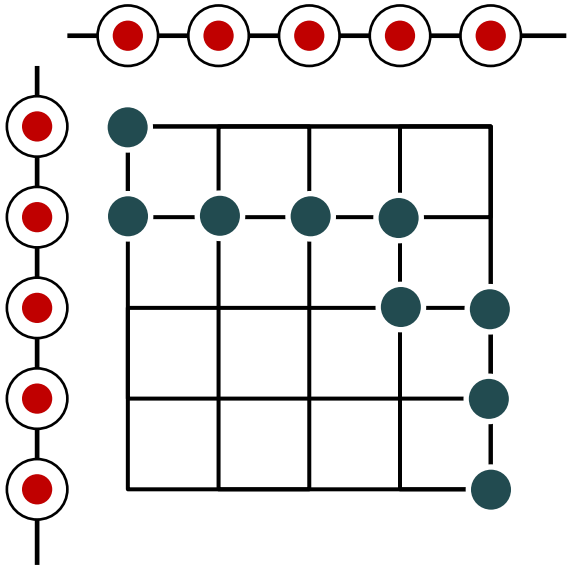
- a domain wall switch?





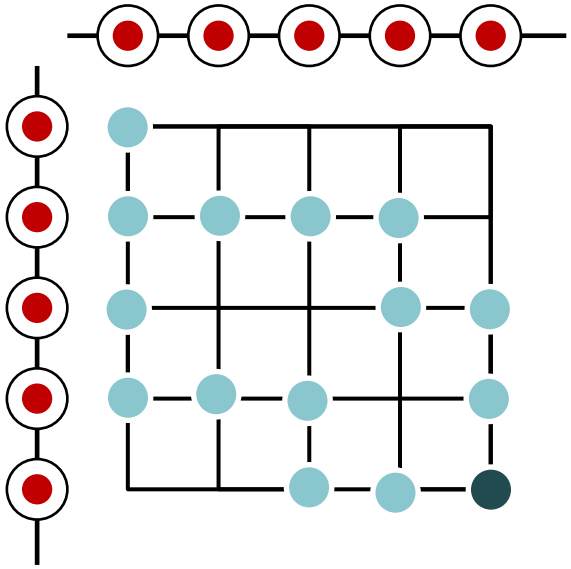
3 2D clocks (with two registers)

- two clocks



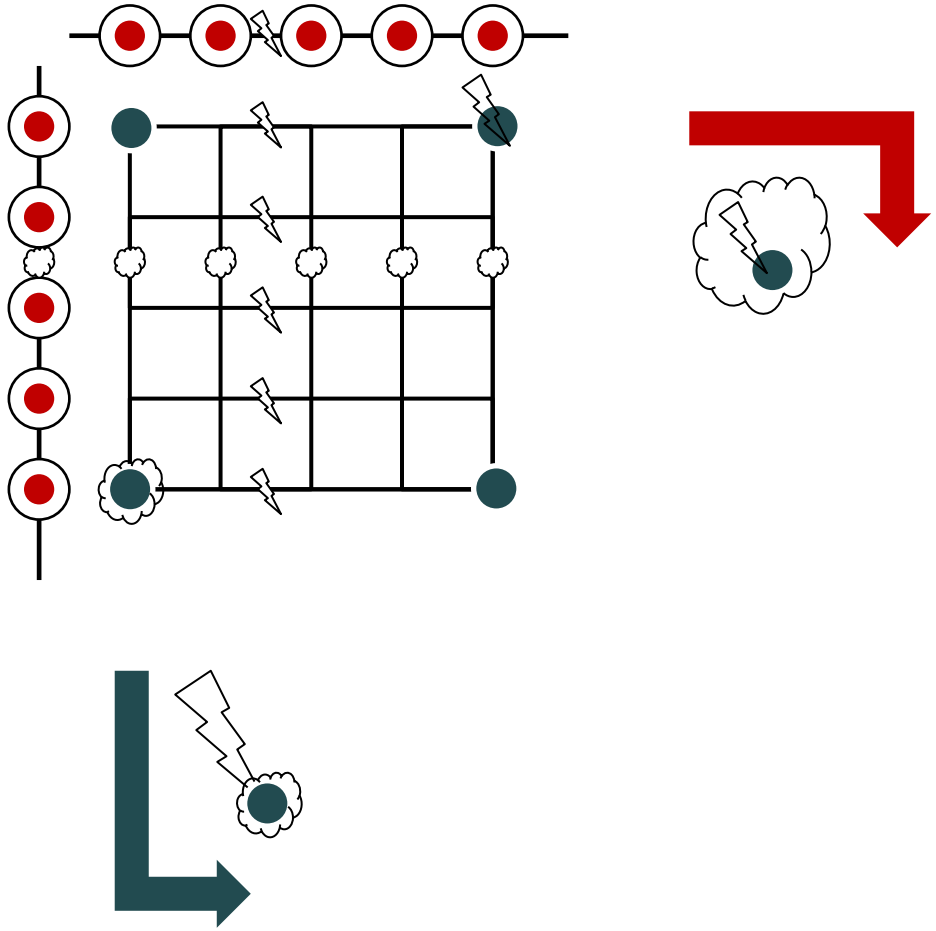
3 2D clocks (with two registers)

- two clocks



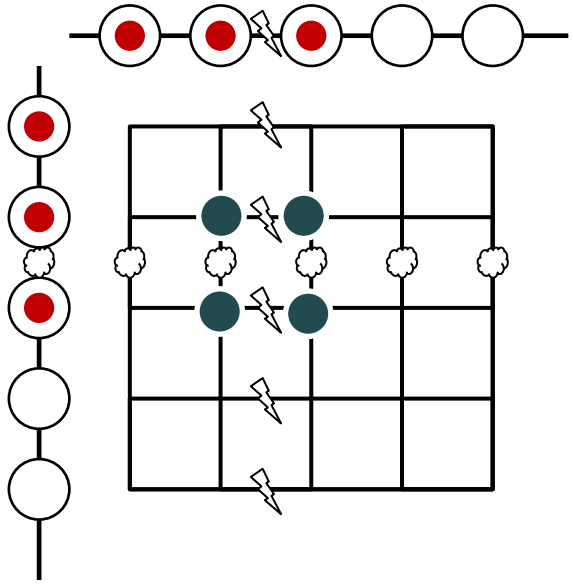
3 2D clocks (with two registers)

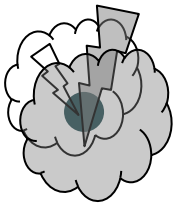
- add non-commuting (data) operations



3 2D clocks (with two registers)

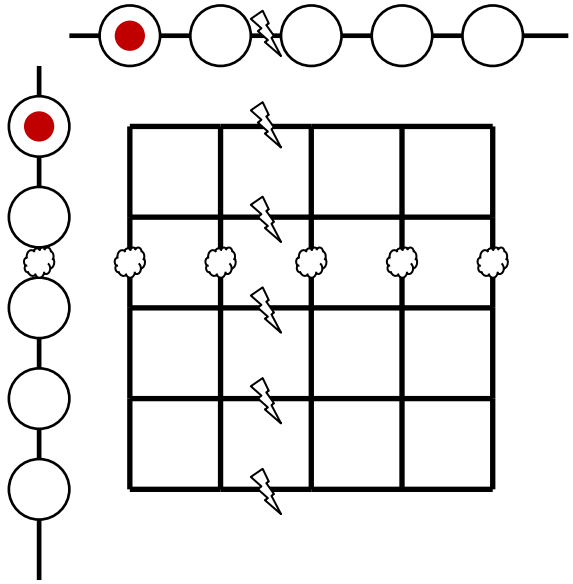
- add non-commuting (data) operations





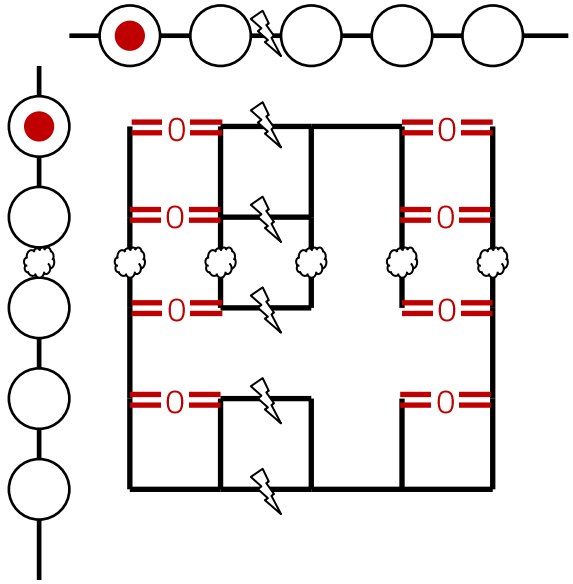
3 2D clocks (with two registers): fixing the mess

- remove some transitions



3 2D clocks (with two registers): fixing the mess

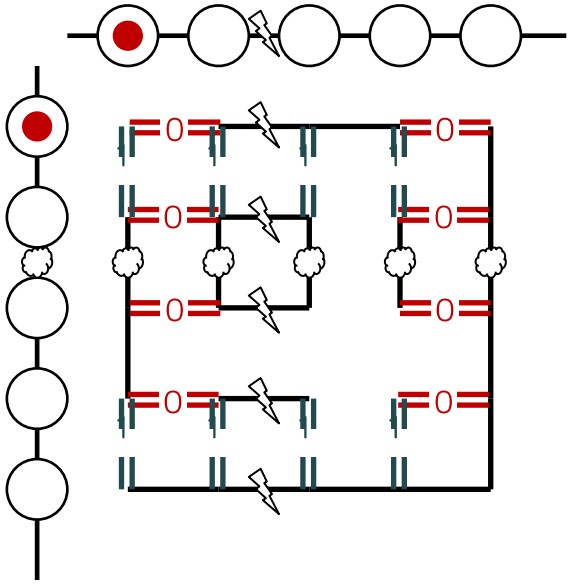
- condition transitions on a data (control) qubit



control: **0**

3 2D clocks (with two registers): fixing the mess

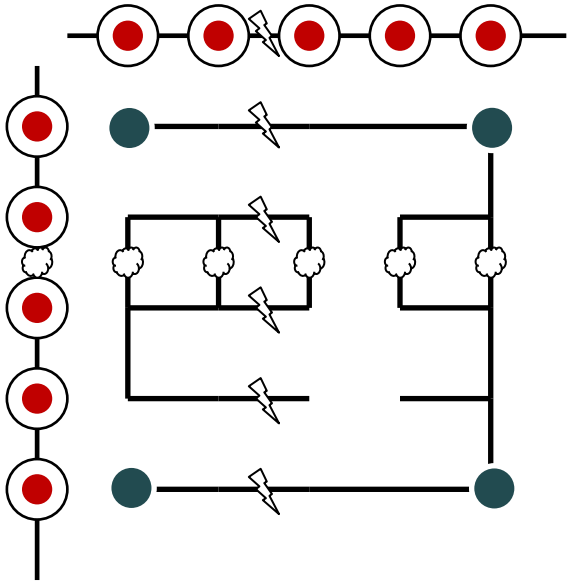
- condition transitions on a data (control) qubit

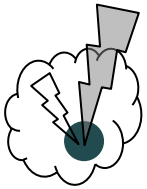


control: **0**,**1**

3 2D clocks (with two registers)

- condition transitions on a data (control) qubit

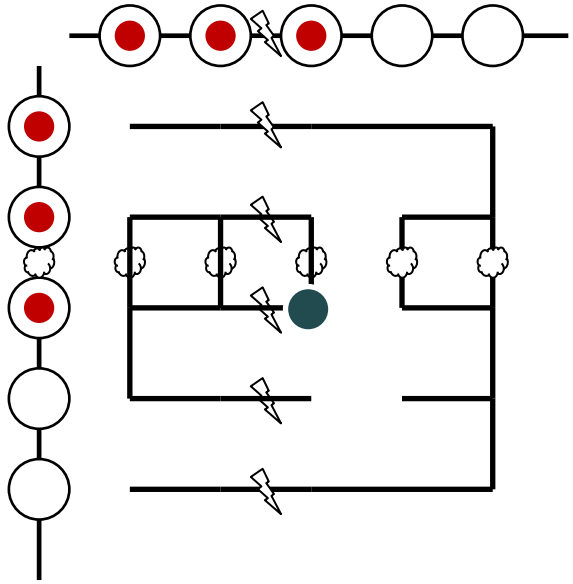


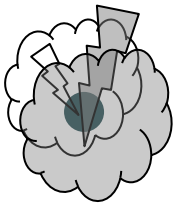


control: **0**

3 2D clocks (with two registers)

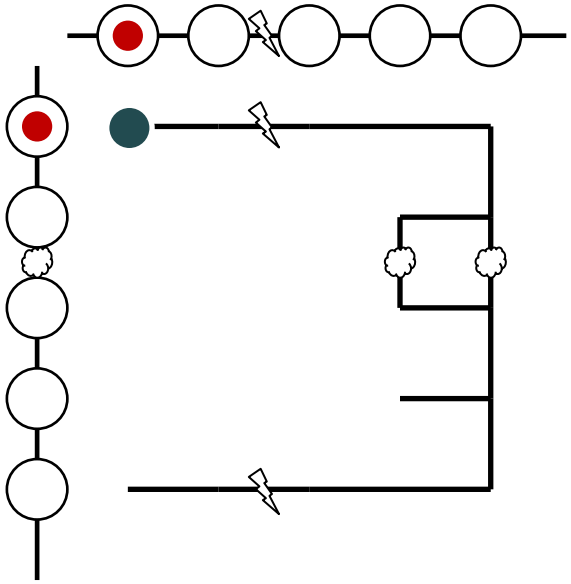
- there is no “bound” ground state (, don’t commute)

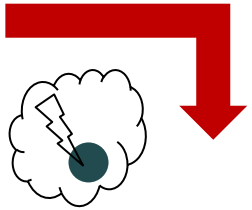




control: **0**

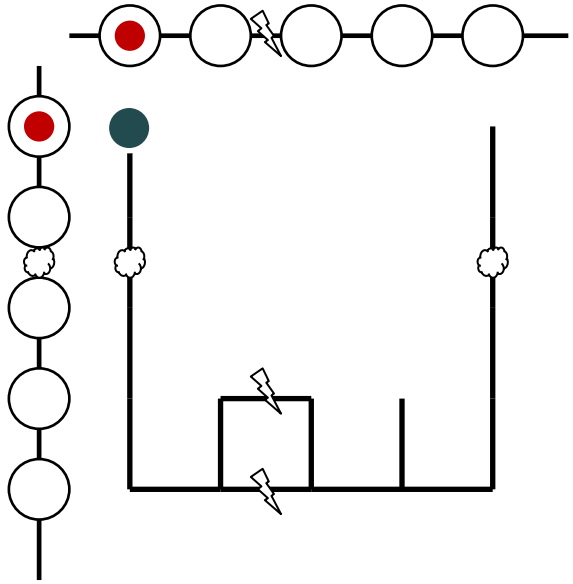
3 2D clocks (with two registers)

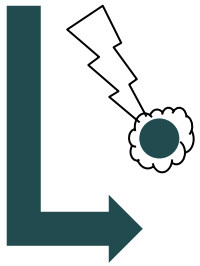




control: **0**

3 2D clocks (with two registers)

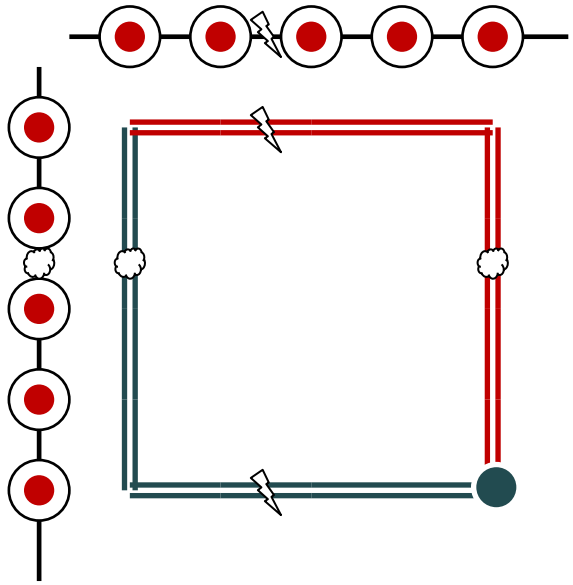


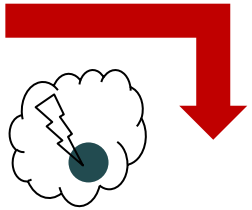


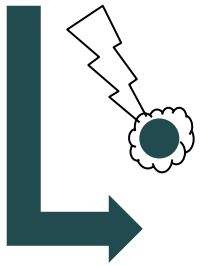
control: 1

3 2D clocks (with two registers)

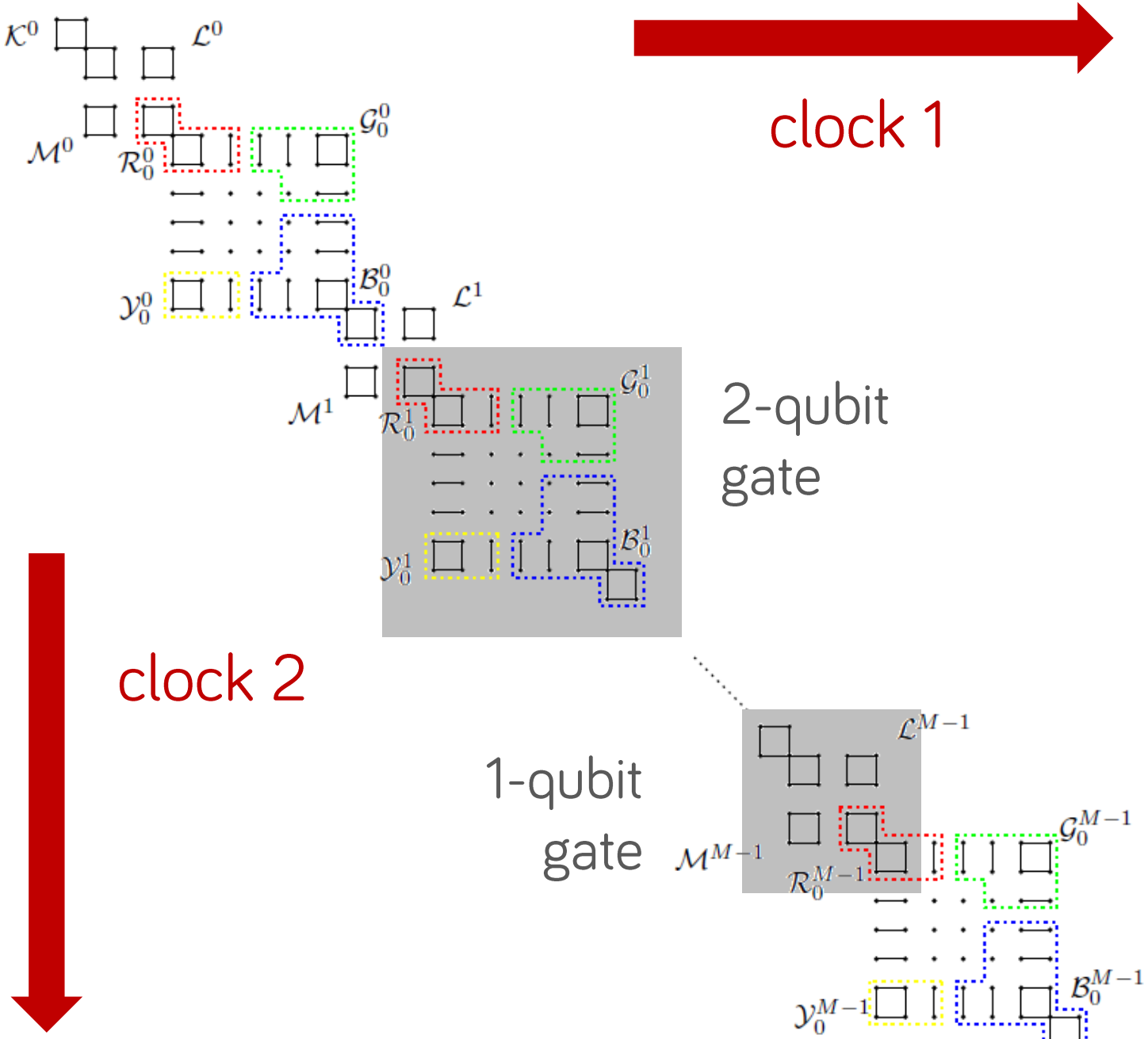
- like a railroad switch… with tracks left behind!

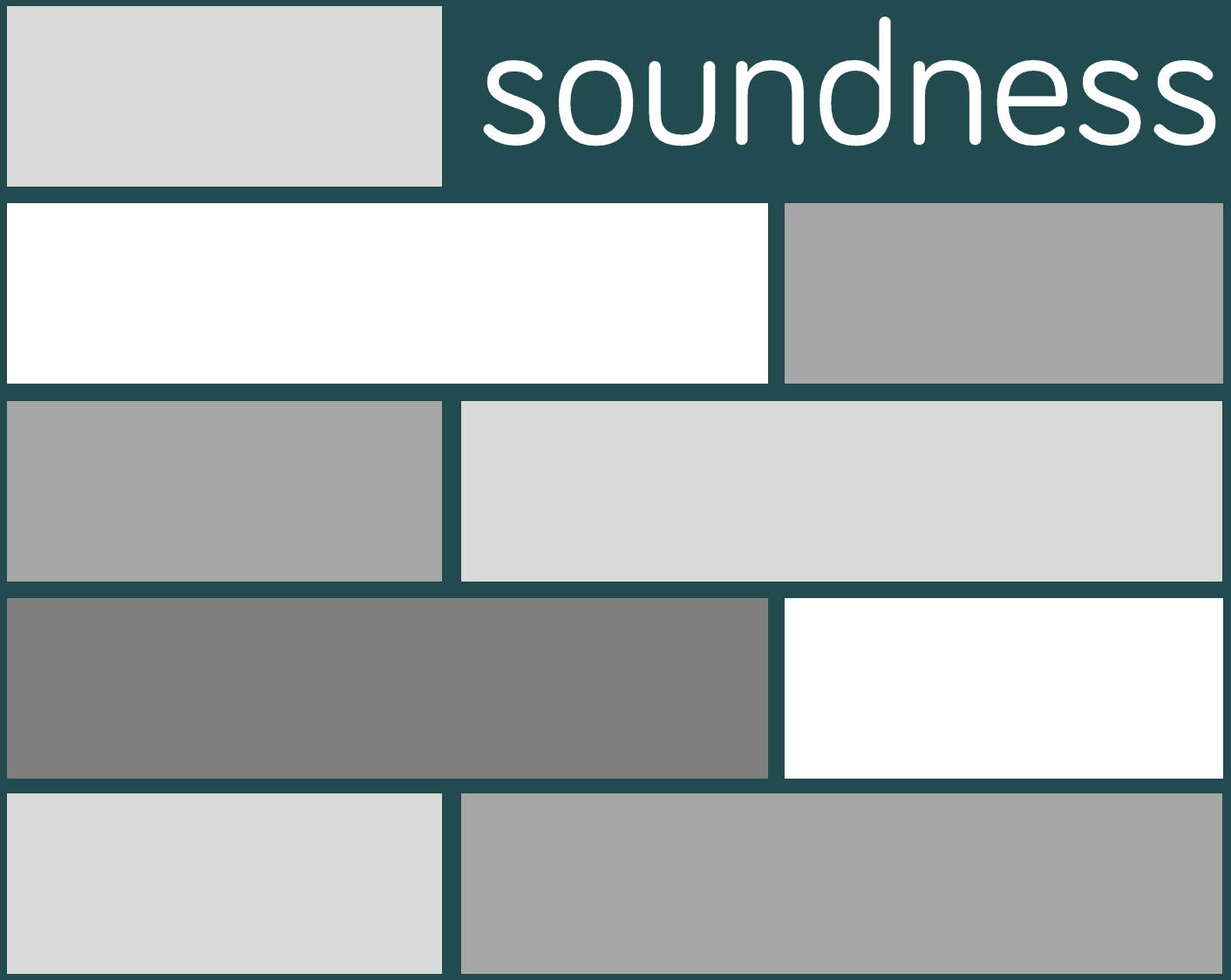






control: 0,1


clock 1
clock 2
2-qubit gate
1-qubit gate


soundness

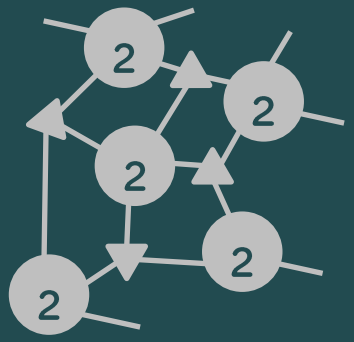
projection lemma

projection lemma

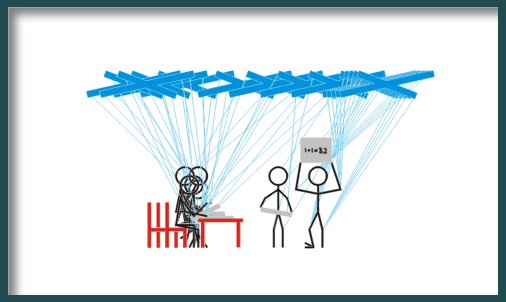
projection lemma

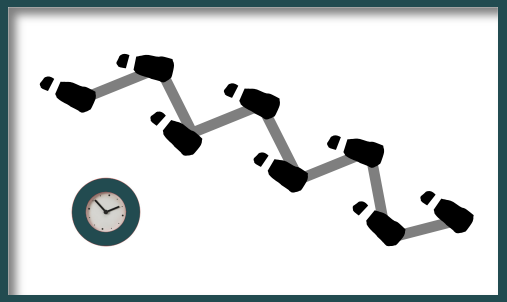
projection lemma

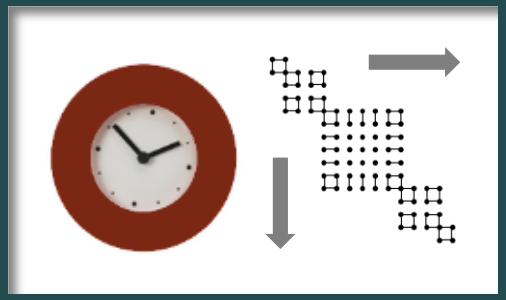
no solution?
all states have
a high energy



unfrustrated q3SAT







$\Rightarrow$ QMA_1 COMPLETE


3
SAT
2
3