QUANTUM MECHANICS IN A DISCRETE SPACE-TIME

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(*Received November 30, 1979*)

A complete description of quantum kinematics in the sense of Mackey and Weyl is presented for the class of systems whose underlying configuration spaces are finite sets equipped with the structure of finite Abelian groups. For a given finite Abelian group there is a unique class of unitarily equivalent, irreducible imprimitivity systems in a finite-dimensional Hilbert space. Schwinger's tensor product decomposition is extended to this class of systems. The finite analogue of the Galilei group over the finite space-time lattice yields a discrete time evolution operator which is proposed to be the free Hamiltonian.

1. Introduction

Models of discrete space-time have attracted the attention of physicists for a long time, either from the fundamental point of view – what is the true nature of the physical space-time manifold [1], or with the aim to get rid of the infinities present in conventional field theories [2].

A possible technique to introduce discreteness into space-time is to assume that the points in a space-time manifold do not form a continuum but a discrete set of points on a lattice. On the other hand, for continuous models this often represents a suitable kind of approximation. Such approximations are successfully used in solid state physics and quantum chemistry, in statistical physics and quantum field theory. For instance, lattice gauge field theories [3] presently provide one of the most promising approaches toward a unified theory of elementary particles since the lattice formulation allows an analysis on a non-perturbative level.

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In this paper we restrict our considerations to finite lattices. In order to formulate quantum kinematics of a system over a configuration space M consisting of a finite number of n points (Sect. 2) we need an extra structure providing momentum observables. One possibility – which we are using in this paper – is a transitive action of a group G on the set M making from M a homogeneous space of G. Since we want to keep in close analogy with the continuous Euclidean situation we equip M with the structure of a finite Abelian group G of order n. The group structure of M allows for the natural G-action on M via group multiplication, which is used in the sequel in constructing quantum kinematics over M.

In the special case of a cyclic group $G = Z_n$ this problem was considered by H. Weyl [4]. He constructed a set of unitary operators in a Hilbert space of dimension n in complete analogy with the Weyl system over the Euclidean configuration space. The structure of these Weyl operators was further investigated by J. Schwinger [5] and T. S. Santhanam [6].

We use G. W. Mackey's theory of quantum kinematics on homogeneous spaces [7] to exhibit the irreducible systems of imprimitivity and the corresponding irreducible Weyl systems for an arbitrary finite Abelian group. Schwinger's tensor product decompositions of the Weyl operators turn out to be a simple consequence of the structure theorem for finite Abelian groups. We are thus able to extend Schwinger's treatment to arbitrary Weyl systems in finite-dimensional Hilbert spaces. As a by-product we obtain the Stone–von Neumann uniqueness theorem which in this case is also a trivial corollary to Mackey's generalization of this theorem to any locally compact Abelian group [8].

To keep the essential features of the usual non-relativistic quantum mechanics, we construct in Sect. 3 a free, discrete-time evolution operator with the property that it combines with the Weyl operators into a representation of a finite analogue of the quantal Galilei group in one dimension. Since this construction is unique, it leads to a unique matrix replacing the Laplace differential operator.

2. Quantum kinematics on finite Abelian groups

2.1. Configuration spaces. We investigate systems over configuration spaces M given by finite sets. As explained in the Introduction, we equip a set M consisting of n elements with the structure of a finite Abelian group of order n. The investigation of the general case can be reduced to a simpler analysis for cyclic groups because of a theorem describing the structure of finite Abelian groups [9].

THEOREM. *Any finite Abelian group is isomorphic to the direct product* $Z_{m_1} \times \ldots \times Z_{m_f}$ *of a finite number of cyclic groups for integers* $m_1, \ldots, m_f$ *greater than* 1, *each of which is a power of a prime* ($m_r = n_r^{a_r}$ *where the primes* n_r, $r = 1, \ldots, f$, *need not be mutually different*).

Hence in the first step it is sufficient to consider only the cyclic groups. Given a positive integer n, let the configuration space M be the finite set

$$M = Z_n = \{\varrho | \varrho = 0, 1, \ldots, n-1\}$$

with additive group law modulo n. Since there is a natural transitive action of Z_n on itself, we may suppose that M is a homogeneous space of $G = Z_n$,

$$G = Z_n = \{j | j = 0, 1, \ldots, n-1\},$$

realized as an additive group modulo n with the action

$$G \times M \to M\colon (j, \varrho) \to \varrho + j (\mathrm{mod}\, n)$$

and the isotropy subgroup $H = \{0\}$.

2.2. Systems of imprimitivity. According to Mackey [7] quantum kinematics of a system on a locally compact space $M = G/H$ is determined by a transitive system of imprimitivity $(\mathscr{U}, \mathscr{E})$ for G based on M in a Hilbert space $\mathscr{H}$. Here $\mathscr{U} = \{U(g) | g \in G\}$ is a unitary representation of G in $\mathscr{H}$ and $\mathscr{E} = \{E(S) | S$ Borel subset of $M\}$ is a projection-valued measure in $\mathscr{H}$ satisfying

$$U(g)\, E(S)\, U(g)^{-1} = E(g^{-1} \cdot S). \tag{1}$$

For the finite set M, (1) simplifies to

$$U(j)\, E(\varrho)\, U(j)^{-1} = E\big(\varrho - j (\mathrm{mod}\, n)\big), \tag{2}$$

where $\varrho \in M$, $j \in G$, $E(\varrho) = E(\{\varrho\})$ and $E(S) = \sum_{\varrho \in S} E(\varrho)$.

Complete classification of transitive systems of imprimitivity up to simultaneous unitary equivalence of both $\mathscr{U}$ and $\mathscr{E}$ is obtained from Mackey's Imprimitivity Theorem [7]. Its application to our system yields:

If $(\mathscr{U}, \mathscr{E})$ acts irreducibly in $\mathscr{H}$, then there is, up to unitary equivalence, only one system of imprimitivity, where

(i) $\mathscr{H}$ is a Hilbert space of dimension n with the inner product

$$(\varphi, \psi) = \sum_{\varrho = 0}^{n-1} \bar{\varphi}_\varrho \psi_\varrho$$

where φ_ϱ, ψ_ϱ, $\varrho = 0, 1, \ldots, n-1$, denote the components of φ, ψ in a standard basis;

(ii) $\mathscr{U}$ is the induced representation $\mathscr{U} = \mathrm{Ind}_H^G I$ called the (right) regular representation

$$[U(j)\psi]_\varrho = \psi_{\varrho + j (\mathrm{mod}\, n)} \qquad (j \in G);$$

its matrix form in the standard basis is

$$(U(j))_{\varrho\sigma} = \delta_{\varrho+j(\mathrm{mod}\ n),\sigma}; \tag{3}$$

(iii) $\mathscr{E}$ is given by

$$[E(\varrho)\psi]_\sigma = \delta_{\varrho\sigma}\psi_\sigma, \quad \text{i.e.} \quad (E(\varrho))_{\sigma\tau} = \delta_{\sigma\varrho}\delta_{\tau\varrho} \quad (\varrho \in M). \tag{4}$$

2.3. Physical interpretation. This unique system of imprimitivity has a simple physical meaning. The one-dimensional projectors $E(\varrho)$ project on the eigenvectors $e^\varrho \in \mathscr{H}$ corresponding to the positions $\varrho = 0, 1, \ldots, n-1$. In the above matrix realization (3), (4) the set $\{e^\varrho\}$ forms the standard basis of $\mathscr{H}$. Then in a normalized state $\psi = (\psi_0, \ldots, \psi_{n-1})$ the probability to measure the position ϱ is equal to

$$(\psi, E(\varrho)\psi) = |\psi_\varrho|^2.$$

The unitary operators $U(j)$ act as displacement operators

$$U(j)e^\varrho = e^{\varrho - j\,(\mathrm{mod}\ n)};$$

they are given by cyclic matrices generated via $U(j) = U(1)^j$ from

$$A = U(1) = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 \\ 1 & 0 & 0 & . & . & 0 \end{bmatrix}.$$

2.4. Coordinate and momentum representations. The above matrix form may be called the coordinate representation. The transition to the momentum representation is provided by a transformation diagonalizing the commuting set $\{U(j)\}$. This Z_n-analogue of the Fourier transformation is given by the unitary Sylvester matrix

$$S_{j\varrho} = \frac{\varepsilon^{j\varrho}}{\sqrt{n}}, \qquad \varepsilon = e^{2\pi i/n},$$

in the form S^{-1}: $\psi \to \psi'$, i.e.

$$\psi'_j = \frac{1}{\sqrt{n}} \sum_{\varrho=0}^{n-1} \varepsilon^{-j\varrho}\psi_\varrho.$$

It is easily checked that $S^{-1}AS = B$, where B is the diagonal matrix

$$B = \mathrm{diag}(1, \varepsilon, \ldots, \varepsilon^{n-1});$$

hence

$$U'(k) = S^{-1}\,U(k)\,S = B^k. \tag{5}$$

The momentum eigenstates – eigenstates of $\{U(k)\}$ with eigenvalues $\{\varepsilon^{kj}\}$, $j = 0, 1, \ldots, n-1$ – are

$$f^j = \frac{1}{\sqrt{n}}(1, \varepsilon^j, \ldots, \varepsilon^{(n-1)j}),$$

and the associated one-dimensional projectors $F(j)$: $\psi \to (f^j, \psi) f^j$ serve as the spectral projectors for $U(k)$,

$$U(k) = \sum_{j=0}^{n-1} \varepsilon^{kj} F(j).$$

The spectral family $F'(j) = S^{-1} F(j) S$ over G in $\mathscr{H}$ may be regarded as belonging to a complementary imprimitivity system over momenta (in the momentum representation). For, suppose that $M = Z_n$ acts on $G = Z_n$ by the action

$$M \times G \to G\colon\ (\varrho, j) \mapsto j - \varrho\,(\mathrm{mod}\, n).$$

Then there is again a unique (up to unitary equivalence) imprimitivity system $(\mathscr{V}, \mathscr{F})$ for M on G in $\mathscr{H}$. In the momentum representation it has a similar form as $(\mathscr{U}, \mathscr{E})$ had in the coordinate representation, i.e.

$$\big(F'(j)\big)_{kl} = \delta_{kj}\,\delta_{lj}, \qquad V'(\sigma) = V'(1)^\sigma, \qquad V'(1) = A^{-1},$$

$$V(\sigma) = S V'(\sigma) S^{-1} = \sum_{\varrho=0}^{n-1} \varepsilon^{\sigma\varrho} E(\varrho). \tag{6}$$

We thus have two representations, the coordinate representation where $E(\varrho)$ and $V(\sigma) = B^\sigma$ are diagonal, and the momentum representation where $F'(j)$ and $U'(k) = B^k$ are diagonal. The transition between the two representations is given by the discrete Fourier transform $S\colon \mathscr{H} \to \mathscr{H}$. The Weyl relations for the continuous case

$$e^{i\xi P}\, e^{i\eta Q} = e^{i\xi\eta}\, e^{i\eta Q}\, e^{i\xi P}$$

are replaced by the discrete Weyl system $(\mathscr{U}, \mathscr{V})$ with

$$U(j)\, V(\varrho) = \varepsilon^{j\varrho}\, V(\varrho)\, U(j) \tag{7}$$

or

$$A^j\, B^\varrho = \varepsilon^{j\varrho}\, B^\varrho\, A^j. \tag{8}$$

Remark. One can also define the Weyl system in terms of unitary matrices

$$W(j,\varrho)=\varepsilon^{-\frac{1}{2}j\varrho}\,U(j)\,V(\varrho)=\varepsilon^{-\frac{1}{2}j\varrho}A^j B^\varrho$$

with the product law

$$W(j,\varrho)\,W(j',\varrho')=\varepsilon^{\frac{1}{2}(j\varrho'-j'\varrho)}\,W(j+j',\varrho+\varrho').$$

These n^2 operators supply, as Schwinger [5] has shown, a complete operator basis for a physical system possessing n states:

$$n^{-1}\,\mathrm{Tr}\big(W(j,\varrho)\,W(j',\varrho')^*\big)=\delta_{jj'}\,\delta_{\varrho\varrho'}$$

and for any matrix Y

$$n^{-1}\sum_{j,\varrho}W(j,\varrho)\,Y\,W(j,\varrho)^*=1\ \mathrm{Tr}\,Y.$$

2.5. Coordinate and momentum operators. In analogy with the continuous case the matrices Q and P can be defined via

$$U(k)=e^{i\frac{2\pi}{n}kP},\qquad V(\sigma)=e^{i\frac{2\pi}{n}\sigma Q}$$

in the coordinate representation. (This definition differs somewhat from [6] where the transition to the continuous case was treated.) The spectral decomposition (6) implies that the eigenvalues of Q are $q_\varrho=\varrho\,(\mathrm{mod}\,n)$; we choose

$$Q=\mathrm{diag}(0,1,\dots,n-1).$$

Similarly, eq. (5) yields the eigenvalues of P, $p_j=j\,(\mathrm{mod}\,n)$. Choosing again

$$P'=S^{-1}PS=\mathrm{diag}(0,1,\dots,n-1),$$

one easily finds

$$P_{\varrho\sigma}=\begin{cases}\dfrac{n-1}{2} & \text{for } \varrho=\sigma,\\[2ex] \dfrac{1}{\varepsilon^{\varrho-\sigma}-1} & \text{for } \varrho\neq\sigma.\end{cases}$$

The commutator is tracefree,

$$[Q,P]_{\varrho,\sigma}=\begin{cases}0 & \text{for } \varrho=\sigma,\\[1ex] \dfrac{\varrho-\sigma}{\varepsilon^{\varrho-\sigma}-1} & \text{for } \varrho\neq\sigma.\end{cases}$$

2.6. Tensor product decompositions. Quantum kinematics on the configuration spaces Z_n derived in the preceding sections can be directly extended to any direct product of the form

$$M = Z_{m_1} \times \ldots \times Z_{m_f}.$$

The corresponding transitive system of imprimitivity for $G = Z_{m_1} \times \ldots \times Z_{m_f}$ on M is the tensor product of the imprimitivity systems $(\mathscr{U}_n, \mathscr{E}_n)$ for Z_{m_n}, and acts in the Hilbert space $\mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_f$ of dimension $m_1 \ldots m_f$. Each such system of imprimitivity is irreducible if and only if each $(\mathscr{U}_n, \mathscr{E}_n)$ is irreducible, hence if irreducible, it is unique up to unitary equivalence. Then, according to the structure theorem for finite Abelian groups (Sect. 2.1) it is sufficient to put $m_r = n_r^{a_r}$ ($n_r > 1$, prime, $r = 1, \ldots, f$) to obtain the quantum kinematics for any finite Abelian group.

In the special case of Z_n the structure theorem actually implies[1] the unique decomposition

$$Z_n = Z_{m_1} \times \ldots \times Z_{m_f} \tag{9}$$

where $m_r = n_r^{a_r}$, and $n = n_1^{a_1} \ldots n_f^{a_f}$ is the unique prime decomposition of n with distinct primes $n_r > 1$. Thus the system of imprimitivity for Z_n is equivalent to the tensor product system with distinct primes n_r, as discovered by Schwinger [5], in the form

$$\mathscr{U} = \mathscr{U}_1 \otimes \ldots \otimes \mathscr{U}_f, \qquad \mathscr{V} = \mathscr{V}_1 \otimes \ldots \otimes \mathscr{V}_f, \tag{10}$$

where each Weyl system $(\mathscr{U}_r, \mathscr{V}_r)$ acts in $\mathscr{H}_r$ of dimension $n_r^{a_r}$. An independent proof of this result is given in Appendix 1.

One may give physical meaning to these tensor product decompositions as if they describe quantum degrees of freedom in relation to the prime periods in a prime decomposition of $n = \dim \mathscr{H}$.

3. Galilean quantum dynamics on finite Abelian groups

3.1. Evolution operators. The reversible time evolution of an isolated quantum system is determined by a strongly continuous one-parameter group of unitary operators $\{L(t) | t \in \boldsymbol{R}\}$ acting in some Hilbert space. According to Stone's theorem there exists exactly one self-adjoint operator H – the Hamiltonian – such that

$$L(t) = e^{-it(H + V1)} \qquad (t \in \boldsymbol{R})$$

where V is an arbitrary real constant.

[1] Indeed, Z_n contains elements of order n and, conversely, if a group G of order n has an element of order n, then $G = Z_n$. This is the case for the direct product (9) only if the primes n_r are mutually different, since if x, y from G, H have orders p, q, respectively, then $(x, y) \in G \times H$ has order equal to the lowest common multiple of p and q.

In the Hilbert space $\mathscr{H}$ of finite dimension n, Hamiltonian time evolutions can be completely classified since each one corresponds to a one-parameter subgroup of the unitary group $U(n)$. Now every such subgroup of $U(n)$ is generated by a Hermitian matrix, hence it is characterized (up to a unitary similarity transformation) by its n real eigenvalues which we denote $E_0, E_1, \ldots, E_{n-1}$. Thus the general form of $L(t)$ is

$$L(t) = Y \exp(-itH_0) Y^{-1},$$

where $Y \in U(n)$, $H_0 = \mathrm{diag}(E_0, E_1, \ldots, E_{n-1})$, and the constant V has been included in H_0.

3.2. Free Hamiltonian. In the following we shall carry on the discrete analogy of the continuous situation in quantum dynamics. The first question which naturally arises concerns the form of Hamiltonian time evolution which could play the role of free evolution. We propose here the discrete analogue of the one-parameter group

$$L(t) = \exp\left(-it\left(\frac{P^2}{2m} + V1\right)\right) \quad (t \in \boldsymbol{R}) \tag{11}$$

describing the Galilean free evolution in the Euclidean space. In this section we shall list the main properties of (11) regarded as evolution operator of a system based on the cyclic group $M = Z_n$.

Let us assume that (11) in fact describes a time evolution in the n-dimensional Hilbert space $\mathscr{H}$, with the operator P defined in Sect. 2.5. Using the spectral decomposition of (11) we can write the action of $L(t)$ on all the eigenvectors f^j

$$L(t) f^j = e^{-it[(p_j^2/2m) + V]} f^j,$$

where $p_j = j (\mathrm{mod}\, n)$. We observe the following properties:

(A) If we restrict V to the values $V = \frac{1}{2m} w$, $w \in \boldsymbol{Z}$, then $L(t)$ is periodic with the period $\tau = 4\pi m$, i.e. $L(\tau)\psi = \psi$ for all $\psi \in \mathscr{H}$.

(B) The ambiguity in the eigenvalues $p_j = j (\mathrm{mod}\, n)$ of P leads to different continuous-time evolutions. This ambiguity can, however, be removed by restricting $t \in \boldsymbol{R}$ to the discrete time values $t_s = \frac{\tau}{n} s$, $s \in \boldsymbol{Z}$, with the time unit $\tau/n = 4\pi m/n$. Then the evolution periodically repeats after every n steps. We shall denote it by

$$T(s) = L(t_s) = e^{-i\frac{2\pi}{n}s(P^2 + w1)}.$$

(C) $s \to T(s)$ is a unitary representation of Z_n, $T(s) = T(1)^s$, $s = 1, \ldots, n-1$, $T(1)^n = T(0) = 1$.

(D) The matrix elements of $T(s)$ in the coordinate representation

$$T(s)_{\varrho\sigma} = \left(e^{\varrho},\ T(s)\, e^{\sigma}\right) = n^{-1} \sum_{j=0}^{n-1} \varepsilon^{-[(j^2+w)s+(\sigma-\varrho)j]} \tag{12}$$

have the property characteristic for a circulant matrix

$$T(s)_{\varrho\sigma} = T(s)_{\varrho+1 (\mathrm{mod}\, n), \sigma+1 (\mathrm{mod}\, n)}.$$

(E) Let us introduce C via $T(1) = \varepsilon^{-w} C$. Then in addition to $A^j B^{\varrho} = \varepsilon^{j\varrho} B^{\varrho} A^j$, the relations

$$A^j C^s = C^s A^j \quad \text{and} \quad B^{\varrho} C^s = \varepsilon^{-\varrho^2 s} A^{2\varrho s} C^s B^{\varrho} \tag{13}$$

hold.

(F) The discrete-time Schrödinger equation for free evolution takes the difference form

$$\psi(s+1) - \psi(s) = \left(T(1) - 1\right) \psi(s).$$

Notice that the time evolution has been made independent of $(\mathrm{mod}\, n)$-ambiguity just by considering only the times t_s. In the next section we shall show the connection of $T(s)$ with the Z_n-analogue of the Galilei group in one dimension.

3.3. Finite Galilei group. In the usual quantum mechanics the solution of the free particle problem is equivalent to the construction of an irreducible unitary representation of the quantal Galilei group G_E. We shall proceed in this way for a system based on $M = Z_n$, and find its free dynamical group.

Let us recall the definition of the quantal Galilei group G_E in one dimension [10]. It is a nilpotent Lie group with the multiplication law

$$(\theta;\ u,\ v,\ t)\,(\theta;\ u',\ v',\ t') = (\theta + \theta' + vu' + \tfrac{1}{2} t' v^2;\ u + u' + t' v,\ v + v',\ t + t').$$

Here θ, u, v and t are real parameters corresponding to the Abelian subgroups of phase transformations, space translations, pure Galilei transformations and time translations, respectively. The group G_E can be written as a semidirect product $G_E = G_1 \otimes G_2$ with $g_1 = (\theta;\ u,\ 0,\ t) \in G_1$, $g_2 = (0;\ 0,\ v,\ 0) \in G_2$, and $(\theta;\ u,\ 0,\ t) \cdot (0;\ 0,\ v,\ 0) = (\theta;\ u,\ v,\ t)$. The irreducible unitary representations of G_E are well known (see e.g. [10]). They are labelled by the invariants $m \in \boldsymbol{R}$ and $V = H - \dfrac{P^2}{2m} \in \boldsymbol{R}$,

$$[U(\theta;\ u,\ v,\ t) f](p) = e^{-i[m\theta + (p^2/2m + V)t - pu]} f(p - mv), \tag{14}$$

where $f \in L^2(\boldsymbol{R}, dp)$.

The finite analogue G_{nE} of G_E is constructed by replacing each of the continuous Abelian subgroups by Z_n. We shall show that

(i) the set of operators $\{\varepsilon^a 1, U(j), V(\varrho), T(s)\}$ generates an irreducible unitary representation of G_{nE} in $\mathscr{H}$, and

(ii) this representation is uniquely determined (within an unphysical constant phase factor) once the Weyl system $(\mathscr{U}, \mathscr{V})$ is given.

The transition to discrete transformations is performed by putting

$$m\theta = \frac{2\pi}{n}a, \qquad u = \frac{2\pi}{n}j, \qquad mv = \varrho, \qquad t = \frac{4\pi m}{n}s, \tag{15}$$

where a, j, ϱ, $s \in Z_n$. In this way we arrive at the finite Galilei group G_{nE} of order n^4 with elements $(a; j, \varrho, s)$ and the multiplication law

$$(a;\, j,\, \varrho,\, s)(a';\, j',\, \varrho',\, s') = (a + a' + \varrho j' + s'\varrho^2;\; j + j' + 2s'\varrho,\; \varrho + \varrho',\; s + s')$$

where all sums are modulo n.

Since $G_{nE} = G_{n1} \otimes G_{n2}$ with $g_1 = (a; j, 0, s) \in G_{n1}$, $g_2 = (0; 0, \varrho, 0) \in G_{n2}$, and $g = (a; j, \varrho, s) = g_1 g_2$, one verifies that

$$U(a;\, j,\, \varrho,\, s) = \varepsilon^{-a}\, U(j)\, T(s)\, V(\varrho)$$

is an irreducible unitary representation of G_{nE} in $\mathscr{H}$. In the momentum representation, this representation is equivalent to the formula obtained from (14) with the aid of (15) and setting $p = k \in Z_n$:

$$[U'(a;\, j,\, \varrho,\, s)\,\psi']_k = \varepsilon^{-a-(j^2+w)s+jk}\, \psi'_{k-\varrho (\mathrm{mod}\ n)}\,.$$

Note that the subgroup of order n^3 formed by all elements $(a; j, \varrho, 0)$ is the discrete Heisenberg group (with the discrete centre Z_n), and that the Weyl system $(\mathscr{U}, \mathscr{V})$ determines its unique irreducible unitary representation in $\mathscr{H}$,

$$U(a;\, j,\, \varrho,\, 0) = \varepsilon^{-a}\, U(j)\, V(\varrho).$$

Given this representation via $U(j) = A^j$ and $V(\varrho) = B^\varrho$ with $AB = \varepsilon BA$, relations

$$AC = CA, \qquad BC = \varepsilon^{-1} A^2 CB \tag{16}$$

determine a unitary C uniquely up to a phase factor $e^{i\alpha}$ (for the proof see Appendix 2). The candidate $\varepsilon^{-a} A^j e^{i\alpha s} C^s B^\varrho$ for $U(a; j, \varrho, s)$ can, however, fulfil the group law of G_{nE} only if $e^{i\alpha} = \varepsilon^{-w}$ for some integer w. This result reminds us of the uniqueness of Nelson's extensions in the continuous case [10].

4. Conclusions

We have shown in this paper that one can formulate a finite space-time lattice analogue of non-relativistic quantum mechanics while preserving the main features of the continuous situation, as embodied in the Galilei dynamical group in one dimension. Our approach was based on the replacement of the continuous Abelian group of configuration space translations by one of the finite Abelian groups. From the transitive group action the unique class of unitarily equivalent, irreducible quantum kinematics was constructed in terms of a transitive imprimitivity system in a finite-dimensional Hilbert space. This method of quantization was developed by Wightman and Mackey, and is closely related to the quantum logic of propositional calculus.

The coordinate and momentum operators arising from this framework then served to define a candidate for the free Hamiltonian evolution operator. We discovered a remarkable fact that this operator emerges naturally and uniquely from the unitary irreducible representation of the finite analogue of the Galilei group over finite space-time lattice. As Appendix 3 shows, the corresponding finite-difference Schrödinger equation turns out to differ substantially from the difference equation with the commonly assumed second order difference operator replacing the Laplacian [2], [11].

Since our approach can accomodate potentials as well as external gauge fields, it can provide a suitable starting point for the approximate solution of the continuous Schrödinger equation. In this connection we found instructive the papers [12] on the numerical solution of the inverse scattering problem, and Montroll's method in molecular and solid state physics [13].

Our results on quantum kinematics may be viewed from a somewhat different angle. Namely, one could apply them to the quantization of classical "spin-up-or-down" systems which represent an important class of models in statistical mechanics [14]. Since the configuration spaces of these systems are essentially $Z_2 \times \ldots \times Z_2$, their Weyl systems take the tensor product form (10) with $A_r B_r + B_r A_r = 0$, generating the two-dimensional representation of the standard Clifford algebra. Thus our results include a uniqueness theorem for anticommutation relations and commutation relations of finite quantum spin systems [15]. Let us mention that the Weyl systems considered in this paper are closely connected with the representations of generalized Clifford algebras [16].

Acknowledgments

One of the authors (J. T.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality extended to him at the International Centre for Theoretical Physics, Trieste, where the work was completed. He also expresses his gratitude to Prof. Dr. H. D. Doebner for his kind hospitality at the Institute for Theoretical Physics, Technical University of Clausthal, where this work was started, and to the Alexander von Humboldt Stiftung for financial support during his stay there. Helpful discussions with Professors H. D. Doebner, D. Finkelstein and T. S. Santhanam are gratefully acknowledged.

Appendix 1

PROPOSITION. *Let the Weyl system* $(\mathcal{U}, \mathcal{V})$ *be given over* $M = Z_n$. *If* $n = n_1^{a_1} \ldots n_f^{a_f}$ *is the unique prime decomposition with mutually distinct primes* $n_r > 1$, *then we have* $\mathcal{U} = \mathcal{U}_1 \otimes \ldots \otimes \mathcal{U}_f$, $\mathcal{V} = \mathcal{V}_1 \otimes \ldots \otimes \mathcal{V}_f$, *where each pair* $(\mathcal{U}_r, \mathcal{V}_r)$ *acts in* $\mathcal{H}_r$ *of dimension* $n_r^{a_r}$.

Proof: Let $n = pr$, $p > 1$, $r > 1$; let $\mathcal{H}_p^{(1)}$, $\mathcal{H}_r^{(2)}$ and $\mathcal{H}_n$ be Hilbert spaces with orthonormal bases $\{e_k^{(1)}\}$, $k = 0, 1, \ldots, p-1$ for $\mathcal{H}_p^{(1)}$, $\{e_l^{(2)}\}$, $l = 0, 1, \ldots, r-1$ for $\mathcal{H}_r^{(2)}$ and $\{e_k^{(1)} \otimes e_l^{(2)}\}$ for $\mathcal{H}_n$; then $\mathcal{H}_n = \mathcal{H}_p^{(1)} \otimes \mathcal{H}_r^{(2)}$. The elements of $\mathcal{H}_n$ are

$$\sum_{k=0}^{p-1} \sum_{l=0}^{r-1} a_{kl} e_k^{(1)} \otimes e_l^{(2)} \quad (a_{kl} \in \boldsymbol{C}).$$

Now, if p, r have no common divisors, then $A = A_p \otimes A_r$, i.e.

$$A e_k^{(1)} \otimes e_l^{(2)} = e_{k-1 (\mathrm{mod}\, p)}^{(1)} \otimes e_{l-1 (\mathrm{mod}\, r)}^{(2)}.$$

This follows from the statement: $A^s e_0^{(1)} \otimes e_0^{(2)}$, $s = 0, 1, \ldots, n-1$, runs just once through all the vectors $e_k^{(1)} \otimes e_l^{(2)}$ if and only if the following sets are the same

$$\{(s (\mathrm{mod}\, p), s (\mathrm{mod}\, r)) | s = 0, 1, \ldots, n-1\} = \{(k, l)|_{l=0,1,\ldots,r-1}^{k=0,1,\ldots,p-1}\}.$$

Obviously the inclusion $\subset$ holds since $n = pr$; it is sufficient to show that the equality $(s (\mathrm{mod}\, p), s (\mathrm{mod}\, r)) = (t (\mathrm{mod}\, p), t (\mathrm{mod}\, r))$ implies $s = t$. Indeed, both p and r are divisible by $|s-t|$ and $|s-t| \in \{0, 1, \ldots, n-1\}$. If p and r have no common divisor, then their lowest common multiple is $pr = n$, hence $|s-t| = 0$. Denoting now $d_s = A^s e_0^{(1)} \otimes e_0^{(2)}$, it follows that $d_0, \ldots, d_{n-1}$ is an orthonormal basis in $\mathcal{H}_n$, and $A d_s = d_{s-1 (\mathrm{mod}\, n)}$ holds.

Appendix 2

Proof that relations (16) determine the unitary matrix C uniquely up to a factor $e^{i\alpha}$, $\alpha \in \boldsymbol{R}$. Since C commutes with the maximal commuting set $\{A^k\}$ it can be written as a linear combination $C = \sum_{k=0}^{n-1} \eta_k A^k$. Then the second equation in (16) takes the form

$$\sum_{k=0}^{n-1} (\varepsilon^{k-1} \eta_{k-2 (\mathrm{mod}\, n)} - \eta_k) A^k = 0,$$

which holds on any element of $\mathcal{H}$ if and only if all coefficients are zero (take e.g. the basis $\{f^j\}$ of $\mathcal{H}$). Now the relation

$$\eta_{k+2 (\mathrm{mod}\, n)} = \varepsilon^{k+1} \eta_k$$

is equivalent to

$$\eta_{2k(\text{mod } n)} = \varepsilon^{(2k-1)+(2k-3)+\ldots+1}\eta_0 = \varepsilon^{k^2}\eta_0$$

and

$$\eta_{2k-1(\text{mod } n)} = \varepsilon^{(2k-2)+(2k-4)+\ldots+2}\eta_1 = \varepsilon^{k(k-1)}\eta_1.$$

Hence the solution is

$$\begin{aligned} &\eta_{2k} = \eta_{n-2k} = \varepsilon^{k^2}\eta_0 && (n \text{ odd}),\\ &\eta_{2k} = 0,\ \eta_{2k+1} = \varepsilon^{k(k+1)}\eta_1 && (n = 2m,\ m \text{ odd}),\\ &\eta_{2k} = \varepsilon^{k^2}\eta_0,\ \eta_{2k+1} = 0 && (n = 2m,\ m \text{ even}). \end{aligned}$$

Since relations (16) imply that C^n commutes with the complete set $\{A^j B^\varrho\}$, C^n is proportional to the unit matrix. By a suitable choice of η_0 (or η_1) $C^n = 1$, hence the eigenvalues $\{\alpha_j\}$ of C belonging to $\{f^j\}$ satisfy $\alpha_j^n = 1$, i.e. α_j's are powers of ε and C is unitary. Relations (16) and unitarity leave only the freedom of multiplying C by $e^{i\alpha}$, $\alpha \in \boldsymbol{R}$. If furthermore $C^n = 1$, $e^{i\alpha}$ is restricted to ε^{-w}, $w \in \boldsymbol{Z}$.

Appendix 3

Time evolution matrices C in the coordinate representation for $n = 2, 3, 4, 5$ and 6:

$n = 2$
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$n = 3$
$$-\frac{i}{\sqrt{3}}\begin{bmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & 1 \end{bmatrix}$$

$n = 4$
$$\frac{1-i}{2}\begin{bmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{bmatrix}$$

$n = 5$
$$\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & \varepsilon^4 & \varepsilon & \varepsilon & \varepsilon^4 \\ \varepsilon^4 & 1 & \varepsilon^4 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon^4 & 1 & \varepsilon^4 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon^4 & 1 & \varepsilon^4 \\ \varepsilon^4 & \varepsilon & \varepsilon & \varepsilon^4 & 1 \end{bmatrix}$$

$n = 6$
$$\frac{i}{\sqrt{3}}\begin{bmatrix} 0 & \varepsilon^4 & 0 & 1 & 0 & \varepsilon^4 \\ \varepsilon^4 & 0 & \varepsilon^4 & 0 & 1 & 0 \\ 0 & \varepsilon^4 & 0 & \varepsilon^4 & 0 & 1 \\ 1 & 0 & \varepsilon^4 & 0 & \varepsilon^4 & 0 \\ 0 & 1 & 0 & \varepsilon^4 & 0 & \varepsilon^4 \\ \varepsilon^4 & 0 & 1 & 0 & \varepsilon^4 & 0 \end{bmatrix}$$

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